The colour adjoint static potential from Wilson loops with generator insertions and its physical interpretation

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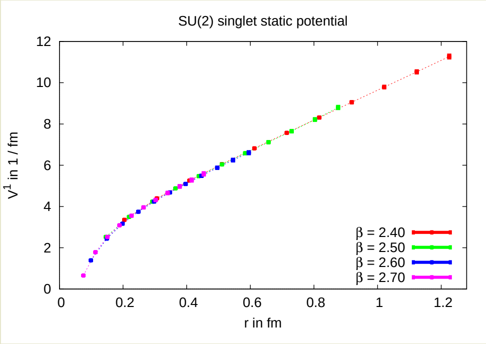
[M. Wagner and O. Philipsen, PoS ConfinementX , 340 (2012) [arXiv:1211.2165 [hep-lat]]]

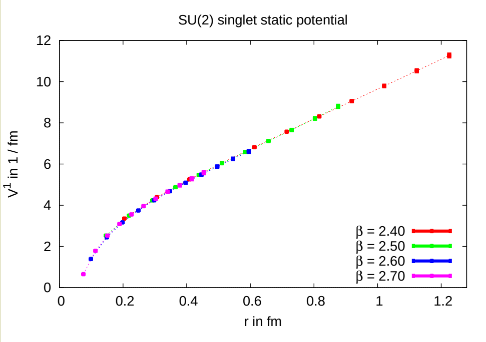
[O. Philipsen and M. Wagner, arXiv:1305.5957 [hep-lat]]

Introduction: singlet static potential (1)

- The (singlet) static potential V^1 is a very common and important observable in lattice gauge theory.
- It is the energy of a static antiquark $\bar{Q}(\mathbf{x})$ and a static quark $Q(\mathbf{y})$ in a colour singlet (i.e. a gauge invariant) orientation as a function of the separation $r \equiv |\mathbf{x} - \mathbf{y}|$.
- The spin of a static quark is irrelevant, i.e. in the following
  - no spin indices or γ matrices,
  - only spinless colour charges,
    $\bar{Q}^a_A(\mathbf{x}) = (Q^{a,\dagger}(\mathbf{x})\gamma_0)_A \rightarrow Q^{a,\dagger}(\mathbf{x})$,
    $Q^a_A(\mathbf{y}) \rightarrow Q^a(\mathbf{y})$,
    where a denotes a colour index and A a spin index.



Introduction: singlet static potential (2)



- The singlet static potential for gauge group $SU(N)$ can be obtained as follows:

  (1) Define a trial state

  $$|\Phi^1\rangle \equiv \bar{Q}(\mathbf{x})U(\mathbf{x},\mathbf{y})Q(\mathbf{y})|0\rangle.$$

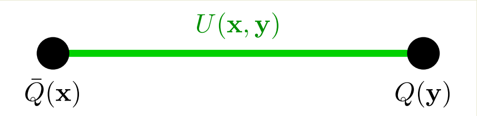
  (2) The temporal correlation function of this trial state simplifies to the well known Wilson loop,

  $$\langle\Phi^1(t_2)|\Phi^1(t_1)\rangle = e^{-2M\Delta t}N\Big\langle W_1(r,\Delta t)\Big\rangle \quad , \quad \Delta t \equiv t_2 - t_1 > 0.$$

  (3) The singlet static potential $V^1 \equiv V_0^1$ can be obtained from the asymptotic exponential behaviour,

  $$\Big\langle W_1(r,\Delta t)\Big\rangle = \sum_{n=0}^{\infty} c_n \exp\Big(-V_n^1(r)\Delta t\Big) \overset{\Delta t\to\infty}{\propto} \exp\Big(-V^1(r)\Delta t\Big)$$

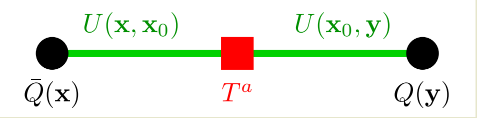
  $$V^1(r) = -\lim_{\Delta t\to\infty} \frac{\langle\dot{W}_1(r,\Delta t)\rangle}{\langle W_1(r,\Delta t)\rangle}.$$



Colour adjoint static potential (1)

- **Goal of this work:** compute and interpret the potential of a static antiquark $\bar{Q}(\mathbf{x})$ and a static quark $Q(\mathbf{y})$ in a colour adjoint (i.e. a gauge variant) orientation in various gauges as a function of the separation $r \equiv |\mathbf{x} - \mathbf{y}|$.
- A colour adjoint orientation of a static antiquark and a static quark can be obtained by inserting the generators of the colour group T^a (e.g. for $SU(3)$, $T^a = \lambda^a/2$), i.e. $\bar{Q} T^a Q|0\rangle$.
- If the static antiquark and the static quark are separated in space, a straightforward generalisation is

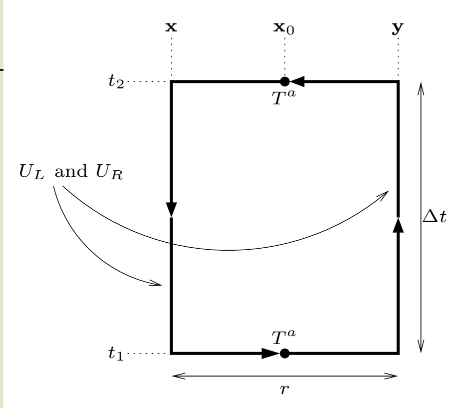
$$|\Phi^{T^a}\rangle \quad \equiv \quad \bar{Q}(\mathbf{x}) U(\mathbf{x}, \mathbf{x}_0) T^a U(\mathbf{x}_0, \mathbf{y}) Q(\mathbf{y})|0\rangle.$$



- A corresponding definition of the colour adjoint static potential has been proposed and used in pNRQCD (a framework based on perturbation theory).

[N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. **77**, 1423 (2005) [hep-ph/0410047]]

Colour adjoint static ... (2)



- We discuss non-perturbative calculations analogous as for the singlet static potential in various gauges,

$$\langle \Phi^{T^a}(t_2) | \Phi^{T^a}(t_1) \rangle = e^{-2M\Delta t} N \Big\langle W_{T^a}(r, \Delta t) \Big\rangle \quad ,$$

$$W_{T^a}(r, \Delta t) \equiv \frac{1}{N} \mathrm{Tr}\Big(T^a U_R T^{a,\dagger} U_L\Big)$$

$$\Big\langle W_{T^a}(r, \Delta t) \Big\rangle = \sum_{n=0}^{\infty} c_n \exp\Big(-V_n^{T^a}(r) \Delta t\Big) \overset{\Delta t \to \infty}{\propto} \exp\Big(-V^{T^a}(r) \Delta t\Big).$$

- In particular we are interested,
  - whether the colour adjoint static potential $V^{T^a} \equiv V_0^{T^a}$ is gauge invariant (i.e. whether the obvious gauge dependence of the correlation function $\langle W_{T^a}(r, \Delta t) \rangle$ only appears in the matrix elements c_n),
  - whether V^{T^a} indeed corresponds to the potential of a static antiquark and a static quark in a colour adjoint orientation, or whether it has to be interpreted differently.

V^{T^a} without gauge fixing

- Without gauge fixing

$$\Big\langle W_{T^a}(r, \Delta t) \Big\rangle \quad = \quad 0,$$

because this correlation function is gauge variant (and does not contain any gauge invariant contribution).

$\rightarrow$ Without gauge fixing the calculation of a colour adjoint static potential fails.

V^{T^a} in Coulomb gauge

- Coulomb gauge: $\nabla \mathbf{A}^g(x) = 0$, which amounts to an independent condition on every time slice t.

- The remaining residual gauge symmetry corresponds to global independent colour rotations $h^{\mathrm{res}}(t) \in SU(N)$ on every time slice t; with respect to this residual gauge symmetry the colour adjoint Wilson loop transforms as

$$\begin{aligned} \Big\langle W_{T^a}(r, \Delta t) \Big\rangle \quad &= \quad \frac{1}{N} \mathrm{Tr}\Big(T^a U_R T^{a,\dagger} U_L\Big) \quad \rightarrow_{h^{\mathrm{res}}} \\ &\rightarrow_{h^{\mathrm{res}}} \quad \frac{1}{N} \mathrm{Tr}\Big(h^{\mathrm{res},\dagger}(t_1) T^a h^{\mathrm{res}}(t_1) U_R h^{\mathrm{res}}(t_2) T^{a,\dagger} h^{\mathrm{res},\dagger}(t_2) U_L\Big). \end{aligned}$$

- Since $h^{\mathrm{res}}(t_1)$ and $h^{\mathrm{res}}(t_2)$ are independent, the situation is analogous to that without gauge fixing, i.e.

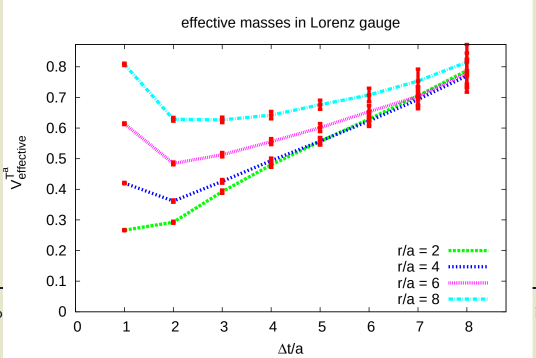
$$\Big\langle W_{T^a}(r, \Delta t) \Big\rangle_{\mathrm{Coulomb\ gauge}} \quad = \quad 0.$$

$\rightarrow$ In Coulomb gauge the calculation of a colour adjoint static potential fails.

V^{T^a} in Lorenz gauge

- Lorenz gauge: $\partial_\mu A^g_\mu(x) = 0$.
- In Lorenz gauge a Hamiltonian or a transfer matrix does not exist.
- Only gauge invariant correlation functions like the ordinary Wilson loop $\langle W_1(r, \Delta t)\rangle$ exhibit an asymptotic exponential behaviour and, therefore, allow the determination of energy eigenvalues.
- The colour adjoint Wilson loop $\langle W_{T^a}(r, \Delta t)\rangle_{\text{Lorenz gauge}}$ does not decay exponentially in the limit of large Δt.

$\rightarrow$ The physical meaning of a colour adjoint static potential determined from $\langle W_{T^a}(r, \Delta t)\rangle_{\text{Lorenz gauge}}$ (as frequently done in perturbation theory) is unclear.



V^{T^a} in temporal gauge (1)

- Temporal gauge: $\partial_\mu A_0^g(x) = 0$ or equivalently $U_0^g(x) = 1$.
- Temporal links gauge transform as

$$U_0^g(t, \mathbf{x}) \quad = \quad g(t, \mathbf{x}) U_0(t, \mathbf{x}) g^\dagger(t + a, \mathbf{x}) \quad , \quad g(t, \mathbf{x}) \in SU(N).$$

- A possible choice to implement temporal gauge is

$$\begin{aligned} g(t = 2a, \mathbf{x}) &= U_0(t = a, \mathbf{x}), \\ g(t = 3a, \mathbf{x}) &= g(t = 2a, \mathbf{x}) U_0(t = 2a, \mathbf{x}) \quad = \quad U_0(t = a, \mathbf{x}) U_0(t = 2a, \mathbf{x}), \\ g(t = 4a, \mathbf{x}) &= g(t = 3a, \mathbf{x}) U_0(t = 3a, \mathbf{x}) \quad = \quad \ldots, \\ \ldots &= \ldots \end{aligned}$$

V^{T^a} in temporal gauge (2)

- By inserting the transformation to temporal gauge $g(t, \mathbf{x})$, the gauge variant colour adjoint Wilson loop turns into a gauge invariant observable:

$$\Big\langle W_{T^a}(r, \Delta t) \Big\rangle_{\text{temporal gauge}} =$$
$$= \frac{1}{N} \Big\langle \mathrm{Tr}\Big(U^{T^a,g}(t_1; \mathbf{x}, \mathbf{y}) U^{T^{a,\dagger},g}(t_2; \mathbf{y}, \mathbf{x}) \Big) \Big\rangle_{\text{temporal gauge}} = \quad \ldots \quad =$$
$$= \frac{2}{N(N^2-1)} \sum_a \sum_b \Big\langle \mathrm{Tr}\Big(T^a U_R T^b U_L \Big) \mathrm{Tr}\Big(T^a U(t_1, t_2; \mathbf{x}_0) T^b U(t_2, t_1; \mathbf{x}_0) \Big) \Big\rangle$$

$(U^{T^a}(\mathbf{x}, \mathbf{y}) = U(\mathbf{x}, \mathbf{x}_0) T^a U(\mathbf{x}_0, \mathbf{y}))$.

- $\mathrm{Tr}(T^a U_R T^b U_L)$: Wilson loop with generator insertions.

- $\mathrm{Tr}(T^a U(t_1, t_2; \mathbf{x}_0) T^b U(t_2, t_1; \mathbf{x}_0))$: propagator of a static adjoint quark.

$\rightarrow$ The colour adjoint Wilson loop in temporal gauge is a correlation function of a gauge invariant three-quark state, one fundamental static quark, one fundamental static anti-quark, one adjoint static quark.

V^{T^a} in temporal gauge (3)

- Equivalently, after defining

$$|\Phi^{Q\bar{Q}Q^{\mathrm{ad}}}\rangle \quad \equiv \quad Q^{\mathrm{ad},a}(\mathbf{x}_0)(\bar{Q}(\mathbf{x})U^{T^a}(\mathbf{x},\mathbf{y})Q(\mathbf{y}))|0\rangle,$$

one can verify

$$\langle\Phi^{Q\bar{Q}Q^{\mathrm{ad}}}(t_2)|\Phi^{Q\bar{Q}Q^{\mathrm{ad}}}(t_1)\rangle \quad \propto \quad \Big\langle W_{T^a}(r,\Delta t)\Big\rangle_{\mathrm{temporal\ gauge}}.$$

$\rightarrow$ V^{T^a} in temporal gauge should not be interpreted as the potential of a static quark and a static anti-quark, which form a colour-adjoint state.

$\rightarrow$ V^{T^a} in temporal gauge is the potential of a colour-singlet three-quark state.

$\rightarrow$ V^{T^a} in temporal gauge does not only depend on the $Q\bar{Q}$ separation $r = |\mathbf{x}-\mathbf{y}|$, but also on the position $s = |\mathbf{x}-\mathbf{x}_0|/2 - |\mathbf{y}-\mathbf{x}_0|/2$ of the static adjoint quark Q^{ad}, i.e. $V^{T^a}(r,s)$ (in the following we work with the symmetric alignment $\mathbf{x}_0 = (\mathbf{x}+\mathbf{y})/2$).

V^{T^a} in temporal gauge (4)

- A different approach, leading to the same result, is the transfer matrix formalism.

  [O. Jahn and O. Philipsen, Phys. Rev. D **70**, 074504 (2004) [hep-lat/0407042]]
  [O. Philipsen, Nucl. Phys. B **628**, 167 (2002) [hep-lat/0112047]]

- One can perform a spectral analysis of the colour adjoint Wilson loop:

$$\Big\langle W_{T^a}(r, \Delta t) \Big\rangle_{\text{temporal gauge}} = \frac{1}{N} \sum_k e^{-(V_k^{T^a}(r) - \mathcal{E}_0)\Delta t} \sum_{\alpha,\beta} \Big| \langle k^a_{\alpha\beta} | U^{T^a}_{\alpha\beta}(\mathbf{x}, \mathbf{y}) | 0 \rangle \Big|^2 ,$$

  where $|k^a_{\alpha\beta}\rangle$ denotes states containing three static quarks (one fundamental static quark, one fundamental static anti-quark, one adjoint static quark).

$\rightarrow$ Again the conclusion is that V^{T^a} in temporal gauge is the potential of a colour-singlet three-quark state.

A gauge invariant definition via B fields?

- In the literature one can also find a proposal of a gauge invariant quantity, from which the colour adjoint static potential can possibly be determined,

$$W_B(r,\Delta t) \equiv \frac{1}{N}\mathrm{Tr}\Big(T^a U_R T^{b,\dagger} U_L\Big)\mathbf{B}^a(\mathbf{x}_0,t_1)\mathbf{B}^b(\mathbf{x}_0,t_2),$$

i.e. open colour indices are saturated by colour magnetic fields.

[N. Brambilla, A. Pineda, J. Soto and A. Vairo, Rev. Mod. Phys. **77**, 1423 (2005) [hep-ph/0410047]]

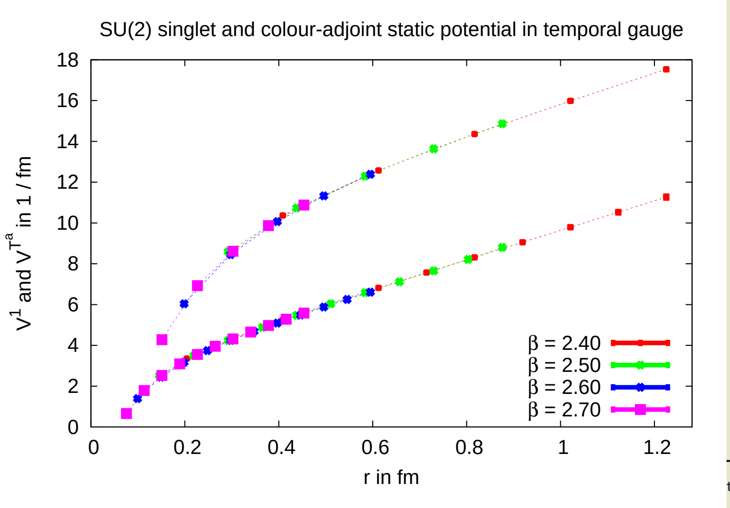
- However, using the transfer matrix formalism one can again perform a spectral analysis and show that only states with a fundamental quark and a fundamental antiquark $|k_{\alpha\beta}\rangle$ (i.e. singlet static potentials) contribute to the correlation function:

$$\Big\langle W_B(r,\Delta t)\Big\rangle = \sum_k e^{-(V_k^{1,-}(r)-\mathcal{E}_0)\Delta t}\sum_{\alpha,\beta}\Big|\langle k_{\alpha\beta}|U_{\alpha\beta}^{T^a\mathbf{B}^a}(\mathbf{x},\mathbf{y})|0\rangle\Big|^2 .$$

$\rightarrow$ $\langle W_B(r,\Delta t)\rangle$ is not suited to extract a colour adjoint static potential.

Numerical lattice results for $SU(2)$

- $SU(2)$ colour group, four different lattice spacings $a = 0.038\,\mathrm{fm} \ldots 0.102\,\mathrm{fm}$.
- In temporal gauge the colour adjoint (or rather $Q\bar{Q}Q^{\mathrm{ad}}$) static potential V^{T^a} is attractive,
  - for small separations stronger than the singlet static potential V^1,
  - for large separations the slope is the same as for the singlet static potential V^1 (indicates flux tube formation between QQ^{ad} and $\bar{Q}Q^{\mathrm{ad}}$).



LO perturbative calculations (1)

- Perturbation theory for static potentials is a good approximation for small quark separations and should agree in that region with corresponding non-perturbative results.

- Singlet static potential (gauge invariant, i.e. the gauge is not important):

$$V^1(r) \quad = \quad -\frac{(N^2-1)g^2}{8N\pi r} + \mathrm{const} + \mathcal{O}(g^4).$$

- Colour adjoint static potential (in Lorenz gauge):

$$V^{T^a}(r) \quad = \quad +\frac{g^2}{8N\pi r} + \mathrm{const} + \mathcal{O}(g^4).$$

  – In Lorenz gauge a Hamiltonian or a transfer matrix does not exist, i.e. the physical meaning is unclear; appears frequently in the literature.

  – The repulsive behaviour is not reproduced by any of the presented non-perturbative considerations or computations.

LO perturbative calculations (2)

- Colour adjoint static potential ("in temporal gauge"; more precisely: perturbative calculation in Lorenz gauge of the gauge invariant observable, which is equivalent to the colour adjoint Wilson loop in temporal gauge):

$$V^{Q\bar{Q}Q^{\mathrm{ad}}}(r, s=0) \quad = \quad -\frac{(4N^2-1)g^2}{8N\pi r} + \mathrm{const} + \mathcal{O}(g^4).$$

  - Attractive and stronger by a factor $4 \ldots 5$ than the singlet static potential (depending on N).
  - Qualitative agreement with numerical lattice results for $SU(2)$.

Conclusions

- We have discussed the non-perturbative definition of a static potential V^{T^a} for a quark antiquark pair in a colour adjoint orientation, based on Wilson loops with generator insertions $\langle W_{T^a}(r, \Delta t) \rangle$ in various gauges:
  - **Without gauge fixing/Coulomb gauge:** $\langle W_{T^a}(r, \Delta t) \rangle = 0$, i.e. the calculation of a potential V^{T^a} fails.
  - **Lorenz gauge:** a Hamiltonian or a transfer matrix does not exist, the physical meaning of a corresponding potential V^{T^a} is unclear.
  - **Temporal gauge:** a strongly attractive potential V^{T^a}, which should be interpreted as the potential of three quarks, i.e. $V^{T^a} = V^{Q\bar{Q}Q^{\mathrm{ad}}}$.

  Clearly the resulting potential V^{T^a} is gauge dependent.
- Saturating open colour indices with $\mathbf{B}^a$, yields a singlet static potential.
- LO perturbation theory in Lorenz gauge has long predicted V^{T^a} to be repulsive; it appears impossible, to reproduce this repulsive behaviour by a non-perturbative computation based on Wilson loops.