String breaking/computing energy levels on the lattice

Marc Wagner

Institut für Physik, Humboldt-Universität zu Berlin, Newtonstraße 15, 12489 Berlin, Germany

September 14, 2007

1 String breaking

Literature: [1].

Definition of the static quark antiquark potential

- $V(R) \;\; = \;\;$ energy of the lowest state containing an infinitely heavy quark antiquark pair at separation R (+ dynamical quarks + gluons).

Pure Yang-Mills theory (gluons, but no dynamical quarks)

- The quark antiquark potential is linear for intermediate and large separations (confinement; cf. Figure 1).
- There is a gluonic flux tube (a string) connecting the quark and the antiquark; the profile of this flux tube is essentially independent of the separation, which leads to a linearly rising quark antiquark potential.

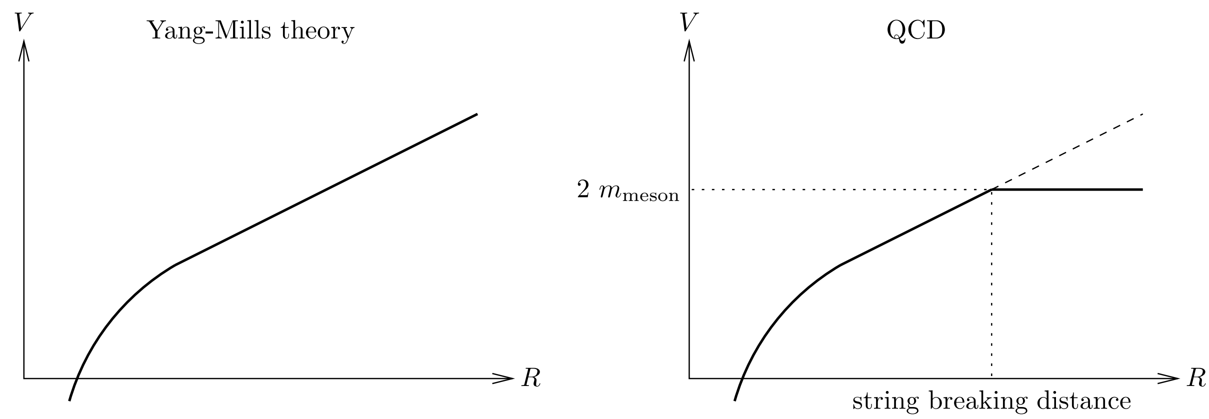

Figure 1: The static quark antiquark potential in Yang-Mills theory and in QCD.

QCD (gluons and dynamical quarks)

- The static quark antiquark potential is linear for intermediate separations, but constant for large separations (cf. Figure 1).
- When increasing the distance of the static quarks beyond a certain value $R_{\textrm{string breaking}}$, a light quark antiquark pair is created; the flux tube breaks and two static-light mesons appear; the attractive force between the static sources is "completely" screened by dynamical quarks, i.e. the potential for $R > R_{\textrm{string breaking}}$ is essentially two times the mass of the lightest static-light meson.
- The static-light spectrum, to be more precise the mass of the lightest static-light meson gives an estimate of the string breaking distance.

2 Computing energy levels on the lattice (theory)

Basic principle

- Compute the temporal correlation of a state $|\phi\rangle = \mathcal{O}|0\rangle$:

$$\begin{aligned} \langle\phi(T)|\phi(0)\rangle \quad &= \quad \langle 0|\mathcal{O}(T)\mathcal{O}(0)|0\rangle \quad = \quad \langle 0|e^{+HT}\mathcal{O}(0)e^{-HT}\mathcal{O}(0)|0\rangle \quad = \\ &= \quad \sum_n \langle\phi(0)|n\rangle e^{-(E_n-E_0)T}\langle n|\phi(0)\rangle \quad = \quad \sum_n \Big|\langle n|\phi(0)\rangle\Big|^2 e^{-(E_n-E_0)T} \end{aligned} \tag{1}$$

  ($|n\rangle$: eigenstates of the Hamiltonian H; E_n: corresponding eigenvalues).

- For large T

$$\langle\phi(T)|\phi(0)\rangle \quad \approx \quad \Big|\langle\Omega|\phi(0)\rangle\Big|^2 e^{-(E_\Omega-E_0)T}, \tag{2}$$

  where $|\Omega\rangle$ is the lowest state with $\langle\Omega|\phi\rangle \neq 0$.

- Then

$$e^{-(E_\Omega-E_0)a} \quad = \quad \frac{e^{-(E_\Omega-E_0)(T+a)}}{e^{-(E_\Omega-E_0)T}} \quad \approx \quad \frac{\langle\phi(T+a)|\phi(0)\rangle}{\langle\phi(T)|\phi(0)\rangle} \tag{3}$$

  and

$$E_\Omega - E_0 \quad \approx \quad \mathcal{E}_\Omega^\phi(T+a) \quad = \quad -\frac{1}{a}\ln\left(\frac{\langle\phi(T+a)|\phi(0)\rangle}{\langle\phi(T)|\phi(0)\rangle}\right). \tag{4}$$

  (a: lattice spacing).

An example of how to choose $|\phi\rangle$ (quark antiquark potential)

- Create a state with a static quark antiquark pair:

$$|\phi\rangle \;=\; \bar{Q}(\mathbf{x})U(\mathbf{x},\mathbf{y})Q(\mathbf{y})|0\rangle. \tag{5}$$

  - One can show that this state contains a static antiquark at $\mathbf{x}$ and a static quark at $\mathbf{y}$, and that it has vanishing overlap to all other states with different "heavy quark number".
  - To preserve gauge invariance, a parallel transporter (a gluonic string) has been included:

$$U(\mathbf{x},\mathbf{y}) \;=\; P\left\{\exp\left(i\int_{\mathbf{x}}^{\mathbf{y}} dz_j\, A_j(\mathbf{z})\right)\right\} \tag{6}$$

    (P denotes path ordering).

- Integrating out the static quarks (the corresponding propagator can be calculated analytically) yields the well known Wilson loop:

$$\langle\phi(T)|\phi(0)\rangle \;=\; \#\left\langle W_{(R,T)}\right\rangle \;=\; \#\frac{1}{3}\mathrm{Tr}\left(P\left\{\exp\left(i\oint dz_\mu\, A_\mu(z)\right)\right\}\right) \tag{7}$$

  (the path of integration is shown in Figure 2).

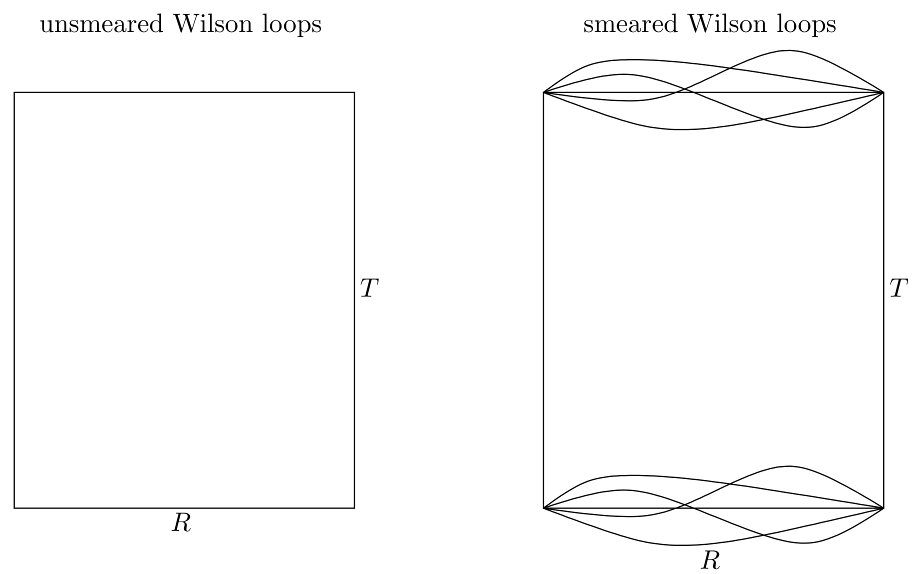

Figure 2: Unsmeared and smeared Wilson loops.

3 Computing energy levels on the lattice (practical issues)

- In lattice calculations it is not possible to measure $\langle\phi(T)|\phi(0)\rangle$ for large T:
  - Lattices have a finite (rather small) extension.
  - Due to the exponential fall off, $\langle\phi(T)|\phi(0)\rangle$ is very small for large T. Therefore, relative statistical errors are huge.
- Solution: choose $|\phi\rangle$ such that it is close to the state you want to measure, i.e. $\langle\Omega|\phi\rangle \approx 1$. Then (2) is already valid for rather small T.
- How can we determine such a state $|\phi\rangle$?
  - **Physical arguments/expectations:** usually one has a rough picture of how the corresponding state might look like; for example for the static quark antiquark potential the lowest state is expected to contain two static point-like sources connected by a gluonic flux tube with a certain diameter; therefore, it is better to use smeared Wilson loops instead of ordinary unsmeared Wilson loops, which are "too thin" (cf. Figure 2).
  - **Use a variational technique.**

A variational technique

- Use a whole set of states $|\phi_j\rangle$, instead of only a single state $|\phi\rangle$, and determine that linear combination

  $$|\psi\rangle \quad = \quad \alpha_j|\phi_j\rangle, \tag{8}$$

  which has maximum overlap to $|\Omega\rangle$ (to be more precise, determine that linear combination, for which $\mathcal{E}^\psi_\Omega(T+a)$ is minimal for given T).
- To determine the "best state" $|\psi\rangle$, compute the correlation matrix

  $$C_{jk}(T) \quad = \quad \langle\phi_j(T)|\phi_k(0)\rangle \tag{9}$$

  and solve the generalized eigenvalue problem

  $$C_{jk}(T+a)\alpha^{(n)}_k \quad = \quad C_{jk}(T)\alpha^{(n)}_k\lambda^{(n)} \quad \text{(no sum over } n\text{)} \tag{10}$$

  (in the following we consider the eigenvalues sorted according to $\lambda^{(1)} \leq \lambda^{(2)} \leq \ldots \leq \lambda^{(N)}$; [reality and positivity can be shown by multiplying (10) with $(\alpha^{(n)}_j)^*$]). $|\psi\rangle$ is given by (8) with $\alpha_j = \alpha^{(N)}_j$ and the corresponding minimal energy is

  $$\mathcal{E}^\psi_\Omega(T+a) \quad = \quad -\frac{1}{a}\ln\left(\frac{\langle\psi(T+a)|\psi(0)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right) \quad = \quad -\frac{1}{a}\ln\left(\frac{(\alpha^{(N)}_j)^*C_{jk}(T+a)\alpha^{(N)}_k}{(\alpha^{(N)}_j)^*C_{jk}(T)\alpha^{(N)}_k}\right) \quad =$$

  $$= \quad -\frac{1}{a}\ln\left(\lambda^{(N)}\right). \tag{11}$$

- "Proof":
  - To determine the best state $|\psi\rangle$, minimize $\mathcal{E}_\Omega^\psi(T+a)$ with respect to α_n^*:

$$\frac{\partial}{\partial\alpha_n^*}\mathcal{E}_\Omega^\psi(T+a) = \frac{\partial}{\partial\alpha_n^*}\left(-\frac{1}{a}\ln\left(\frac{\langle\psi(T+a)|\psi(0)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)\right) =$$
$$= \frac{\partial}{\partial\alpha_n^*}\left(-\frac{1}{a}\ln\left(\frac{\alpha_j^* C_{jk}(T+a)\alpha_k}{\alpha_j^* C_{jk}(T)\alpha_k}\right)\right) =$$
$$= -\frac{1}{a}\frac{C_{nk}(T+a)\alpha_k}{\langle\alpha|C(T+a)|\alpha\rangle} + \frac{1}{a}\frac{C_{nk}(T)\alpha_k}{\langle\alpha|C(T)|\alpha\rangle} = 0. \quad (12)$$

    This is equivalent to

$$C_{nk}(T+a)\alpha_k = C_{nk}(T)\alpha_k \underbrace{\frac{\langle\alpha|C(T+a)|\alpha\rangle}{\langle\alpha|C(T)|\alpha\rangle}}_{=\lambda}. \quad (13)$$

    For $\lambda =$ constant this is a generalized eigenvalue problem. The solutions $\alpha_k^{(n)}$, $\lambda^{(n)}$, $n = 1, \ldots, N$, are consistent with the full equation, since

$$\frac{\langle\alpha^{(n)}|C(T+a)|\alpha^{(n)}\rangle}{\langle\alpha^{(n)}|C(T)|\alpha^{(n)}\rangle} = \lambda^{(n)}. \quad (14)$$

    Note that this line of reasoning is not waterproof, since it does not include solutions with non-constant λ.

- Further remarks:
  - Determine the best state $|\psi\rangle$ at rather small T. This is more stable from a numerical point of view.
  - In principle, the whole low lying spectrum can be obtained via

$$E_{\Omega^{(n)}} - E_0 \approx -\frac{1}{a}\ln\left(\lambda^{(N-n)}\right), \quad (15)$$

    if sufficiently many states $|\phi_j\rangle$ are used ($\Omega^{(n)}$ are low lying states, which have non-vanishing overlap to span$\{|\phi_j\rangle\}$, and $E_{\Omega^{(0)}} < E_{\Omega^{(1)}} < E_{\Omega^{(2)}} < \ldots$ are their corresponding energies).
  - In the limit $N \to \infty$ the generalized eigenvalue problem (10) is just a diagonalization of the Hamiltonian.
  - To get an idea of the quality of the extracted energy $\mathcal{E}_\Omega^\psi$, one can compute the overlap of $|\psi\rangle$ and $|\Omega\rangle$ by considering the following expression for large T:

$$\frac{|\langle\Omega|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{|\langle 0|\psi\rangle|^2 e^{-(E_\Omega - E_0)(T+a)}}{\langle\psi|\psi\rangle e^{-(E_\Omega - E_0)(T+a)}} \approx \frac{\langle\psi(T+a)|\psi(0)\rangle}{\langle\psi|\psi\rangle e^{-((E_\Omega - E_0)a)((T+a)/a)}} \approx$$
$$\approx \frac{\langle\psi(T+a)|\psi(0)\rangle^{(T+a)/a}}{\langle\psi|\psi\rangle\langle\psi(T+a)|\psi(0)\rangle^{T/a}} \quad (16)$$

    ((2) and (4) have been used). Note that this is only possible, if $|\psi\rangle$ is normalized, since $\langle\psi|\psi\rangle$ is hardly accessible in numerical computations.

4 Numerical results for the static quark antiquark potential

- Computation of the static quark antiquark potential:
  - Twisted mass formulation, 2+1+1 flavors, 89 configurations at $\beta = 3.5$, $L^3 \times T = 24^3 \times 48$, $\kappa = 0.16480$, $\mu = 0.004$, $\mu = 0.11$, $\epsilon = 0.09$.
  - Wilson loops with five different smearing levels, i.e. $N = 5$.
- Figure 3a shows $\mathcal{E}_\Omega^\psi$ as a function of T for different quark separations R (the best state $|\psi\rangle$ has been determined at $T = 2$).
- Figure 3b shows the static quark antiquark potential V as a function of R ($V(R) = \mathcal{E}_\Omega^\psi(R, T = 5)$) together with a least squares fit with $V(R) = V_0 - \alpha/R + \sigma R$, from which one can determine e.g. r_0.
- Figure 3c shows the overlap of $|\psi\rangle$ and $|\Omega\rangle$ according to (16) as a function of T for different quark separations R.
- Note that the rather weak overlap for large R (cf. Figure 3c) is consistent with the lower plateaux quality in Figure 3a.

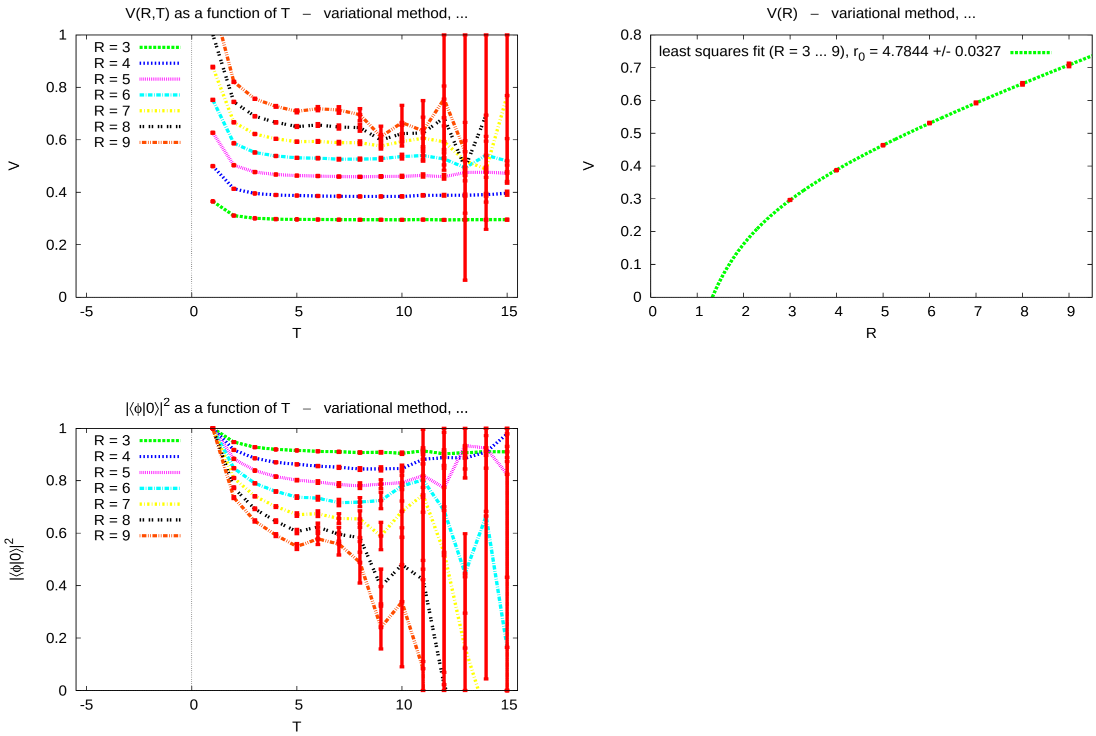

Figure 3: **a)** $\mathcal{E}_\Omega^\psi$ as a function of T for different quark separations R. **b)** V as a function of R. **c)** Overlap of $|\psi\rangle$ and $|\Omega\rangle$ as a function of T for different quark separations R.

Where is string breaking?

- With the method discussed above (using smeared Wilson loops to extract the static quark antiquark potential) one does not observe string breaking. The reason is a bad choice of states $|\phi_j\rangle$: the overlap of $|\phi_j\rangle$ and $|\Omega\rangle$ is acceptable as long as $R \lesssim R_{\text{string breaking}} \approx 2.2 \times r_0$, but there is essentially no overlap, as soon as the string is broken (the corresponding state $|\Omega\rangle$ is expected to be a two meson state then).

- Solution: do not only use "string states" (5) but also "two meson states", e.g.

$$|\phi_j\rangle \quad = \quad \bar{Q}(\mathbf{x})\gamma_5 q(\mathbf{x})\bar{q}(\mathbf{y})\gamma_5 Q(\mathbf{y})|0\rangle \tag{17}$$

  and smeared versions therefrom; when using the variational method, there should be sufficient overlap both below and above the string breaking distance $R_{\text{string breaking}}$ (the corresponding correlation matrix is sketched in Figure 4).

$C_{jk} =$

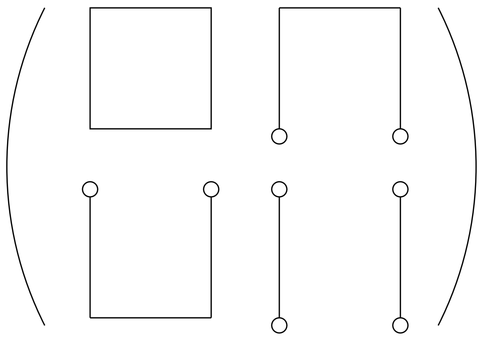

Figure 4: The correlation matrix to compute string breaking.

References

[1] G. S. Bali, H. Neff, T. Duessel, T. Lippert and K. Schilling [SESAM Collaboration], "Observation of string breaking in QCD," Phys. Rev. D **71**, 114513 (2005) [arXiv:hep-lat/0505012].