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Final Project: Orbits around black holes

1 Introduction

General Relativity is currently the most accepted description of gravitational phenomena. This theory, which identifies gravity as a geometrical feature of space and time, provides an elegant unification of concepts such as space and time and energy and momentum, but most importantly, all the observational and experimental tests performed so far have confirmed its predictions, from explaining Mercury's perihelion precession, recognized as a problem in 1859 by Le Verrier, to the detection of gravitational radiation in 2015 by the LIGO collaboration.

One of the predictions of General Relativity is the existence of black holes. These can be roughly described as collapsed stars with a gravitational field so strong that not even light can escape from them; however, the physics in the vicinity of these astrophysical bodies is much richer (and counter intuitive) than what this description lets us see.

Though indirect evidence of the existence of black holes was known since the last decades of the XXth century, the first direct confirmation of their existence was the detection of gravitational radiation from a merger of two black holes in September 2015.

There is indirect evidence that, in addition to stellar-mass black holes originated from the collapse of some types of stars, supermassive black holes of $10^6 - 10^{10}$ solar masses are located at the center of galaxies. There is currently an ongoing effort by the Event Horizon Telescope Collaboration (EHTC) to observe for the first time the shadow of the black hole at the center of our own Galaxy.

In this project you will build a code that will allow you to explore the behavior of light and matter in the vicinity of a black hole, by simulating the trajectories of massive particles, which can represent planets or stars depending on the size of the black hole, as well as the deflection of light rays around this bodies.

Note on the units: In computer simulations of physical systems it is common to work using dimensionless units. This allows all the quantities to be of order unity, thus minimizing the effect of finite floating point precision. Appropriate dimensionless units can be obtained by dividing physical units by the relevant scales in the problem, and quantities in physical units can be recovered by multiplying back by the corresponding factors. In particular, in General Relativity it is very convenient to work using the *geometrized system of units*. In these units, once the length unit has been chosen, code time and mass are related to the physical ones by $t = ct_{\rm phys}$ and $M = GM_{\rm phys}/c^2$, where c is the speed of light and G is the gravitation constant. All the formulas presented here are already written in code units. Though this choice is left to you, we strongly advise to choose the length unit so that the mass of the black hole is $M = 1$.

2 The equations

The motion of a particle in the vicinity of a spherically symmetric (Schwarzschild) black hole of mass M is described by the Lagrangian[1]

[1] In General Relativity, particles in absence of forces follow curves of extremal length, called *geodesics*. The Lagrangian (1) is therefore nothing more than the square of the line element of the *Schwarzschild spacetime* (see https://en.wikipedia.org/wiki/Schwarzschild_metric).

$$L = g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2 \tag{1}$$

where r, θ and ϕ are spatial coordinates similar to the spherical coordinates of flat spacetime and t is the time coordinate which corresponds to the time measured by observers at rest infinitely far away from the black hole. The dot denotes derivation to a parameter τ, which for massive particles can be interpreted as the proper time, i.e., the time measured by a clock moving with the particle. The functions $g_{\mu\nu}$ can be written explicitly as

$$g_{tt} = -\left(1 - \frac{2M}{r}\right)\,, \quad g_{rr} = \left(1 - \frac{2M}{r}\right)\,, \quad g_{\theta\theta} = r^2 \quad \text{and} \quad g_{\phi\phi} = r^2 \sin^2\theta\,. \tag{2}$$

Due to spherical symmetry, the motion of the particle will be confined to a plane, and coordinates can be adapted to make it coincide with the equatorial plane. Therefore, $\dot{\theta} = 0$ and $g_{\phi\phi} = r^2$. Symmetry with respect to translations in time and rotations respect to the polar axis yield two conserved quantities, namely energy $\tilde{E}$ and angular momentum $\tilde{p}_\phi$.

The Lagrangian (1) gives the equations of motion

$$\begin{aligned}
\frac{dt}{d\tau} &= \left(1 - \frac{2M}{r}\right)^{-1} E \\
\frac{dr}{d\tau} &= \left(1 - \frac{2M}{r}\right) p_r \\
\frac{d\phi}{d\tau} &= \frac{p_\phi}{r^2} \\
\frac{dp_r}{d\tau} &= \frac{p_\phi^2}{r^3} - \frac{M}{r^2}\left[\left(1 - \frac{2M}{r}\right)^{-2} E^2 + p_r^2\right].
\end{aligned} \tag{3}$$

Where the meaning of E, p_r and p_ϕ depends on whether the particles are massive or massless. In addition to be a solution of equations (3), to represent a physical trajectory, the motion of particles with mass $m \neq 0$ must satisfy at all times the normalization condition

$$g_{tt}{}^{-1}(E)^2 + g_{rr}{}^{-1}(p_r)^2 + g_{\phi\phi}{}^{-1}(p_\phi)^2 = -1\,, \tag{4}$$

where $E = \tilde{E}/m$ is the *specific* energy and $p_r = \tilde{p}_r/m$ and $p_\phi = \tilde{p}_\phi/m$ are the *specific* radial and angular momenta. Conversely, the motion of massless particles such as photons must satisfy

$$g_{tt}{}^{-1}(E)^2 + g_{rr}{}^{-1}(p_r)^2 + g_{\phi\phi}{}^{-1}(p_\phi)^2 = 0\,, \tag{5}$$

where E, p_r and p_ϕ are the energy and the radial and angular momenta of the particle. As it will be explained in the next section, the parameters E and p_ϕ determine completely the shape of the orbit. [2]

[2]For further information on the orbits in the Schwarzschild spacetime, see https://en.wikipedia.org/wiki/Schwarzschild_geodesics. A very pedagogical explanation can be found in Section 11.1 of Bernard Schutz's *A First Course in General Relativity*, though it requires some basic knowledge of General Relativity. For more advanced readers, Sections 5 and 6 of the classic textbook *Gravitation*, by Misner, Thorne & Wheeler contain a more complete and rigorous discussion.

3 The physics: exploring the parameter space

For massive particles, the normalization condition (4) can be used to derive a very useful quantity for the qualitative study of orbits: the effective potential $V(r)$. Rewriting (4), we obtain

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - V(r)^2 \Rightarrow V(r)^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{p_\phi^2}{r^2}\right) . \tag{6}$$

As an example, a plot of this quantity for $M = 1$, $p_\phi = 4.1$ is shown in Figure 1. For this choice of parameters, the effective potential posses a local maximum $V^2_{\rm max}$ and a local minimum $V^2_{\rm min}$. It is important to notice that an orbit with $E^2 < V^2(r)$ is not physical.

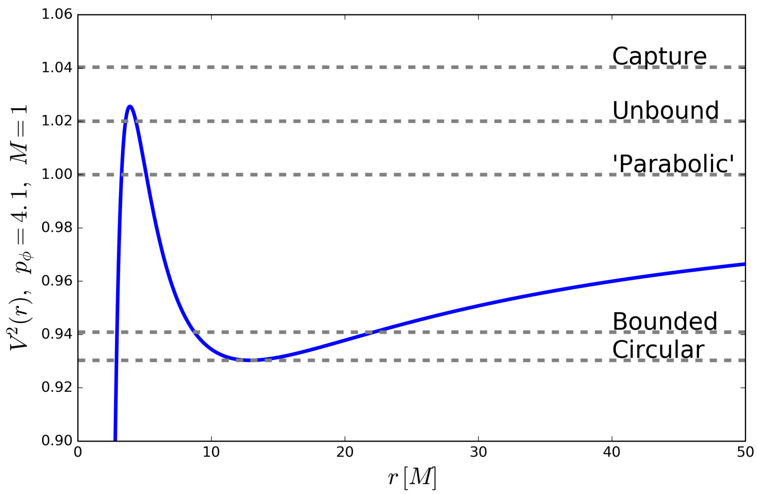

Figure 1: Effective potential for the parameters $M = 1$, $p_\phi = 4.1$ and types of orbits according to the energy.

The qualitative features of the orbit will depend on the relation between E^2 and V^2.

- If $E^2 = V^2_{\rm min}$, the only possible position of the particle will be the circle with the radius that minimizes V^2.
- For $V^2_{\rm min} < E^2 < 1$, the orbit will be bounded by two radii.
- For $E^2 = 1$, the particle will perform an unbound orbit and come to rest at infinity. Even though the trajectory is not a parabola, this type of orbit is called 'parabolic'.
- For $1 < E^2 < V^2_{\rm max}$, the particle will move on unbound orbits. An unstable circular orbit occurs at $E^2 = V^2_{\rm max}$.
- For $E^2 > V^2_{\rm max}$ a particle coming from infinity will surpass the potential barrier and will be captured by the black hole.

For massless particles, the normalization gives a different effective potential:

$$V^2(r) = \frac{p_\phi^2}{r^2}\left(1 - \frac{2M}{r}\right) . \tag{7}$$

In this case, $V^2(r)$ has no local minima, and only unbound and capture orbits exist, being the only exception an unstable circular orbit at $r = 3\,M$ which is call the *light ring*.

4 Numerical integration of ordinary differential equations

To find the trajectory of the particle in spacetime, we have to integrate equations (3) with respect to the parameter τ. We then must face the problem of integrating ordinary differential equations (ODEs) numerically. From the definition of derivative,

$$\dot{y} = \frac{dy}{d\tau} = \lim_{\Delta\tau \to 0} \frac{y(\tau + \Delta\tau) - y(\tau)}{\Delta\tau} \tag{8}$$

If our ODE states that $\dot{y} = g(y, \tau)$ (g is said to be the *right-hand-side* of the equation), and we define $y_i = y(\tau_i)$ and $y_{i+1} = y(\tau_i + \Delta\tau)$ then we can take a finite approximation to the limit of equation (8) and write

$$y_{i+1} \approx y_i + \Delta\tau \, g(y_i, \tau_i) \, . \tag{9}$$

This allows us to obtain an approximation of the value of y_i at τ_{i+1} from y_i and $g(y_i, \tau_i)$. As you might think, the approximation improves as $\Delta\tau$ becomes small. The iterative application of equation (9) to find successive values of $y(\tau)$ at $\tau_0 + \Delta\tau$, $\tau_0 + 2\Delta\tau$, etc. is known as the *Euler method*. Unfortunately the price to pay for its simplicity is a rapid growth of numerical errors, which makes it useless for any practical application.

An alternative that is still very easy to implement and at the same time accurate enough for many applications is the *fourth-order Runge-Kutta method* (RK4) [3]. In this method, the evaluation of $g(y_i, \tau_i)$ in equation (9) is replaced by a weighted average of different evaluations of the right-hand-side.

$$\bar{g}(y_i, \tau_i) = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \tag{10}$$

where

$$\begin{aligned}
k_1 &= g\left(y_i, \tau_i\right) \\
k_2 &= g\left(y_i + \frac{1}{2}k_1\Delta\tau, \tau_i + \frac{1}{2}\Delta\tau\right) \\
k_3 &= g\left(y_i + \frac{1}{2}k_2\Delta\tau, \tau_i + \frac{1}{2}\Delta\tau\right) \\
k_4 &= g\left(y_i + k_3\Delta\tau, \tau_i + \Delta\tau\right)
\end{aligned} \tag{11}$$

When applying this method to the solution of a system of ODEs, it is important to keep in mind that all of the variables must be updated at the simultaneously and the function evaluations at all fractional time steps must be consistent.

5 Implementation

The work of this project can be separated in two parts. The first one consists on coding an integrator for systems of ODEs and the second on using it to solve the actual physics problem. Building the code as a sequence of steps as independent as possible that can be tested successively can be helpful to avoid wasting time trying to find the source of potential problems. A possible work plan for the completion of the project is described in what follows.

[3] An excellent reference for commonly used methods for numerical integration of ordinary differential equations is Chapter 17 of *Numerical Recipes, Third Edition* by Press, Teukolsky, Vetterling and Flannery. Section 17.1 is specifically concerned with the RK4 method. The interested student is encouraged to implement also the Runge-Kutta algorithm with adaptive stepsize presented in Section 17.2

- As a warm-up exercise, try to build a program to integrate the ODE $\dot{y} = -y$ using the two methods described above and compare the results with the analytical solution.
- Though it is necessary to advance the state by small time steps to keep the solution accurate, writing the data to a file every time step can slow down the execution and create data files heavier than is necessary. Add a condition to decide when to dump data to the output.
- Now you have a RK4 integrator that works for a single variable. However, in the problem that concerns this project, the state is described by several variables. Generalize the function that advances one step in order to allow it to receive all the necessary variables as input and return all the necessary variables as output. Make it flexible so that it can accept arbitrary right-hand-sides for the ODEs. *Hint:* Remember the lectures about arrays and pointers.
- Now it is time to consider the physics. Think of which quantities will be the parameters that you will give as input to your code. Then, ensure that the initial state is physically possible, i.e., that it satisfies the normalization conditions (4) or (5).
- Now you need to add conditions to stop the execution of the code. This is necessary in order to avoid wasting computational resources e.g., when the evolution is no longer interesting (the particle is gone too far away, it has fallen through the black hole horizon at $r = 2M$ or the state is no longer physical), or when a given number of iterations has been exceeded. Some of this conditions are more expensive to evaluate and you might not want to do it at every single step.
- Print the normalization conditions to see to which extent they are fulfilled during evolution. This can give you an idea of how trustworthy is the solution.

6 Using the code

- Try different parameters in your code to obtain at least one example of each type of orbit for massive particles (except the unstable one).
- For a quasi-circular orbit, test the formula for the periastron precession. Unlike Newtonian bound orbits, which in absence of perturbations are perfect ellipses, in General Relativity successive periastra (points of closest approach of the particle to the star or black hole), differ by an angle given by

$$\Delta\phi = 2\pi(1 - 6\,M/r_0)^{-1/2}\,, \tag{12}$$

  where r_0 is the radius of the circular orbit.
- Compare the proper time τ of an observer moving with the particle to that measured by an observer at rest at infinity, t. For whom time runs faster?
- Change the normalization condition in order to simulate photon trajectories. Launch photons at different directions from the same point. Check the validity of Synge's formula for the angular size of the black hole (its *shadow*) in the sky of an observer at rest at $r = r_O$. The angular radius α of the shadow should be given by

$$\sin^2\alpha = \frac{27}{4}\left(\frac{2M}{r_O}\right)^3\left(\frac{r_O}{2M} - 1\right)\,. \tag{13}$$