Numerische Methoden der Physik

WiSe 2023-2024 – Prof. Marc Wagner

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Exercise sheet 6

To be handed in on 22.11.2023 and discussed on 24.11.2023 and 27.11.2023.

Exercise 1 [*The Poisson equation*] (1+2+3+5+7+2=20 pts.)

Consider a non moving electrically charged particle with negative charge $Q = -q$ in d-dimensional space, which is positioned in the centre of a grounded sphere of radius R. The electrostatic potential ϕ produced by the particle can be calculated solving the Poisson equation

$$\Delta\,\phi(\mathbf{r}) = \frac{q}{\varepsilon_0}\,\delta^{(n)}(\mathbf{r})$$

with the boundary condition $\phi(\mathbf{r}) = 0$ for $|\mathbf{r}| \geqslant R$ (here ε_0 is the vacuum permittivity).

(i) Rewrite the Poisson equation in d dimensions in a way such that only dimensionless quantities are used.

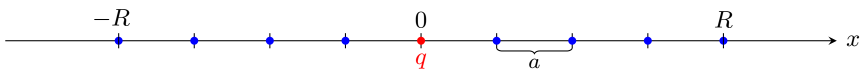
(ii) Calculate the potential $\phi(\mathbf{r})$ analytically for $d = 1$ (a 1-dimensional sphere is a set of two points at positions $x = \pm R$) and for $d = 2$.

(iii) Consider the problem in $d = 1$ dimension. A possibility to solve it numerically is to discretize the domain of the solution (in our case the interval $I \equiv [-R, R]$) introducing a (uniform) lattice and to approximate the derivatives with finite differences. In this way, the differential equation is mapped onto a system of linear equations, which can be solved using standard numerical techniques as discussed in the lecture.

Let us introduce a lattice

$$x \to x_j = j\,a \qquad \text{with} \qquad j \in [-n, n] \subset \mathbb{Z} \qquad \text{and} \qquad a = \frac{R}{n} \quad ,$$

which consists of $2n+1$ lattice points. The potential ϕ at the lattice points $x = \pm R$ is fixed by the boundary conditions, i.e. there are $2n-1$ unknown variables $\phi_j \equiv \phi(x_j)$, which have to be determined by solving a system of linear equations in task (iv).



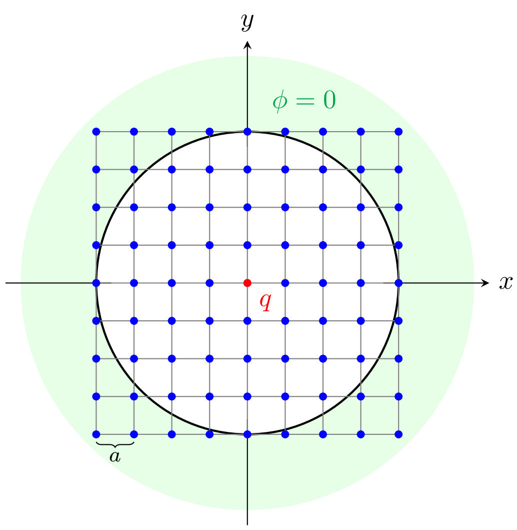
Rewrite the Poisson equation in the form $A_{ij}\phi_j = b_i$, where $\phi_j \equiv \phi(x_j)$ is the unknown potential of the problem discretized on the lattice (now a vector with $2n+1$ components), where A and $\mathbf{b}$ are a matrix and a vector, respectively, which have to be specified before you can start the numerical implementation. Make use of the approximation

$$\frac{\partial^2 \phi_j}{\partial x^2} \approx \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{a^2}$$

and think of how to properly treat the δ-function.

(iv) Solve the obtained system of linear equations with one of the numerical techniques discussed in the lecture (e.g. LU-decomposition), obtaining in this way the electrostatic potential $\phi(x)$. Study the dependence of your result on the parameter n, directly connected to the number of lattice points and compare your solution with the analytical solution.

(v) Consider now the above problem in two dimensions with the particle in the centre of the sphere. Repeat tasks (iii) and (iv) introducing now a $2d$-lattice.



(vi) Still in $d = 2$ dimensions, place the particle at position $(x, y) = (R/2, 0)$ and solve again the Poisson equation. Can this problem still be solved analytically? If not, explain why.