UNRUH EFFECT

Marc Wagner
mcwagner@theorie3.physik.uni-erlangen.de
http://theorie3.physik.uni-erlangen.de/∼mcwagner
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Outline

- Introduction
- Accelerated motion in Minkowski spacetime
- Coordinates of an accelerated frame
- Rindler spacetime
- Massless scalar field in Minkowski and in Rindler spacetime
- Lightcone coordinates
- Bogolyubov transformation (relation between creation and annihilation operators in Minkowski and in Rindler spacetime)
- Particle numbers of the Minkowski vacuum in an accelerated frame
- Unruh temperature

Introduction

Literature

- V. F. Mukhanov and S. Winitzki, *Introduction to Quantum Fields in Classical Backgrounds*, 2005.
  `www.theorie.physik.uni-muenchen.de/∼serge/T6/`

What is the Unruh effect?

- Vacuum state in Minkowski spacetime: state where no particles are present; lowest energy eigenstate.
- Unruh effect: an accelerated observer moving through the Minkowski vacuum detects particles (Minkowski vacuum and vacuum in the frame of the accelerated observer are different quantum states).
- In this talk: observer moves in 1+1 dimensions with constant acceleration.

Accelerated motion (1)

Three different frames

- Laboratory frame (inertial frame; coordinates $x^\mu = (t, x)$): the usual inertial frame.
- Accelerated frame or proper frame (non inertial frame; coordinates (τ, ξ)): the frame where the accelerated observer is at rest.
- Comoving frames (inertial frames; coordinates $x'^\mu = (t', x')$): frames where the accelerated observer is momentarily at rest.

Accelerated motion (2)

Constant acceleration (i)

- Constant acceleration: constant four acceleration in the comoving frames.
- Any inertial frame:

$$u_\mu u^\mu \;=\; 1$$
$$\rightarrow \quad 0 \;=\; \frac{d}{d\tau}\left(u_\mu u^\mu\right) \;=\; 2u_\mu \frac{d}{d\tau}u^\mu \;=\; 2u_\mu a^\mu. \quad (1)$$

- Comoving frame (at that time when the accelerated observer is at rest):

$$u'_\mu \;=\; (1,0) \quad \rightarrow \quad a'^\mu \;=\; (0,A). \quad (2)$$

Accelerated motion (3)

Constant acceleration (ii)

- A is the ordinary three acceleration a' in the comoving frame (at the time when the accelerated observer is at rest):

$$u'^0 \;=\; \frac{dt'}{d\tau} \quad \rightarrow \quad \frac{d}{d\tau} \;=\; \frac{dt'}{d\tau}\frac{d}{dt'} \;=\; u'^0 \frac{d}{dt'} \quad (3)$$

$$A \;=\; \frac{d^2}{d\tau^2}x' \;=\; \frac{d}{d\tau}\left(u'^0 \frac{d}{dt'}x'\right) \;=\; \frac{d}{d\tau}\left(u'^0 v'\right) \;=$$
$$=\; \left(\frac{d}{d\tau}u'^0\right)v' + u'^0\left(\frac{d}{d\tau}v'\right) \;=$$
$$=\; a'^0 v' + \left(u'^0\right)^2\left(\frac{d}{dt'}v'\right) \;=\; a'. \quad (4)$$

- Constant three acceleration in the laboratory frame (or any inertial frame) is impossible. $a = dv/dt =$ constant would imply that the observer can move faster than the speed of light.

Accelerated motion (4)

Trajectory of the accelerated observer

- Two equations:

$$\left(u^0\right)^2 - \left(u^1\right)^2 \;=\; u_\mu u^\mu \;=\; 1 \quad (5)$$

$$\left(\frac{d}{d\tau}u^0\right)^2 - \left(\frac{d}{d\tau}u^1\right)^2 \;=\; a_\mu a^\mu \;=\; a'_\mu a'^\mu \;=\; -a'^2. \quad (6)$$

- Solution:

$$u^0 \;=\; \cosh(a'\tau) \quad , \quad u^1 \;=\; \sinh(a'\tau). \quad (7)$$

- Trajectory (initial conditions $x^\mu(0) = (0, 1/a')$, $u^\mu(0) = (1,0)$, e. g. at $t = \tau = 0$ the particle is at rest at $x = 1/a'$):

$$t \;=\; \frac{1}{a'}\sinh(a'\tau) \quad , \quad x \;=\; \frac{1}{a'}\cosh(a'\tau). \quad (8)$$

Accelerated frame (coordinates) (1)

- To describe quantum fields in the accelerated frame and to compare them with quantum fields in the laboratory frame we need coordinates (τ, ξ) in the accelerated frame and transformation laws $t = t(\tau, \xi)$ and $x = x(\tau, \xi)$.
- $\tau =$ proper time of the observer (or anybody moving along the trajectory $\xi = 0$).
- $\xi =$ spatial distance from the observer at $\xi = 0$.
- Consider a measuring stick of length ξ_0 in the accelerated frame. In the current comoving frame it is represented by the four vector $s'^\mu = (0, \xi_0)$ (the measuring stick is momentarily at rest in the current comoving frame).
- Four vector of the measuring stick in the laboratory system:

$$s^\mu \;=\; \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \xi_0 \end{pmatrix} \;=$$
$$=\; \begin{pmatrix} u^0 & u^1 \\ u^1 & u^0 \end{pmatrix}\begin{pmatrix} 0 \\ \xi_0 \end{pmatrix} \;=\; \begin{pmatrix} u^1\xi_0 \\ u^0\xi_0 \end{pmatrix}. \quad (9)$$

Accelerated frame (coordinates) (2)

- The far end of the measuring stick has proper coordinates (τ, ξ_0). From that, (8) and (9) the transformation law between laboratory coordinates (t, x) and proper coordinates (τ, ξ) can be derived:

$$t = \frac{1}{a'}\sinh(a'\tau) + u^1\xi = \frac{1+a'\xi}{a'}\sinh(a'\tau) \quad (10)$$

$$x = \frac{1}{a'}\cosh(a'\tau) + u^0\xi = \frac{1+a'\xi}{a'}\cosh(a'\tau). \quad (11)$$

- Inverse transformation law:

$$\tau = \frac{1}{2a'}\ln\left(\frac{x+t}{x-t}\right) \quad , \quad \tau \in (-\infty, \infty) \quad (12)$$

$$\xi = \sqrt{x^2 - t^2} - \frac{1}{a'} \quad , \quad \xi \in [-1/a', \infty). \quad (13)$$

Rindler spacetime (1)

- (10) and (11):

$$dt = \frac{dt}{d\tau}d\tau + \frac{dt}{d\xi}d\xi = $$
$$= (1+a'\xi)\cosh(a'\tau)d\tau + \sinh(a'\tau)d\xi \quad (14)$$

$$dx = \frac{dx}{d\tau}d\tau + \frac{dx}{d\xi}d\xi = $$
$$= (1+a'\xi)\sinh(a'\tau)d\tau + \cosh(a'\tau)d\xi. \quad (15)$$

- Rindler spacetime:

$$ds^2 = dt^2 - dx^2 = (1+a'\xi)^2 d\tau^2 - d\xi^2. \quad (16)$$

Rindler spacetime (2)

Conformally flat Rindler spacetime (i)

- Quantising fields in conformally flat spacetime in 1+1 dimensions is as easy as quantising fields in Minkowski spacetime.
- To get a conformally flat metric we need a coordinate transformation $\xi = \xi(\tilde{\xi})$ with

$$d\xi = (1+a'\xi)d\tilde{\xi}. \quad (17)$$

- Separation of variables yields

$$\tilde{\xi} = \int d\xi \frac{1}{1+a'\xi} = \frac{1}{a'}\ln(1+a'\xi) \quad ,$$
$$\tilde{\xi} \in (-\infty, \infty). \quad (18)$$

- This is a rescaling of the spatial coordinate ξ. $\tilde{\xi}$ is not the spatial distance but parameterises the spatial distance ξ.
- Conformally flat Rindler spacetime:

$$ds^2 = e^{2a'\tilde{\xi}}\left(d\tau^2 - d\tilde{\xi}^2\right). \quad (19)$$

Rindler spacetime (3)

Conformally flat Rindler spacetime (ii)

- Transformation law between laboratory coordinates (t, x) and conformally flat Rindler coordinates $(\tau, \tilde{\xi})$:

$$t = \frac{e^{a'\tilde{\xi}}}{a'}\sinh(a'\tau) \quad (20)$$

$$x = \frac{e^{a'\tilde{\xi}}}{a'}\cosh(a'\tau). \quad (21)$$

Massless scalar field (1)

- Action of a minimally coupled massless scalar field:

$$S[\phi] \quad = \quad \int d^2x \, \sqrt{-g}\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi). \qquad (22)$$

- Laboratory frame (Minkowski spacetime):

$$S[\phi] \quad = \quad \int dt \, dx \, \frac{1}{2}\Big((\partial_t\phi)^2 - (\partial_x\phi)^2\Big). \qquad (23)$$

- Accelerated frame (conformally flat Rindler spacetime; $\sqrt{-g} = e^{2a'\tilde{\xi}}$, $g^{\mu\nu} = \mathrm{diag}(e^{-2a'\tilde{\xi}}, e^{-2a'\tilde{\xi}})$):

$$S[\phi] \quad = \quad \int d\tau \, d\tilde{\xi} \, \frac{1}{2}\left((\partial_\tau\phi)^2 - \Big(\partial_{\tilde{\xi}}\phi\Big)^2\right). \qquad (24)$$

- In 1+1 dimensions minimal coupling is equivalent to conformal coupling. Therefore the action in conformally flat Rindler spacetime is identical to the action in Minkowski spacetime. Quantising the field ϕ in conformally flat Rindler spacetime is therefore as easy as in Minkowski spacetime.

Massless scalar field (2)

Quantisation in Minkowski spacetime

- Field operator in laboratory coordinates ($a(k)$ = annihilation operators, $a^\dagger(k)$ = creation operators):

$$\phi(t,x) \quad = \quad \frac{1}{\sqrt{2\pi}}\int dk \, \frac{1}{\sqrt{2|k|}} \Big(e^{-i|k|t+ikx}a(k) + e^{i|k|t-ikx}a^\dagger(k)\Big). \qquad (25)$$

- Minkowski vacuum:

$$a(k)|0_M\rangle \quad = \quad 0. \qquad (26)$$

- Expectation values of certain operators, e. g. H (energy), P (momentum), $T^{\mu\nu}$ (energy momentum tensor, i. e. energy and momentum density), allow a physical interpretation of a-particle states.
- Example: $a^\dagger(k)|0_M\rangle$ represents a particle with definite momentum k.

Massless scalar field (3)

Quantisation in Rindler spacetime

- Field operator in conformally flat Rindler coordinates ($b(k)$ = annihilation operators, $b^\dagger(k)$ = creation operators):

$$\phi(\tau,\tilde{\xi}) \quad = \quad \frac{1}{\sqrt{2\pi}}\int dk \, \frac{1}{\sqrt{2|k|}} \Big(e^{-i|k|\tau+ik\tilde{\xi}}b(k) + e^{i|k|\tau-ik\tilde{\xi}}b^\dagger(k)\Big). \qquad (27)$$

- Rindler vacuum:

$$b(k)|0_R\rangle \quad = \quad 0. \qquad (28)$$

- Physical interpretation of b-particle states: analogous to physical interpretation of a-particle states (the action which is identical in both cases determines the physical meaning of all quantum states).

Massless scalar field (4)

- The field operators represent the same quantum field, i. e. $\phi(t,x) = \phi(\tau,\tilde{\xi})$.
- The creation and annihilation operators $a(k)$, $a^\dagger(k)$ and $b(k)$, $b^\dagger(k)$ are different, i. e. they create or annihilate different field excitations.
- Therefore the Minkowski vacuum $|0_M\rangle$ and the Rindler vacuum $|0_R\rangle$ are different quantum states.

Lightcone coordinates

- Get the relation between $a(k)$, $a^\dagger(k)$ and $b(k)$, $b^\dagger(k)$ by comparing the left and right hand side of

$$\phi(t,x) \quad = \quad \phi(\tau,\tilde{\xi}). \tag{29}$$

- Lightcone coordinates will simplify this procedure:

$$u \quad = \quad t-x \quad , \quad v \quad = \quad t+x \tag{30}$$

$$\tilde{u} \quad = \quad \tau-\tilde{\xi} \quad , \quad \tilde{v} \quad = \quad \tau+\tilde{\xi}. \tag{31}$$

- Relation between (u,v) and $(\tilde{u},\tilde{v})$:

$$\begin{aligned} u \quad &= \quad t-x \quad = \quad \frac{e^{a'\tilde{\xi}}}{a'}\sinh(a'\tau) - \frac{e^{a'\tilde{\xi}}}{a'}\cosh(a'\tau) \quad = \\ &= \quad -\frac{1}{a'}e^{a'(\tilde{\xi}-\tau)} \quad = \quad -\frac{1}{a'}e^{-a'\tilde{u}} \end{aligned} \tag{32}$$

$$\begin{aligned} v \quad &= \quad t+x \quad = \quad \frac{e^{a'\tilde{\xi}}}{a'}\sinh(a'\tau) + \frac{e^{a'\tilde{\xi}}}{a'}\cosh(a'\tau) \quad = \\ &= \quad \frac{1}{a'}e^{a'(\tilde{\xi}+\tau)} \quad = \quad \frac{1}{a'}e^{a'\tilde{v}} \end{aligned} \tag{33}$$

- Lightcone coordinates do not mix: $u = u(\tilde{u})$, $v = v(\tilde{v})$.

Bogolyubov transformation (1)

- Field operator in (u,v)-coordinates:

$$\begin{aligned} \phi(u,v) \quad &= \\ &= \quad \frac{1}{\sqrt{2\pi}}\int dk\, \frac{1}{\sqrt{2|k|}} \\ &\quad \left(e^{-i|k|t+ikx}a(k) + e^{i|k|t-ikx}a^\dagger(k)\right) \quad = \\ &= \quad \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega(t-x)}a(\omega) + e^{i\omega(t-x)}a^\dagger(\omega)\right) + \\ &\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 d\omega\, \frac{1}{\sqrt{-2\omega}} \\ &\quad \left(e^{i\omega(t+x)}a(\omega) + e^{-i\omega(t+x)}a^\dagger(\omega)\right) \quad = \\ &= \quad \frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}}\Big(e^{-i\omega u}a(\omega) + e^{i\omega u}a^\dagger(\omega) + \\ &\quad e^{-i\omega v}a(-\omega) + e^{i\omega v}a^\dagger(-\omega)\Big) \quad = \\ &= \quad A(u) + B(v). \end{aligned} \tag{34}$$

- Advantage: $\phi(u,v)$ now is a sum of a u-dependent part and a v-dependent part.

Bogolyubov transformation (2)

- Field operator in $(\tilde{u},\tilde{v})$-coordinates:

$$\begin{aligned} \phi(\tilde{u},\tilde{v}) \quad &= \\ &= \quad \frac{1}{\sqrt{2\pi}}\int_0^\infty d\Omega\, \frac{1}{\sqrt{2\Omega}}\Big(e^{-i\Omega\tilde{u}}b(\Omega) + e^{i\Omega\tilde{u}}b^\dagger(\Omega) + \\ &\quad e^{-i\Omega\tilde{v}}b(-\Omega) + e^{i\Omega\tilde{v}}b^\dagger(-\Omega)\Big) \quad = \\ &= \quad P(\tilde{u}) + Q(\tilde{v}). \end{aligned} \tag{35}$$

- The u-dependent parts of (34) and (35) must be equal:

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega u(\tilde{u})}a(\omega) + e^{i\omega u(\tilde{u})}a^\dagger(\omega)\right) \quad = \\ &= \quad \frac{1}{\sqrt{2\pi}}\int_0^\infty d\Omega\, \frac{1}{\sqrt{2\Omega}}\left(e^{-i\Omega\tilde{u}}b(\Omega) + e^{i\Omega\tilde{u}}b^\dagger(\Omega)\right). \end{aligned} \tag{36}$$

- Can be solved for $b(\Omega)$ and $b^\dagger(\Omega)$ by performing a Fourier transformation on both sides:

$$\frac{1}{\sqrt{2\pi}}\int d\tilde{u}\, e^{i\Omega\tilde{u}} \ldots. \tag{37}$$

Bogolyubov transformation (3)

- Right hand side:

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int d\tilde{u}\, e^{i\Omega\tilde{u}}\frac{1}{\sqrt{2\pi}}\int_0^\infty d\Omega'\, \frac{1}{\sqrt{2\Omega'}} \\ &\quad \left(e^{-i\Omega'\tilde{u}}b(\Omega') + e^{i\Omega'\tilde{u}}b^\dagger(\Omega')\right) \quad = \\ &= \quad \int_0^\infty d\Omega'\, \frac{1}{\sqrt{2\Omega'}}\frac{1}{2\pi}\int d\tilde{u}\, e^{i\Omega\tilde{u}} \\ &\quad \left(e^{-i\Omega'\tilde{u}}b(\Omega') + e^{i\Omega'\tilde{u}}b^\dagger(\Omega')\right) \quad = \\ &= \quad \int_0^\infty d\Omega'\, \frac{1}{\sqrt{2\Omega'}}\left(\delta(\Omega-\Omega')b(\Omega') + \delta(\Omega+\Omega')b^\dagger(\Omega')\right) \quad = \\ &= \quad \left\{\begin{array}{rl} b(\Omega)/\sqrt{2\Omega} & \text{for } \Omega > 0 \\ b^\dagger(-\Omega)/\sqrt{-2\Omega} & \text{for } \Omega < 0 \end{array}\right. . \end{aligned} \tag{38}$$

Bogolyubov transformation (4)

- Left hand side:

$$\frac{1}{\sqrt{2\pi}} \int d\tilde{u}\, e^{i\Omega\tilde{u}} \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}} \Big(e^{-i\omega u(\tilde{u})} a(\omega) + e^{i\omega u(\tilde{u})} a^\dagger(\omega)\Big) =$$
$$= \int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}} \bigg(a(\omega) \int d\tilde{u}\, \frac{1}{2\pi} e^{i\Omega\tilde{u} - i\omega u(\tilde{u})} + a^\dagger(\omega) \int d\tilde{u}\, \frac{1}{2\pi} e^{i\Omega\tilde{u} + i\omega u(\tilde{u})}\bigg) =$$
$$= \int_0^\infty d\omega\, \frac{1}{\sqrt{2\omega}} \bigg(a(\omega) \int d\tilde{u}\, \frac{1}{2\pi} \exp\bigg(i\Omega\tilde{u} + i\omega\frac{1}{a'} e^{-a'\tilde{u}}\bigg) + a^\dagger(\omega) \int d\tilde{u}\, \frac{1}{2\pi} \exp\bigg(i\Omega\tilde{u} - i\omega\frac{1}{a'} e^{-a'\tilde{u}}\bigg)\bigg). \quad (39)$$

Bogolyubov transformation (5)

- Result ($\Omega > 0$):

$$b(\Omega) = \int_0^\infty d\omega\, \Big(\alpha_{\omega,\Omega} a(\omega) + \beta_{\omega,\Omega} a^\dagger(\omega)\Big) \quad (40)$$
$$\alpha_{\omega\Omega} = \sqrt{\frac{|\Omega|}{\omega}} \int d\tilde{u}\, \frac{1}{2\pi} \exp\bigg(i\Omega\tilde{u} + i\omega\frac{1}{a'} e^{-a'\tilde{u}}\bigg) \quad (41)$$
$$\beta_{\omega\Omega} = \sqrt{\frac{|\Omega|}{\omega}} \int d\tilde{u}\, \frac{1}{2\pi} \exp\bigg(i\Omega\tilde{u} - i\omega\frac{1}{a'} e^{-a'\tilde{u}}\bigg). \quad (42)$$

- Result ($\Omega < 0$; follows from an analogous calculation with the v-dependent parts of (34) and (35)):

$$b(\Omega) = \int_0^\infty d\omega\, \Big(\alpha_{\omega,-\Omega} a(-\omega) + \beta_{\omega,-\Omega} a^\dagger(-\omega)\Big). \quad (43)$$

- Transformations like (40) and (43) which relate two different sets of creation and annihilation operators are called Bogolyubov transformations. The coefficients (41) and (42) are called Bogolyubov coefficients.

Particle numbers

- Expectation value of the number of b-particles with "momentum" Ω in the Minkowski-vacuum ($\Omega > 0$):

$$\langle 0_M | b^\dagger(\Omega) b(\Omega) | 0_M \rangle =$$
$$= \langle 0_M | \int_0^\infty d\omega\, \Big(\alpha^*_{\omega,\Omega} a^\dagger(\omega) + \beta^*_{\omega,\Omega} a(\omega)\Big) \int_0^\infty d\omega'\, \Big(\alpha_{\omega',\Omega} a(\omega') + \beta_{\omega',\Omega} a^\dagger(\omega')\Big) | 0_M \rangle =$$
$$= \int_0^\infty d\omega \int_0^\infty d\omega'\, \beta^*_{\omega,\Omega} \beta_{\omega',\Omega} \langle 0_M | a(\omega) a^\dagger(\omega') | 0_M \rangle =$$
$$= \int_0^\infty d\omega\, |\beta_{\omega,\Omega}|^2. \quad (44)$$

- The integral on the right hand side of (44) can be solved (Mukhanov et al., page 112 and 113):

$$\langle 0_M | b^\dagger(\Omega) b(\Omega) | 0_M \rangle = \frac{1}{e^{2\pi\Omega/a} - 1}. \quad (45)$$

- An analogous calculation for $\Omega < 0$ can be carried out. The result for arbitrary Ω is

$$\langle 0_M | b^\dagger(\Omega) b(\Omega) | 0_M \rangle = \frac{1}{e^{2\pi|\Omega|/a} - 1}. \quad (46)$$

Unruh temperature

- Comparing (46) with the Bose distribution

$$n(\Omega) = \frac{1}{e^{|\Omega|/T} - 1} \quad (47)$$

yields the Unruh temperature

$$T = \frac{a}{2\pi}. \quad (48)$$

- Conclusion: An accelerated observer moving through the Minkowski vacuum has the impression of moving through a thermal bath of b-particles with temperature T.