Advanced quantum mechanics

SS 2019 – Prof. Dr. Marc Wagner

Organization: Room GSC 0|21
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Exercise sheet 11

To be handed in 04.07.19 before the lecture.
To be discussed in the week of 08.07.19.
27.06.19

Exercise 1 [*Rapidity*] (2 pts.)

Show that the composition of two boosts with rapidities χ_1 and χ_2 along the same axis is equal to a single boost along this axis with rapidity $\chi = \chi_1 + \chi_2$.

Exercise 2 [*Properties of $SO(3)$, $SU(2)$, $U(1)$ and the Poincare group*] (2+2+3+4=11 pts.)

(a) Show graphically that the group $SO(3)$ is non-Abelian, e.g. draw the effect of a rotation around the x-axis followed by a rotation around the y-axis on a vector, and vice versa.

(b) Show that the matrices $R_j(\alpha) = 1 + i\alpha J_j$ with $J_j = \sigma_j/2$, which were defined in the lecture, are infinitesimal $SU(2)$ matrices.

(c) Explain in simple words, or by a suitable diagram, why the generators of the Poincare group fulfill the relations $[J_x, P_x] = 0$ and $[J_x, P_y] \propto P_z$.

(d) $U(1)$ is the group of unitary 1×1-matrices where the operation is a simple multiplication.

- Show that $U(1)$ fulfills the defining properties of a group.
- Is $U(1)$ an Abelian or a non-Abelian group?
- What are infinitesimal $U(1)$ transformations?
- What are the generators and the algebra of $U(1)$?

Exercise 3 [*Further properties of $SU(2)$*] (3+2+2=7 pts.)

In the lecture it was shown that $SU(2)$-matrices can be written as $g = \exp(i\alpha_j J_j)$ with $J_j = \sigma_j/2$.

(a) Typically for calculations, the equivalent form

$$g = \cos\left(\frac{|\vec{\alpha}|}{2}\right) + i\sin\left(\frac{|\vec{\alpha}|}{2}\right)\frac{\alpha_j}{|\vec{\alpha}|}\sigma_j \tag{1}$$

is more convenient. Show that both expressions for g are indeed equivalent.

(b) A rotation of a spinor $\psi = (\psi_1, \psi_2)$ with 2 components is described by an $SU(2)$-matrix. Explain why a 720° rotation, instead of only a 360° rotation, is needed to transform such a spinor into itself. Show that this is not the case for the expectation value of the spin $\psi^\dagger(\vec{\sigma}/2)\psi$.

(c) Show that an element of $SU(2)$ can be written as $g = g_0 + i g_j \sigma_j$, where $(g_0)^2 + \sum_j (g_j)^2 = 1$. Relate g_0 and g_j to the parameters α_j in eq. (1). What does this imply for the geometrical form of the parameter space of $SU(2)$?