Operators for meson creation in the continuum and on the lattice

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- Goal: construct operators creating meson-like states, which have well defined quantum numbers with respect to total angular momentum J, parity P and charge conjugation C, when applied to the QCD vacuum.

- Building blocks for such "meson creation operators":

$$\bar{Q}(\mathbf{x})U(\mathbf{x};\mathbf{y})\Gamma q(\mathbf{y}) \tag{1}$$

(Q and q denote possibly different quark flavors and Γ is one of 16 independent γ-matrix combinations, yielding well defined quantum numbers with respect to spin S, (spin) parity P^S and, in the case of identical quark flavors, (spin) charge conjugation C^S [the 16 combinations are listed e.g. in [1], page 22, and [2], page 50]).

- A meson creation operator is a suitable linear combination of building blocks (1), yielding well defined quantum numbers with respect to total angular momentum J, parity P and, in the case of identical quark flavors, charge conjugation C (the "spatial structure of the meson" U [angular momentum L, (angular momentum) parity P^L and (angular momentum) charge conjugation C^L) and its spin part Γ have to be combined properly).

Parity and charge conjugation

- Parity P and charge conjugation C are discrete symmetries of the "standard Lagrangian"

$$\mathcal{L} \quad = \quad \bar{\psi}\gamma_\mu\partial_\mu\psi + m\bar{\psi}\psi. \tag{2}$$

To be more precise, they have been chosen such that they are symmetries of (2).

  - Parity: $\mathbf{x} \rightarrow -\mathbf{x}$, no effect on spin orientations; by writing ψ in terms of creation and annihilation operators one can show for the chiral representation

$$P\bar{\psi}P \quad = \quad \eta^*\bar{\psi}\gamma^0 \quad , \quad P\psi P \quad = \quad \eta\gamma^0\psi, \tag{3}$$

    where η is an undetermined phase (Minkowski version; cf. [2], page 64 to 71).

  - Charge conjugation: takes a fermion with a given spin orientation into an antifermion with the same spin orientation; by writing ψ in terms of creation and annihilation operators one can show for the chiral representation

$$C\bar{\psi}C \quad = \quad \Big(-i\gamma^0\gamma^2\psi\Big)^T \quad , \quad C\psi C \quad = \quad \Big(-i\bar{\psi}\gamma^0\gamma^2\Big)^T \tag{4}$$

    (Minkowski version; cf. [2], page 64 to 71).

- Like the standard Lagrangian the QCD Lagrangian is also P- and C-invariant. Therefore, the corresponding eigenstates can be classified according to P and C.
  - When using different quark flavors, states obtained by meson creation operators (1) do not have a well defined behavior under charge conjugation. One has to symmetrize the operator in a symmetric/antisymmetric way by adding/subtracting the same expression with quark flavors exchanged. On the other hand, it is easy to show that this leads to degenerate masses with respect to charge conjugation (in simple words: a $\bar{Q}q$-meson and a $\bar{q}Q$-meson have the same mass). Therefore, charge conjugation is not a useful quantum number, when different quark flavors are considered.

Momentum

- Consider only mesons with vanishing momentum by averaging over space:

$$\begin{aligned}
\mathcal{O}(\mathbf{x})|0\rangle \quad &\rightarrow \quad \frac{1}{L^3}\int d^3x\,\mathcal{O}(\mathbf{x})|0\rangle \quad = \quad \frac{1}{L^3}\underbrace{\int d^3x\,e^{\pm i\mathbf{P}\mathbf{x}}}_{=(2\pi)^3\delta^{(3)}(\mathbf{P})}\mathcal{O}(0)\underbrace{e^{\mp i\mathbf{P}\mathbf{x}}}_{=1}|0\rangle \quad = \\
&= \quad \frac{(2\pi)^3}{L^3}\sum_n\int d^3p\,|n,\mathbf{p}\rangle\langle n,\mathbf{p}|\delta^{(3)}(\mathbf{P})\mathcal{O}(0)|0\rangle \quad = \\
&= \quad \frac{(2\pi)^3}{L^3}\sum_n|n,\mathbf{p}=0\rangle\langle n,\mathbf{p}=0|\mathcal{O}(0)|0\rangle. \qquad (5)
\end{aligned}$$

- Reasons for excluding non-vanishing momenta:
  - Obtain a better signal by avoiding contamination by heavier $\mathbf{p}\neq 0$-mesons.
  - Angular momentum is the same with respect to any axis, i.e. angular momentum is definitely not connected to non-vanishing momentum (see below).

Angular momentum

- Wigner-Eckart theorem: cf. e.g. [3].
- The operator

$$\mathcal{O}_{LM}(\mathbf{x}) \quad = \quad \frac{1}{4\pi}\int d\Omega\,Y_{LM}(\hat{\mathbf{z}}(\Omega))\underbrace{\bar{Q}(\mathbf{x})U(\mathbf{x};\mathbf{x}+\mathbf{z}(\Omega))\Gamma q(\mathbf{x}+\mathbf{z}(\Omega))}_{=F(\mathbf{x},\mathbf{x}+\mathbf{z}(\Omega))} \qquad (6)$$

($\mathbf{z}(\Omega)$ is a vector of fixed length pointing in "Ω-direction") is a spherical tensor operator, which, when applied to $|0\rangle$, creates a state with angular momentum quantum numbers L and M (for the moment we neglect spin, e.g. by choosing $\Gamma=\gamma_5$, i.e. we only consider the spatial part of the meson creation operator):

$$
\begin{aligned}
R\mathcal{O}_{LM}(\mathbf{x})R^\dagger \quad &= \quad \frac{1}{4\pi}\int d\Omega\, Y_{LM}(\hat{\mathbf{z}}(\Omega))F(\mathbf{x},\mathbf{x}+\mathbf{z}(R(\Omega))) \quad = \\
&= \quad \frac{1}{4\pi}\int \underbrace{d(R^{-1}(\Omega))}_{=d\Omega}\, Y_{LM}(\hat{\mathbf{z}}(R^{-1}(\Omega)))F(\mathbf{x},\mathbf{x}+\mathbf{z}(\underbrace{R(R^{-1}(\Omega))}_{=\Omega})) \quad = \\
&= \quad \frac{1}{4\pi}\int d\Omega\, Y_{LM}(\hat{\mathbf{z}}(R^{-1}(\Omega)))F(\mathbf{x},\mathbf{x}+\mathbf{z}(\Omega)) \quad = \\
&= \quad (R^{-1})_{L,M;L',M'}\frac{1}{4\pi}\int d\Omega\, Y_{L'M'}(\hat{\mathbf{z}}(\Omega))F(\mathbf{x},\mathbf{x}+\mathbf{z}(\Omega)),
\end{aligned} \tag{7}
$$

where R is a rotation around $\mathbf{x}$ and $R_{L,M;L',M'}$ is the "effect of the same rotation on spherical harmonics", i.e. an element of an irreducible representation of the group of rotations (for the sake of simplicity, we have also assumed that $F(\mathbf{x},\mathbf{y})$ is rotationally invariant with respect to the axis $\mathbf{x}-\mathbf{y}$; if this is not the case, one has to include another rotation around that axis).

- Since we consider mesons with momentum zero, angular momentum is independent of the axis of rotation. This can be made plausible by the following classical calculation:
  - Let $\mathbf{P} = \sum_j \mathbf{p}_j = 0$ be the momentum of a set of particles.
  - The angular momentum with respect to the origin is given by

    $$\mathbf{L}_0 \quad = \quad \sum_j \mathbf{r}_j \times \mathbf{p}_j. \tag{8}$$

    It is identical to the angular momentum with respect to any other point $\mathbf{d}$:

    $$\mathbf{L}_\mathbf{d} \quad = \quad \sum_j \Big(\mathbf{r}_j - \mathbf{d}\Big) \times \mathbf{p}_j \quad = \quad \mathbf{L}_0 - \underbrace{\mathbf{d} \times \sum_j \mathbf{p}_j}_{=0} \quad = \quad \mathbf{L}_0. \tag{9}$$

  Therefore, no harm is done in placing the antiquark $\bar{Q}$ at the "center" $\mathbf{x}$ and the quark q on a sphere surrounding the antiquark.

Spin and angular momentum

- The spin generated by the two fermion operators in (1) can be either 0 or 1 and is determined by Γ.
  - Example: $\Gamma = \gamma_5$ corresponds to $S^{P^S C^S} = 0^{-+}$.
    * $\bar{\psi}\gamma_5\psi$ is a scalar. Therefore, $S = 0$.
    * $P^S = -$, because

      $$\bar{\psi}\gamma_5\psi \quad \rightarrow \quad P\bar{\psi}P\gamma_5 P\psi P \quad = \quad \bar{\psi}\gamma_0\gamma_5\gamma_0\psi \quad = \quad -\bar{\psi}\gamma_5\psi. \tag{10}$$

- $C^S = +$, because

$$\begin{aligned}
\bar{\psi}\gamma_5\psi \quad \rightarrow \quad C\bar{\psi}C\gamma_5 C\psi C \quad &= \quad \Big(- i\gamma_0(i\gamma_2)\psi\Big)^T \gamma_5 \Big(- i\bar{\psi}\gamma_0(i\gamma_2)\Big)^T \quad = \\
&= \quad (\gamma_0)_{AB}(\gamma_2)_{BC}\psi_C(\gamma_5)_{AD}\bar{\psi}_E(\gamma_0)_{EF}(\gamma_2)_{FD} \quad = \\
&= \quad -\bar{\psi}_E(\gamma_0)_{EF}(\gamma_2)_{FD}(\gamma_5^T)_{DA}(\gamma_0)_{AB}(\gamma_2)_{BC}\psi_C \quad = \\
&= \quad -\bar{\psi}\gamma_0\gamma_2 \underbrace{\gamma_5^T}_{=\gamma_5} \gamma_0\gamma_2\psi \quad = \quad \bar{\psi}\gamma_5\psi
\end{aligned} \tag{11}$$

(note that $\gamma_5 = \gamma_5^T$ in the chiral representation; moreover two additional i have been included to obtain the Euclidean version of (4)).

- Example: $\Gamma = \gamma_j$ corresponds to $S^{P^S C^S} = 1^{--}$.
  - $\bar{\psi}\gamma_j\psi$ is a vector. Therefore, $S = 1$.
  - $P^S = -$ (same calculation as (10)).
  - $C^S = -$ (similar calculation as (11); note that in the chiral representation γ_0, γ_2 and γ_5 are symmetric, while γ_1 and γ_3 are antisymmetric).
- For a complete list cf. e.g. [1], page 22.

- Since we have both well defined spin $S \in \{0, 1\}$ and well defined angular momentum $L \in \{0, 1, 2, \ldots\}$, we can couple them to states with well defined total angular momentum J via Clebsch-Gordan coefficients. Putting everything together (spatial averaging, angular momentum, spin) yields the general form of a meson creation operator:

$$\begin{aligned}
\mathcal{O}_{JJ_z}^{PC} \quad &= \quad \# \int d^3x \sum_{M,S_z} \mathcal{C}(J, J_z; L, M, S, S_z) \\
&\int d\Omega\, Y_{LM}(\hat{\mathbf{z}}(\Omega))\bar{Q}(\mathbf{x})U(\mathbf{x}; \mathbf{x} + \mathbf{z}(\Omega))\Gamma_{SS_z}^{P^S C^S} q(\mathbf{x} + \mathbf{z}(\Omega))
\end{aligned} \tag{12}$$

with $P = (-1)^L P^S$ and $C = (-1)^L C^S$ (the C-quantum number of the spatial part of the meson creation operator is the same as its P-quantum number; the reason is that charge conjugation simply exchanges quark and antiquark).

Lattice operators

- Crucial point: the sphere of the quark field q and, therefore, the spherical harmonics in (12) have to be approximated by a finite number of lattice sites:

$$\begin{aligned}
\mathcal{O}_{JJ_z}^{PC} \quad &= \quad \# \sum_{\mathbf{x}} \sum_{M,S_z} \mathcal{C}(J, J_z; L, M, S, S_z) \\
&\sum_{n=1}^{N} Y_{LM}(\hat{\mathbf{z}}_n)\bar{Q}(\mathbf{x})U(\mathbf{x}; \mathbf{x} + \mathbf{z}_n)\Gamma_{SS_z}^{P^S C^S} q(\mathbf{x} + \mathbf{z}_n).
\end{aligned} \tag{13}$$

- To analyze the angular momentum content generated by such a lattice meson creation operator, it is convenient to transform it back to a continuum-like expression:

$$\mathcal{O}_{JJ_z}^{PC} \quad =$$

$$
\begin{aligned}
&= \# \sum_{\mathbf{x}} \sum_{M,S_z} \mathcal{C}(\ldots) \\
&\quad \sum_{n=1}^{N} \int d^3 z\, \delta^{(3)}(\mathbf{z} - \mathbf{z}_n) Y_{LM}(\hat{\mathbf{z}}_n) \bar{Q}(\mathbf{x}) U(\mathbf{x}; \mathbf{x} + \mathbf{z}_n) \Gamma_{SS_z}^{P^S C^S} q(\mathbf{x} + \mathbf{z}_n) \quad = \\
&= \# \sum_{\mathbf{x}} \sum_{M,S_z} \mathcal{C}(\ldots) \\
&\quad \sum_{n=1}^{N} \int dr\, r^2 \int d\Omega\, \delta^{(3)}(\mathbf{z} - \mathbf{z}_n) Y_{LM}(\hat{\mathbf{z}}) \bar{Q}(\mathbf{x}) U(\mathbf{x}; \mathbf{x} + \mathbf{z}) \Gamma_{SS_z}^{P^S C^S} q(\mathbf{x} + \mathbf{z}) \quad = \\
&= \# \sum_{\mathbf{x}} \sum_{M,S_z} \mathcal{C}(\ldots) \\
&\quad \int d\Omega \underbrace{\sum_{n=1}^{N} \delta^{(2)}(\hat{\mathbf{z}} - \hat{\mathbf{z}}_n) Y_{LM}(\hat{\mathbf{z}})}_{=\sum_{l,m} \alpha_{lm}^{LM} Y_{lm}(\hat{\mathbf{z}})} \bar{Q}(\mathbf{x}) U(\mathbf{x}; \mathbf{x} + \mathbf{z}) \Gamma_{SS_z}^{P^S C^S} q(\mathbf{x} + \mathbf{z}).
\end{aligned}
$$

The coefficients α_{lm}^{LM} are given by

$$
\alpha_{lm}^{LM} \quad = \quad \int d\Omega\, Y_{lm}^*(\hat{\mathbf{z}}) \sum_{n=1}^{N} \delta^{(2)}(\hat{\mathbf{z}} - \hat{\mathbf{z}}_n) Y_{LM}(\hat{\mathbf{z}}) \quad = \quad \sum_{n=1}^{N} Y_{lm}^*(\hat{\mathbf{z}_n}) Y_{LM}(\hat{\mathbf{z}_n}). \tag{14}
$$

By applying this formula and by doing a Clebsch-Gordan decomposition one can determine, which total angular momentum states are generated by $\mathcal{O}_{JJ_z}^{PC}$.

- For example in the continuum there are three "useful" ways to couple $L = 1$ and $\Gamma = \gamma_j$ ($S^{P^S C^S} = 1^{--}$):
  - To $\mathcal{O}_{00}^{++}$ via
    $$
    \sum_{M,S_z} \mathcal{C}(0,0;1,M,1,S_z) Y_{1M} \Gamma_{1S_z}^{--} \quad = \quad \#\Big(\hat{z}_1\gamma_1 + \hat{z}_2\gamma_2 + \hat{z}_3\gamma_3\Big). \tag{15}
    $$
    * Continuum total angular momentum: $J = 0$.
    * Lattice angular momentum according to (14): $J = 0, 4, \ldots$ (alternatively, this result can be obtained by means of discrete group theory; the corresponding representation of the discrete rotation group O_h is called A_1).
  - To $\mathcal{O}_{1J_z}^{++}$ via
    $$
    \sum_{M,S_z} \mathcal{C}(1,J_z;1,M,1,S_z) Y_{1M} \Gamma_{1S_z}^{--} \quad =
    $$
    $$
    = \left\{ \begin{array}{cl} \#\Big(\Big(\hat{z}_1 - i\hat{z}_2\Big)\gamma_3 - \hat{z}_3\Big(\gamma_1 - i\gamma_2\Big)\Big) & \text{if} \quad J_z = -1 \\ \#\Big(\hat{z}_1\gamma_2 - \hat{z}_2\gamma_1\Big) & \text{if} \quad J_z = 0 \\ \#\Big(\Big(\hat{z}_1 + i\hat{z}_2\Big)\gamma_3 - \hat{z}_3\Big(\gamma_1 + i\gamma_2\Big)\Big) & \text{if} \quad J_z = +1 \end{array} \right. . \tag{16}
    $$
    * Continuum total angular momentum: $J = 1$.

* Lattice angular momentum according to (14): $J = 1, 3, 4, \ldots$ (alternatively, this result can be obtained by means of discrete group theory; the corresponding representation of the discrete rotation group O_h is called T_1).

– To $\mathcal{O}^{++}_{2J_z}$ via

$$\sum_{M,S_z} \mathcal{C}(2, J_z; 1, M, 1, S_z) Y_{1M} \Gamma^{--}_{1S_z} =$$

$$= \left\{ \begin{array}{cl} \#\Big(\hat{z}_1\gamma_1 - \hat{z}_2\gamma_2 - i\Big(\hat{z}_1\gamma_2 + \hat{z}_2\gamma_1\Big)\Big) & \text{if} \quad J_z = -2 \\ \#\Big(\Big(\hat{z}_1 - i\hat{z}_2\Big)\gamma_3 + \hat{z}_3\Big(\gamma_1 - i\gamma_2\Big)\Big) & \text{if} \quad J_z = -1 \\ \#\Big(\hat{z}_1\gamma_1 + \hat{z}_2\gamma_2 - 2\hat{z}_3\gamma_3\Big) & \text{if} \quad J_z = 0 \\ \#\Big(\Big(\hat{z}_1 + i\hat{z}_2\Big)\gamma_3 + \hat{z}_3\Big(\gamma_1 + i\gamma_2\Big)\Big) & \text{if} \quad J_z = +1 \\ \#\Big(\hat{z}_1\gamma_1 - \hat{z}_2\gamma_2 + i\Big(\hat{z}_1\gamma_2 + \hat{z}_2\gamma_1\Big)\Big) & \text{if} \quad J_z = +2 \end{array} \right. . \qquad (17)$$

* Continuum total angular momentum: $J = 2$.
* Lattice angular momentum according to (14): $J = 2, 3, 4, \ldots$ or $J = 2, 4, \ldots$ (alternatively, this result can be obtained by means of discrete group theory; the corresponding representations of the discrete rotation group O_h are called T_2 and E).

The discrete rotation group O_h

- Literature: [4].
- Continuous rotation group SO(3): an infinite number of representations (dimensions $2J+1$, $J = 0, 1, 2, \ldots$).
- Discrete rotation group O_h: two 1-dimensional representations A_1 and A_2, one 2-dimensional representation E and two 3-dimensional representations T_1 and T_2.
- For the special case of "$N = 6$-lattice spherical harmonics" A_1, T_1 and E are shown in Figure 1 (T_2 corresponding to $\hat{z}_1\hat{z}_2$, $\hat{z}_2\hat{z}_3$ and $\hat{z}_3\hat{z}_1$, and A_2 corresponding to $\hat{z}_1\hat{z}_2\hat{z}_3$ obviously do not exist for $N = 6$).
- The possible angular momentum content of these discrete representations up to $J = 4$ is

$$\begin{array}{cc} J = 0 & A_1 \\ J = 1 & T_1 \\ J = 2 & E, T_2 \\ J = 3 & A_2, T_1, T_2 \\ J = 4 & A_1, E, T_1, T_2 \\ \ldots & \ldots \end{array} \qquad (18)$$

(this table can be obtained by applying (14)). **Note that such discrete representations do not necessarily contain all these angular momenta.** For example continuum spherical harmonics also form representations A_1, A_2, E, T_1 and T_2, but, of course, have "fixed angular momentum".

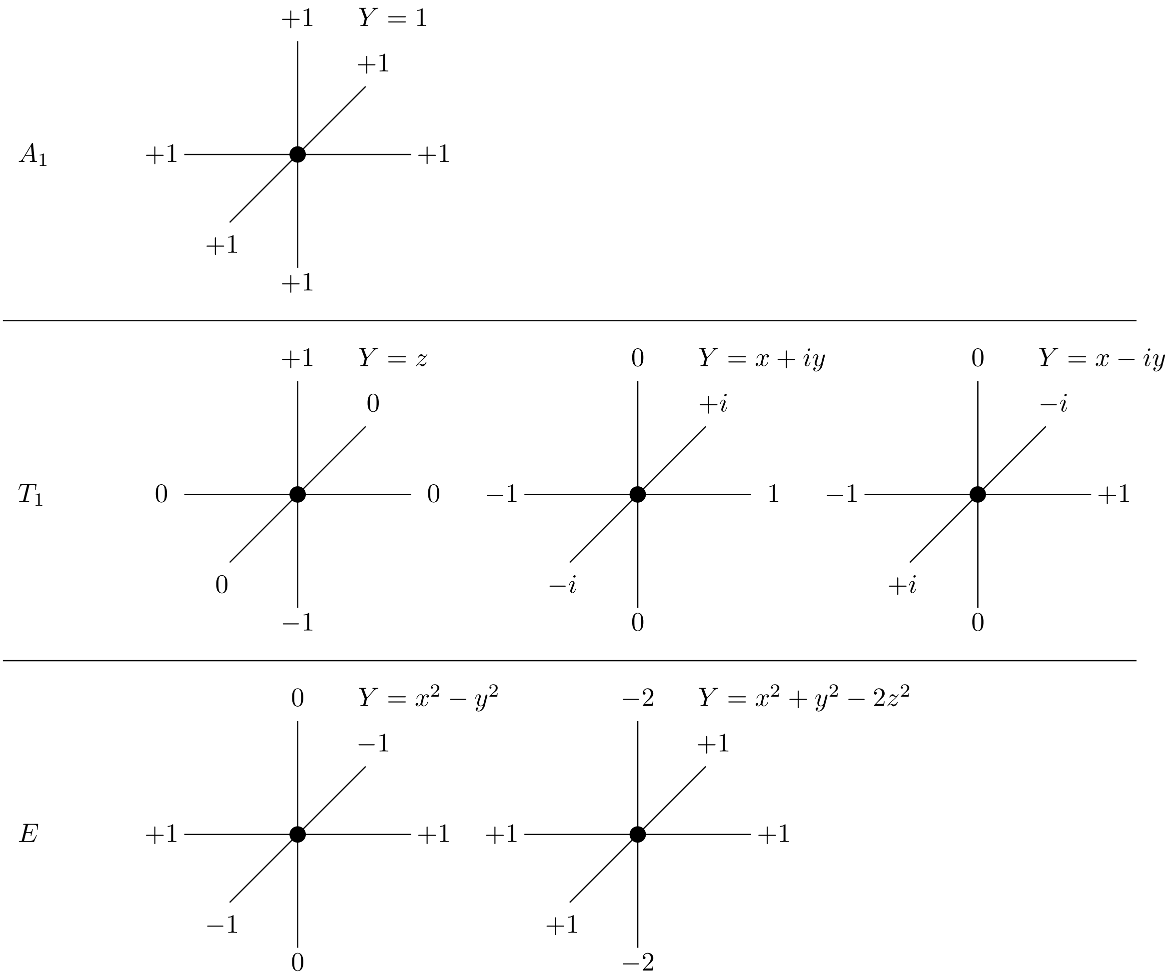

Figure 1: A_1, T_1 and E for $N = 6$.

- These discrete representations can be coupled forming again discrete representations (this is similar to Clebsch-Gordan coupling in the continuum). Via (18) one can immediately see, which total angular momentum states may be present in the result. **Note again that these states do not necessarily appear.**

- To identify the total angular momentum of a state computed in a lattice simulation, it is necessary to check different discrete representations for degenerate partners (in general, one has to compute excited states for that). For example a spin-0 meson generated via a T_2-angular momentum representation could have $J = 2, 3, 4, \ldots$; however, if there is no degenerate partner, i.e. no state with the same mass in either A_2 or T_1, $J = 3$ can be excluded.

- By choosing straight lines for the parallel transporters we have considered the simplest case ($N = 6$). In general, one also considers arbitrary diagonal paths ($N = 48$) to have the opportunity to study the T_2 or A_2 representation for the spatial part of the meson creation operator.

- For spin-1/2 objects, e.g. for baryons, one has to consider double cover representations of O_h. This gives rise to three more discrete representations, which are called G_1, G_2 and H.
- For very large spheres, i.e. spheres, where many lattice points are essentially right on it, and where the direction dependence of the parallel transporters due to the lattice discretization is negligible, another strategy is to consider different sets of 48 points. Then the weight of the J-contributions will also be different. By choosing suitable linear combinations certain J-contributions can be eliminated. Note that using an infinite number of such 48-point sets, while the lattice spacing $a \to 0$, yields the continuum limit, where a meson creation operator has a well defined angular momentum.

References

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