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Exercise sheet 2

To be discussed on 27.04.2021 and 28.04.2021

Exercise 1 [*Third order Runge-Kutta method*] (20 pts.)

Consider the differential equation $\dot{y}(t) = f(t, y)$, with f being at least 2-times differentiable. The Runge-Kutta method is a numerical procedure to iteratively obtain an approximate solution for $y(t)$. Show that for a given time t the third-order Runge-Kutta expression for the new time $t + h$ for small h

$$y(t+h) = y(t) + \frac{h}{6}\,(k_1 + 4k_2 + k_3) \qquad \text{with} \qquad \begin{cases} k_1 = f(t,\, y) \\ k_2 = f\Big(t + \dfrac{h}{2},\, y + \dfrac{h}{2}\, k_1\Big) \\ k_3 = f(t + h,\, y - h\, k_1 + 2\, h\, k_2) \end{cases}$$

is equivalent to the Taylor expansion

$$y(t+h) = y(t) + h\,\frac{dy}{dt} + \frac{h^2}{2}\,\frac{d^2y}{dt^2} + \frac{h^3}{6}\,\frac{d^3y}{dt^3} + \mathcal{O}(h^4)\ .$$

Use the Taylor expansion of a function g of two variables (u, v) around a given point (a, b)

$$g(u,v) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(u-a)^n\,(v-b)^m}{n!\,m!} \left(\frac{\partial^{n+m} g}{\partial u^n \partial v^m}\right)_{(u,v)=(a,b)}$$

and consider to use the simplified notation

$$y(t) \equiv y\,, \qquad f(t,y) \equiv f\,, \qquad \frac{\partial f}{\partial t} \equiv f_t \quad \text{and} \quad \frac{\partial f}{\partial y} \equiv f_y\ .$$

For example,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\,\frac{dy}{dt} \equiv f_t + f_y\, f\ .$$