Exercise sheet 3

To be discussed on 18, 21 May and 25 May[1]

Exercise 1 [*The Kepler problem*]

Consider the so-called Kepler problem, a particular case of a two-body problem, in which the two bodies interact by a central force $\mathbf{F}(\mathbf{r})$ that varies in strength as the inverse square of the distance r between them. It can be shown that this is equivalent to describing the motion of a single body of mass m in a central, attractive ($\alpha > 0$) potential $V(\mathbf{r}) = -\alpha/r$, with $r = |\mathbf{r}|$. Given that the total energy E is negative and after a suitable choice of coordinates, the body trajectory will lie in the (x, y)-plane and it will be an ellipse of eccentricity e and semi-major axis a.

First part

(i) Write down the equations of motion. In order to handle them numerically in a convenient way as well as to be able to approach the problem in the most general way (e.g. to avoid to solve the problem for each value of a), it is recommended to make use of dimensionless quantities. Rewrite the equations of motion as

$$\frac{\mathrm{d}^2}{\mathrm{d}\hat{t}^2}\,\hat{\mathbf{r}} = -\frac{\hat{\mathbf{r}}}{\hat{r}^3}\,,$$

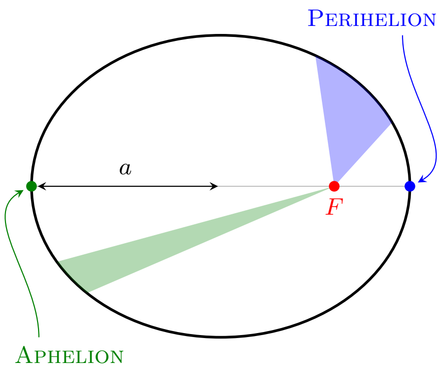
where $\hat{\mathbf{r}} \equiv \mathbf{r}/a$ and $\hat{t}$ has been properly defined as dimensionless time.

(ii) Choose the initial conditions

$$\hat{\mathbf{r}}_0 \equiv \hat{\mathbf{r}}\big|_{\hat{t}=0} \qquad \text{and} \qquad \hat{\mathbf{v}}_0 \equiv \left.\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}\hat{t}}\right|_{\hat{t}=0}$$

such that $E < 0$ and in a way to have the body at the perihelion at $\hat{t} = 0$ and moving counter-clockwise.

(iii) Solve the considered initial value problem numerically, using a fourth order Runge-Kutta method and making use of an uniform time step. Plot the so obtained body trajectory in the $(\hat{x}, \hat{y})$-plane, where $\hat{x} \equiv x/a$ and $\hat{y} \equiv y/a$. In order to qualitatively grasp how small the time step shall be in order to have an accurate result, add to your plot the analytically known positions of the extrema of the minor and major axis of the ellipse. Is the numerically obtained trajectory crossing these points?



(iv) Check numerically the validity of the second law of Kepler.

Second part

(v) Improve now your program adding the possibility to use an adaptive step size in the Runge-Kutta method[2], using the strategy discussed in the lecture. Repeat task (iii) using an adaptive step size.

(vi) Is it possible to think of an alternative way to have an adaptive step size method taking advantage of the fact that the total energy of the system is a conserved quantity?

(vii) Always using an adaptive step size, solve again the initial value problem, imposing the initial condition such to have an ellipse with eccentricity $e_1 = 0.1$ and $e_2 = 0.9$. How does the time step during the integration of the equations of motion along the trajectory in the two cases? Interpret your results.

(viii) Plot the $\hat{y}$-coordinate as function of $\hat{t}$. Think of a way to calculate the orbital period of the body in your program and check the validity of the third law of Kepler.

[1] Due to the public holiday (Pfingstmontag) on 21 May, the second part will be discussed on the Friday 18 May tutorial only.
[2] Please, think of an handy method to switch between fixed and adaptive step size, do not hard code it.