Definitions of a static SU(2) color triplet potential

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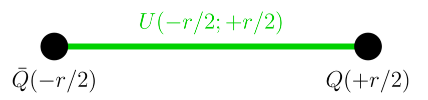
Abstract

We discuss possibilities and problems to non-perturbatively define and compute a static color triplet potential in SU(2) gauge theory. Numerical lattice results are presented and compared to analytical perturbative results.

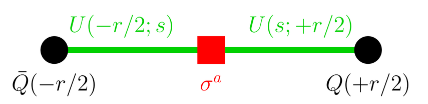
Calculating the static potential: basic principle

- Singlet potential: based on trial states



$$|\Phi^{\text{singlet}}\rangle \quad \equiv \quad \bar{Q}(-r/2)U(-r/2;+r/2)Q(+r/2)|\Omega\rangle.$$

- Triplet potential: in the literature, e.g. [N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B **566**, 275 (2000) [hep-ph/9907240]], based on trial states



$$|\Phi^{\text{triplet},a}\rangle \quad \equiv \quad \bar{Q}(-r/2)U(-r/2;s)\sigma^a U(s;+r/2)Q(+r/2)|\Omega\rangle.$$

- $\pm r/2 \equiv (0,0,\pm r/2)$, Q and $\bar{Q}$ are static quark/antiquark operators, U are spatial parallel transporters (on a lattice products of links), σ^a denote Pauli matrices in color space.
- From the asymptotic behavior of the correlation function,

$$\langle\Phi^X(t_2)|\Phi^X(t_1)\rangle \quad = \quad \sum_{n=0}^{\infty} c_n \exp\Big(-V_n^X(r)\underbrace{(t_2-t_1)}_{=\Delta t>0}\Big) \quad \propto_{\Delta t\to\infty} \quad \exp\Big(-V_0^X(r)\Delta t\Big), \qquad (1)$$

the (ground state) static potential $V_0^X(r)$ can be extracted (on a periodic lattice more complicated; see below); it is a state containing a static quark at $+r/2$ and a static antiquark at $-r/2$; further properties and the interpretation of this state depend on, whether the singlet or triplet trial state is used, and, in case of the triplet trial state, on the choice of gauge.

Without gauge fixing

- Perturbative calculations not possible (gluonic propagators require gauge fixing).
- **Non-perturbative computations (lattice), singlet potential:**

$$\langle\Phi^{\text{singlet}}(t_2)|\Phi^{\text{singlet}}(t_1)\rangle \quad \propto \quad W(r,\Delta t) \quad = \quad \text{Tr}\bigg(P\bigg\{\exp\bigg(ig\oint dz_\mu\,A_\mu(z)\bigg)\bigg\}\bigg),$$

where the integration is along a rectangular path with spatial extension r and temporal extension Δt; the potential can then be obtained via eq. (1),

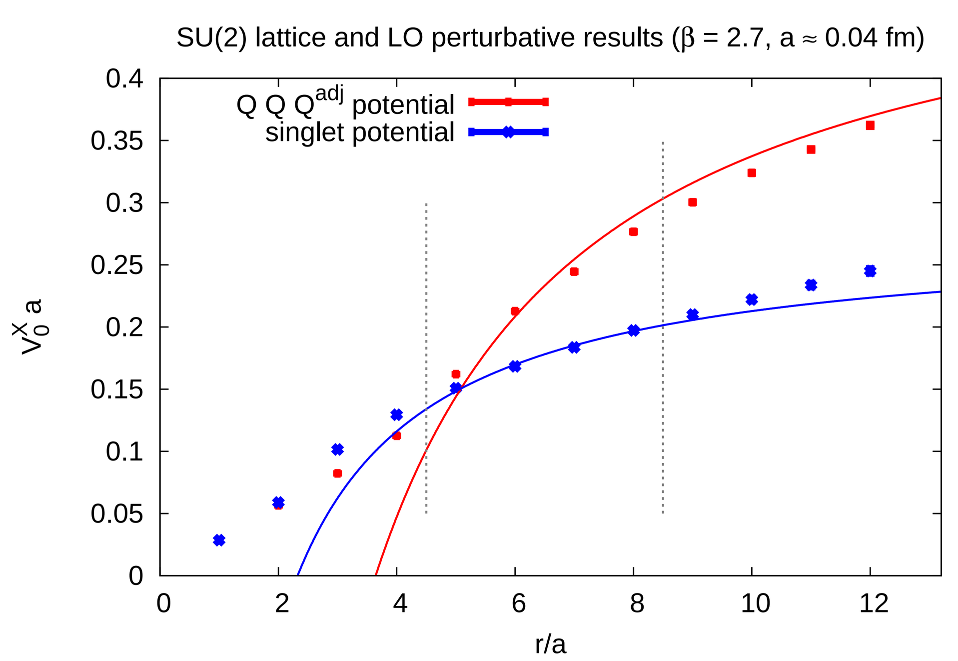
$$V^{\text{singlet}}(r) \quad = \quad -\lim_{\Delta t\to\infty}\frac{\dot{W}(r,\Delta t)}{W(r,\Delta t)}$$

(cf. Figure [bottom, left], blue dots).

- **Non-perturbative computations (lattice), triplet potential:**

$$\langle\Phi^{\text{triplet},a}(t_2)|\Phi^{\text{triplet},a}(t_1)\rangle \quad = \quad 0 \quad (\text{no sum over } a),$$

because $\langle\Phi^{\text{triplet},a}(t_2)|\Phi^{\text{triplet},a}(t_1)\rangle$ is not gauge invariant (does not contain any gauge invariant contribution).



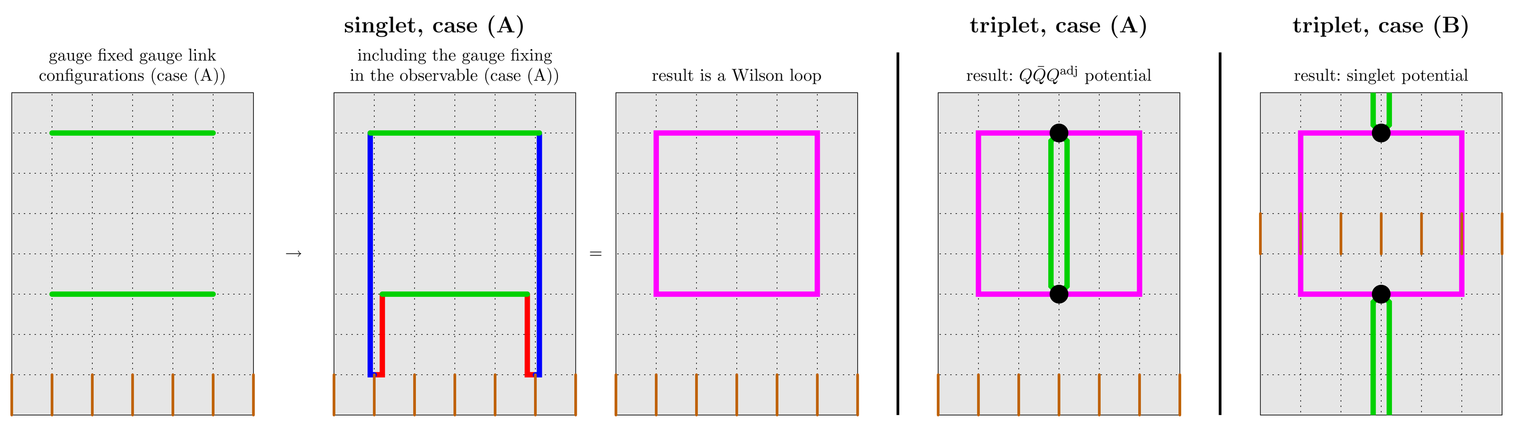
Gauge fixing, temporal gauge ($A_0 = 0$)

- Gauge transformations and gauge fixing on a lattice:
  - Gauge transformation of a temporal link: $U_0(t,\mathbf{x}) \to U_0^g(t,\mathbf{x}) = g(t,\mathbf{x})U_0(t,\mathbf{x})g^\dagger(t+a,\mathbf{x})$, where $g(t,\mathbf{x}) \in \text{SU}(2)$.
  - Temporal gauge $A_0^g = 0$ in the continuum corresponds to $U_0^g(t,\mathbf{x}) = 1$ on a lattice.
  - On a lattice with finite periodic temporal extension it is not possible to fix to temporal gauge everywhere; there will be a slice of links, where $U_0 \neq 0$; in the following wlog. $U_0^g(t=0,\mathbf{x}) \neq 1$, while $U_0^g(t=1\ldots T-1,\mathbf{x}) = 1$ (T is the periodic temporal extension of the lattice).
  - A possible choice for the corresponding gauge transformation $g(t,\mathbf{x})$ is
    $g(t=2a,\mathbf{x}) = U_0(t=a,\mathbf{x})$,
    $g(t=3a,\mathbf{x}) = g(t=2a,\mathbf{x})U_0(t=2a,\mathbf{x}) = U_0(t=a,\mathbf{x})U_0(t=2a,\mathbf{x})$,
    $g(t=4a,\mathbf{x}) = g(t=3a,\mathbf{x})U_0(t=3a,\mathbf{x}) = U_0(t=a,\mathbf{x})U_0(t=2a,\mathbf{x})U_0(t=3a,\mathbf{x})$,
    $g(t=5a,\mathbf{x}) = \ldots$
- **Non-perturbative computations (lattice), singlet potential:**
  - Trial states $|\Phi^{\text{singlet}}\rangle$ are gauge invariant; therefore the result is identical to the result without gauge fixing (cf. Figure [bottom, left], blue dots).
  - **Including the gauge fixing in the observable:**
    - Case (A), $1 \leq t_1 < t_2 < T$, computation on gauge fixed gauge link configurations:

$$\langle\Phi^{\text{singlet}}(t_2)|\Phi^{\text{singlet}}(t_1)\rangle \quad = \quad \Big\langle\text{Tr}\Big(U^g(t_1,-r/2;t_1,+r/2)U^g(t_2,+r/2;t_2,-r/2)\Big)\Big\rangle.$$

    - Replacing U^g by the original links U and the gauge transformation g yields a manifestly gauge invariant observable:

$$\langle\Phi^{\text{singlet}}(t_2)|\Phi^{\text{singlet}}(t_1)\rangle \quad = \quad \Big\langle\text{Tr}\Big(U(t_1,-r/2;t_1,+r/2)\underbrace{g^\dagger(t_1,+r/2)g(t_2,+r/2)}_{U(t_1,+r/2;t_2,+r/2)}$$
$$U(t_2,+r/2;t_2,-r/2)\underbrace{g^\dagger(t_2,-r/2)g(t_1,-r/2)}_{U(t_2,-r/2;t_1,-r/2)}\Big)\Big\rangle \quad = \quad W(r,\Delta t).$$



    - Case (B), $0 = t_1 < t_2 < T$ and $1 \leq t_2 < t_1 < T$, slightly more complicated, but same result.
    - This technique of transforming a non-gauge invariant observable into an equivalent manifestly gauge invariant observable will be helpful for interpreting the triplet potential.
  - **Transfer matrix formalism:**
    [O. Philipsen, Nucl. Phys. B **628**, 167 (2002) [hep-lat/0112047]]
    [O. Jahn and O. Philipsen, Phys. Rev. D **70**, 074504 (2004) [hep-lat/0407042]]
    - On the lattice a spectral decomposition of correlation functions is done in terms of eigenstates and eigenvalues of the transfer matrix.
    - Without gauge fixing the transfer matrix is $\hat{T} = e^{-Ha}$ (lattice discretization errors neglected),

$$\hat{T}|\psi^{(n)}\rangle \quad = \quad \lambda^{(n)}|\psi^{(n)}\rangle;$$

    $\lambda^{(n)} = e^{-E^{(n)}a}$, where $E^{(n)}$ are the energies of gauge invariant states (e.g. vacuum, glueballs).
    - The transfer matrix in temporal gauge is $\hat{T}_0 = e^{-\hat{H}_0 a}$; similarly

$$\hat{T}_0|\psi_0^{(n)}\rangle \quad = \quad \lambda_0^{(n)}|\psi_0^{(n)}\rangle.$$

    - In temporal gauge remaining gauge degrees of freedom are time-independent gauge transformations $g(\mathbf{x})$; one can show $[\hat{T}_0, g(\mathbf{x})] = 0$, i.e. eigenstates of $\hat{T}_0$ can be classified according to SU(2) color quantum numbers $(j(\mathbf{x}), m(\mathbf{x}))$ at each $\mathbf{x}$.
    - $\lambda_0^{(n)} = e^{-E_0^{(n)}a}$, where $E_0^{(n)}$ are the masses/energies of the gauge invariant states discussed above as well as of additional non-gauge invariant states with $j(\mathbf{x}) \neq 0$; such states can be interpreted as states containing static color charges (= static quarks).
    - Notation of energy eigenvalues $E_0^{(n)}$:
      - Gauge invariant states, i.e. no static quarks: $\mathcal{E}_n$ ($j(\mathbf{x}) = 0$ for all $\mathbf{x}$).
      - A static quark/antiquark at $-r/2$ and at $+r/2$: $V_n^{\text{singlet}}(r)$ ($j(-r/2) = j(+r/2) = 1/2$).
      - An adjoint static quark at s: $\mathcal{E}_n^{Q^{\text{adj}}}$ ($j(s) = 1$).
      - A static quark/antiquark at $-r/2$ and at $+r/2$, an adjoint static quark at s: $V_n^{Q\bar{Q}Q^{\text{adj}}}(r)$ ($j(-r/2) = j(+r/2) = 1/2$, $j(s) = 1$).
    - One can derive

$$\langle\Phi^{\text{singlet}}(t_2)|\Phi^{\text{singlet}}(t_1)\rangle \quad = \quad \sum_k e^{-V_k^{\text{singlet}}(r)\Delta t}\sum_m e^{-\mathcal{E}_m(T-\Delta t)}\sum_{\alpha,\beta}\Big|\langle k,\alpha\beta|\hat{U}_{\alpha\beta}(-r/2;+r/2)|m\rangle\Big|^2;$$

    $\alpha \equiv m(-r/2) = \pm 1/2$ and $\beta \equiv m(+r/2) = \pm 1/2$ are color indices at $\pm r/2$.
    - This correlation function is suited to extract the common singlet potential $V_0^{\text{singlet}}(r)$.
- **Non-perturbative computations (lattice), triplet potential:**
  - **Including the gauge fixing in the observable:**
    - Again one has to distinguish two cases, which this time yield different results.
    - (s,t_1) and (s,t_2), the spacetime positions of the “triplet generators’ σ^a, are connected by an adjoint static propagator: $\text{Tr}(\sigma^a U(t_1,s;t_2,s)\sigma^b U(t_2,s;t_1,s))$.
  - **Transfer matrix formalism:**
    - One can derive for case (A)

$$\langle\Phi^{\text{triplet},a}(t_2)|\Phi^{\text{triplet},a}(t_1)\rangle \quad =$$
$$= \quad \sum_k e^{-V_k^{Q\bar{Q}Q^{\text{adj}}}(r)\Delta t}\sum_m e^{-\mathcal{E}_m(T-\Delta t)}\sum_{\alpha,\beta}\Big|\langle k,\alpha\beta,m(s)=a|\hat{U}_{\alpha\beta,a}(-r/2;s;+r/2)|m\rangle\Big|^2$$

    and for case (B)

$$\langle\Phi^{\text{triplet},a}(t_2)|\Phi^{\text{triplet},a}(t_1)\rangle \quad =$$
$$= \quad \sum_k e^{-V_k^{\text{singlet}}(r)\Delta t}\sum_m e^{-\mathcal{E}_m^{Q^{\text{adj}}}(T-\Delta t)}\sum_{\alpha,\beta}\Big|\langle k,\alpha\beta|\hat{U}_{\alpha\beta,a}(-r/2;s;+r/2)|m,m(s)=a\rangle\Big|^2.$$

  - Conclusion: one can either extract a three-quark potential (one quark at $+r/2$, one antiquark at $-r/2$, one adjoint quark at s) (case (A)) or the ordinary singlet potential (case (B)).

Gauge fixing, Lorentz gauge ($\partial_\mu A_\mu = 0$)

- Perturbative calculations:
  - Leading order result for trial states $\bar{Q}(-r/2)U(-r/2;0)\Sigma U(0;+r/2)Q(+r/2)|\Omega\rangle$:
    - Singlet potential $\Sigma = 1$: $V_0^{\text{singlet}}(r) = -3g^2/16\pi r$, i.e. attractive.
      - Can be compared to the non-perturbative lattice result (in any gauge), since the the trial state is gauge invariant; qualitative agreement is found (cf. Figure [bottom, left], solid blue line).
    - Triplet potential $\Sigma = \sigma^a$: $V_0^{\text{triplet}}(r) = +g^2/16\pi r$, i.e. repulsive.
      - In Lorentz gauge a transfer matrix does not exist; the physical interpretation is unclear ...
  - One can also calculate the gauge invariant triplet diagram obtained by using temporal gauge (cf. Figure [center], “triplet, case (A)”): $V_0^{Q\bar{Q}Q^{\text{adj}}}(r) = -9g^2/16\pi r$ (for $s = 0$), i.e. attractive.
    - Qualitative agreement with the lattice result is found (cf. Figure [bottom, left], red dots and solid red line).

Conclusions

- The singlet potential corresponds to a gauge invariant trial state $\bar{Q}(-r/2)U(-r/2;+r/2)Q(+r/2)|\Omega\rangle$; it is the same in any gauge and its interpretation as a static quark antiquark potential is clear.
- The “triplet potential” corresponding to trial states $\bar{Q}(-r/2)U(-r/2;s)\sigma^a U(s;+r/2)Q(+r/2)|\Omega\rangle$ is gauge dependent:
  - Without gauge fixing it cannot be calculated/computed.
  - In temporal gauge it corresponds to a three-quark potential and not to a potential between a quark and an antiquark in a color triplet state, i.e. the name “triplet potential” is rather misleading.
  - In Lorentz gauge a perturbative calculation yields a repulsive potential; since a transfer matrix does not exist, the physical interpretation is unclear ...