Numerische Methoden der Physik

WiSe 2023-2024 – Prof. Marc Wagner

Michael Eichberg: eichberg@itp.uni-frankfurt.de
Lasse Müller: lmueller@itp.uni-frankfurt.de

Exercise sheet 4

To be handed in on 08.11.2023 and discussed on 10.11.2023 and 13.11.2023.

Exercise 1 [*The Kepler problem*] (2+2+4+7+5=20 pts.)

Consider the so-called Kepler problem, a particular case of a two-body problem, in which the two bodies interact by a central force $\mathbf{F}(\mathbf{r})$ that varies in strength as the inverse square of the distance r between them. It can be shown that this is equivalent to describing the motion of a single body of mass m in a central, attractive ($\alpha > 0$) potential $V(\mathbf{r}) = -\alpha/r$, with $r = |\mathbf{r}|$. Given that the total energy E is negative and after a suitable choice of coordinates, the body trajectory will lie in the (x, y)-plane and it will be an ellipse of semi-major axis a and eccentricity

$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}.$$

(i) Write down the equations of motion. In order to handle them numerically in a convenient way as well as to be able to approach the problem in the most general way (e.g. to avoid to solve the problem for each value of a), it is recommended to make use of dimensionless quantities. Rewrite the equations of motion as

$$\frac{\mathrm{d}^2}{\mathrm{d}\hat{t}^2}\,\hat{\mathbf{r}} = -\frac{\hat{\mathbf{r}}}{\hat{r}^3}\,,$$

where $\hat{\mathbf{r}} \equiv \mathbf{r}/a$ and $\hat{t}$ has been properly defined as dimensionless time.

(ii) Choose the initial conditions

$$\hat{\mathbf{r}}_0 \equiv \hat{\mathbf{r}}|_{\hat{t}=0} \qquad \text{and} \qquad \hat{\mathbf{v}}_0 \equiv \left.\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\hat{t}}\right|_{\hat{t}=0}$$

such that $E < 0$, and to have the body at the perihelion at $\hat{t} = 0$ and moving counter-clockwise on an ellipse of eccentricity e and semi-major axis a. For this, use

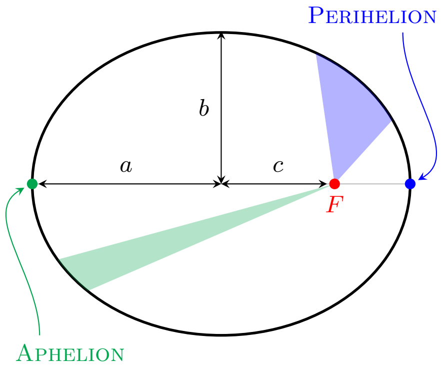
$$E = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{\alpha}{r} \qquad \text{with} \qquad L = m(\mathbf{r} \times \mathbf{p})$$

and consider points on the ellipse where $\dot{r} = 0$.

(iii) Solve the considered initial value problem numerically for the cases $e_1 = 0.1$ and $e_2 = 0.9$, using a fourth order Runge-Kutta method and making use of a uniform time step. Plot the so obtained body trajectory in the $(\hat{x}, \hat{y})$-plane, where $\hat{x} \equiv x/a$ and $\hat{y} \equiv y/a$. Add to your plot the expected elliptical orbit and check if your time step is small enough so that the numerically calculated trajectory qualitatively agrees with it.

(iv) Improve now your program adding the possibility to use an adaptive step size in the Runge-Kutta method[1], using the strategy discussed in the lecture. Repeat task (iii) using using this method. Discuss, how the step size changes along the trajectory during the integration for the two cases $e_1 = 0.1$ and $e_2 = 0.9$.

(v) Plot the $\hat{y}$-coordinate as function of $\hat{t}$. Find a way to calculate the orbital period of the body $\hat{T}$ in your program (just printing the plot and measuring with a ruler is not sufficient). Determine the orbital period of both Earth and Mars by transforming $\hat{T}$ back into a dimensionful physical quantity. Take all necessary physical input (e.g. masses, eccentricity and natural constants) from the internet.



[1]Please, think of an handy method to switch between fixed and adaptive step size, do not hard code it.