Lattice investigation of an inhomogeneous phase of the 2+1-dimensional Gross-Neveu model in the limit of infinitely many flavors

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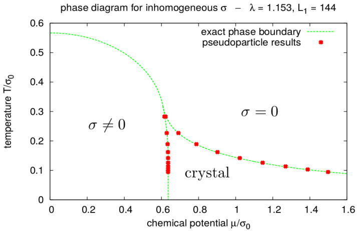
Introduction (1)

- **Long-term goal**: Compute the phase diagram of QCD.
  - Extremely difficult ...
  - ... e.g. "sign problem" in lattice QCD for chemical potential $\mu \neq 0$, computations very challenging/impossible.
- **QCD-inspired models in the $N_f \to \infty$ limit**:

  QCD-inspired = symmetries similar as in QCD, e.g. chiral symmetry
  $N_f \to \infty$ limit = infinite number of flavors
  - Inhomogeneous phases at large μ and small temperature T.

    inhomogeneous phase = phase with a spatially non-constant order parameter
  - Analytical results for the Gross-Neveu (GN) model in 1+1 dimensions.

    [O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **314**, 425 (2004) [hep-th/0402014]]
- **Are there inhomogeneous phases in QCD?**



Introduction (2)

- Project "**Inhomogeneous phases at high density**" of the CRC-TR 211 "**Strong-interaction matter under extreme conditions**" (universities of Bielefeld, Darmstadt, Frankfurt):
  - Goals:
    - Study the phase diagrams of various QCD-inspired models (GN, chiral GN, Nambu-Jona-Lasinio (NJL), ...) with particular focus on inhomogeneous phases.
    - Are there inhomogeneous phases in 2+1 or 3+1 dimensions?
    - Are there inhomogeneous phases with 2- or 3-dimensional modulations?
    - Determine the spatial modulation of the condensates (= order parameters) without using specific ansätze (e.g. no restriction to a chiral density wave).
    - Phase structure not only with respect to μ and T but also isospin and strangeness chemical potential μ_I, μ_S.
    - Are there inhomogeneous phases at finite N_f?
  - Methods:
    - Lattice field theory.
      * Lattice field theory computations in the $N_f \to \infty$ limit (this talk).
      * Lattice field theory simulations at finite N_f (talk by Julian Lenz on Friday).
    - Functional Renormalization Group (poster by Martin Steil on Thursday).

Outline

1. GN model in 2+1 dimensions
2. Problems
   - Discrete symmetry $\sigma \rightarrow -\sigma$ and fermion representation
   - Fermion discretization
   - Efficient computation of $\det(Q)$ and minimization of S_{eff}/N
   - Inhomogeneous phases and finite volume
3. Numerical results, 1+1-dimensional GN model
4. Numerical results, 2+1-dimensional GN model

GN model in 2+1 dimensions (1)

- At the moment we study the GN model 2+1 dimensions.
  - Do inhomogeneous phases exist in 2+1 dimensions?
  - Is the phase diagram in 2+1 dimensions similar to the analytically known phase diagram in 1+1 dimensions?
  - Are there inhomogeneous phases with 2-dimensional modulations?
- Action:

$$S = \int d^3x \left(\sum_{j=1}^{N_f} \bar{\psi}_j \Big(\gamma_\nu \partial_\nu + \gamma_0 \mu \Big) \psi_j - \frac{g^2}{2} \left(\sum_{j=1}^{N_f} \bar{\psi}_j \psi_j \right)^2 \right).$$

- After introducing a scalar field σ (= condensate) and performing the integration over fermionic fields

$$S_{\text{eff}} = N_f \left(\frac{1}{2\lambda} \int d^2x \, \sigma^2 - \ln \Big(\det(\underbrace{\gamma_\nu \partial_\nu + \gamma_0 \mu + \sigma}_{=Q}) \Big) \right)$$

$$Z = \int D\sigma \, e^{-S_{\text{eff}}},$$

where $\lambda = Ng^2$.

GN model in 2+1 dimensions (2)

- $N \to \infty$: only "a single field configuration" important in $\int D\sigma\, e^{-S_{\text{eff}}}$ (minimum of S_{eff}/N).
- For numerical treatment the degrees of freedom have to be reduced to a finite number

  $\to$ finite volume and discretization needed.
  - For example lattice field theory.
  - There are other possibilities to discretize, e.g. finite mode discretization, discretization by piecewise polynomial functions, etc.

    [M. Wagner, Phys. Rev. D **76**, 076002 (2007) [arXiv:0704.3023]]
- Challenges, problems:
  - Discrete symmetry $\sigma \to -\sigma$ and fermion representation: 2-component irreducible versus 4-component reducible representation? ...
  - Fermion discretization: Fermion doubling problem? Explicit breaking of chiral symmetry? Unphysical zero modes? ...
  - Efficient computation of $\det(Q)$ and minimization of S_{eff}/N: After discretization, Q is a large matrix ...
  - Inhomogeneous phases and finite volume: Not just exponentially small corrections (size of the inhomogeneous structures versus size of the volume) ...

Discrete symmetry $\sigma \to -\sigma$ and fermion representation

- One can show $S_{\text{eff}}[+\sigma] = S_{\text{eff}}[-\sigma]$ (i.e. S_{eff} has a discrete symmetry).
- One can show $\sigma \propto \langle \sum_{j=1}^{N_f} \bar{\psi}_j \psi_j \rangle$.
- **1+1 dimensions**:
  - A possible irreducible 2×2 representation for the γ matrices is

$$\gamma_0 = \sigma_1 \quad , \quad \gamma_1 = \sigma_2 .$$

  - $\sigma \neq 0$ would indicate spontaneous breaking of the symmetry $\psi_j \to \sigma_3 \psi_j$.
  - Since σ_3 anticommutes with γ_0 and γ_1, it is appropriate to define $\gamma_5 = \sigma_3$ and to interpret the symmetry as discrete chiral symmetry.
- **2+1 dimensions**:
  - A possible irreducible 2×2 representation for the γ matrices is

$$\gamma_0 = \sigma_1 \quad , \quad \gamma_1 = \sigma_2 \quad , \quad \gamma_2 = \sigma_3 .$$

  - It is impossible to find a corresponding appropriate γ_5 matrix, i.e. a matrix, which anticommutes with γ_0, γ_1 and γ_2.
  - Consequently, a non-vanishing σ cannot be interpreted as a signal for chiral symmetry breaking.
  - A possibility to retain the interpretation of σ as chiral order parameter is to use a reducible 4×4 representation.
  - One can show that the phase diagrams for the irreducible 2×2 representation and the reducible 4×4 representation are identical.

Fermion discretization (1)

- Various discretizations tested.
- Expansion in a set of basis functions, e.g. plane waves,

$$\psi(x,t) \;\rightarrow\; \sum_{m_t,m_x} c_{m_t,m_x} e^{i(p_{m_t} t + p_{m_x} x)} \quad , \quad \sigma(x) \;\rightarrow\; \sum_{m_x} d_{m_x} e^{i p_{m_x} x}$$

with $p_{m_t} = 2\pi(m_t - 1/2)/L_t$, $p_{m_x} = 2\pi m_x/L_x$, $d_{m_x} = (d_{-m_x})^*$.

[M. Wagner, Phys. Rev. D **76**, 076002 (2007) [arXiv:0704.3023]]

(−) Requires $\det(Q) = \det(Q^\dagger)$, not the case e.g. for $\mu_I \neq 0$ or $\mu_s \neq 0$.
  - $\det(Q) \rightarrow \det(\langle f_n|Q|f_{n'}\rangle)$, where f_n are basis functions, e.g. $f_{m_t,m_x} = e^{i(p_{m_t} t + p_{m_x} x)}$.
  - Problem: $\text{span}\{f_n\} \neq \text{span}\{Q f_n\}$, which causes artificially small eigenvalues or zero modes in $\langle f_n|Q|f_{n'}\rangle$ not present in Q.
    $\rightarrow$ Wrong and weird results.
  - Increasing the number of basis functions does not cure the problem.
  - Solution: $\ln(\det(Q)) \rightarrow (1/2)\ln(\det(Q^\dagger Q))$ (requires $\det(Q) = \det(Q^\dagger)$).

(−) Number of spatial modes in $\psi(x,t)$ should be larger than number of modes in $\sigma(x)$.
  - Q depends on $\sigma(x)$; basis functions representing $\psi(x,t)$ must be able to resolve more detail for an accurate approximation of $\det(Q)$.

(+) No fermion doubling.
(+) Resulting condensates $\sigma(x)$ are continuous functions.
(When using lattice field theory, $\sigma(x)$ is represented by a set of points σ_x.)

Fermion discretization (2)

- Lattice discretization:
  - Naively discretized fermions.

$$\psi(x,t) \quad \rightarrow \quad \psi_{x,t} \quad , \quad \partial_x \psi(x,t) \quad \rightarrow \quad \frac{\psi_{x+a,t} - \psi_{x-a,t}}{2a} \quad , \quad \ldots$$

    ($x, t = 0, a, 2a, \ldots$; a: lattice spacing).

    ($-$) Fermion doubling.
  - Naively discretized with non-symmetric derivatives.

    ($-$) No fermion doubling, but other severe problems.
  - Staggered fermions.

    [P. de Forcrand and U. Wenger, PoS LATTICE **2006**, 152 (2006) [hep-lat/0610117]]

    ($-$) Fermion doubling still present.
  - ...
- Most promising seems to be a combination of two approaches:
  - Plane wave expansion in t direction.

    ($+$) Easy analytical simplifications possible, e.g. $\det(Q)$ factorizes.
  - Naive lattice discretization in x direction.

    ($+$) Fermion doubling not a problem in the large-N limit ("$2 \times \infty = \infty$").

$$\psi(x,t) \quad \rightarrow \quad \psi_x(t) \quad = \quad \sum_m \psi_{x,m} e^{ip_m t} \quad , \quad \sigma(x) \quad \rightarrow \quad \sigma_x .$$

Efficient computation of $\det(Q)$ and ...

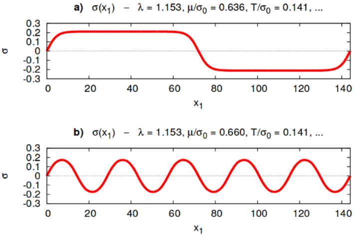
- $Q = \gamma_\nu \partial_\nu + \gamma_0 \mu + \sigma$ is a large matrix, e.g. $\mathcal{O}(10^5) \times \mathcal{O}(10^5)$ entries.
- Efficient computation of $\det(Q)$ and minimization of

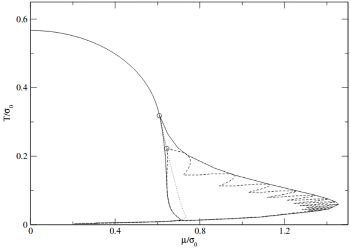
$$\frac{S_{\text{eff}}}{N_f} = \left(\frac{1}{2\lambda} \int d^2x \, \sigma^2 - \ln\Big(\det(Q)\Big)\right)$$

  need
  - preparatory analytical simplifications, e.g. to factorize $\det(Q)$,
  - efficient algorithms and codes.
- Work in progress.
- Details are rather technical, beyond the scope of this presentation.

Inhomogeneous phases and finite volume (1)

- Periodic modulation of the inhomogeneous condensate, wave length λ depends on (μ, T) (left figure [for 1+1 dimensions]).
- Extent of the finite volume L typically fixed.
- If L is a multiple of λ, i.e. $L \approx n\lambda$, $n \in \mathbb{N}_+$
  $\rightarrow$ no particular problems with the finite volume, correct results.
- If $L \approx (n - 1/2)\lambda$, $n \in \mathbb{N}_+$
  $\rightarrow$ modulation of the inhomogeneous condensate does not fit into the finite volume, severely distorted results (see right figure, oscillating dashed line).





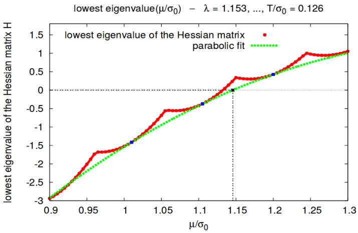
right figure [for 1+1 dimensions] from [P. de Forcrand and U. Wenger, PoS LATTICE **2006**, 152 (2006) [hep-lat/0610117]]

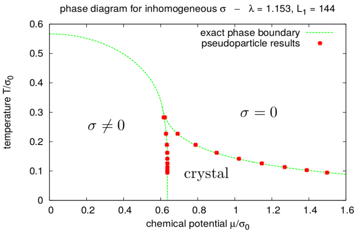
Inhomogeneous phases and finite volume (2)

- Infinite volume phase boundaries can be extracted from finite volume results.
- Phase boundary between inhomogeneous phase and restored phase:
  - Characterized by the appearance/disappearance of negative eigenvalues of the Hessian matrix

$$H_{xy} = \frac{\partial}{\partial \sigma_x} \frac{\partial}{\partial \sigma_y} S_{\text{eff}} \bigg|_{\sigma=0} .$$

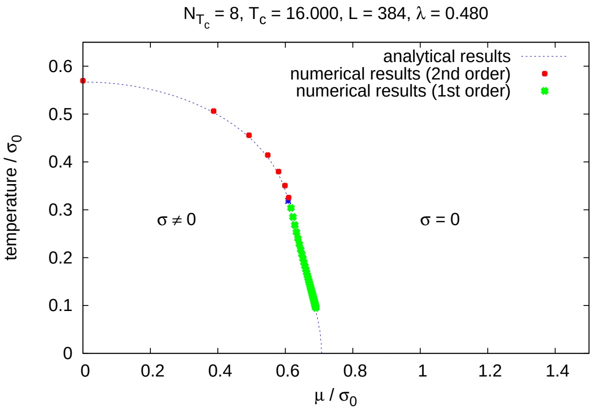
  - Lowest eigenvalue of H as a function of μ oscillates in a finite volume (red curve in left figure):
    - Minima: $L \approx n\lambda$, $n \in \mathbb{N}_+$, identical to the infinite volume result.
    - Maxima: $L \approx (n - 1/2)\lambda$, $n \in \mathbb{N}_+$, significantly different from the infinite volume result.
  - Fitting a smooth curve (e.g. a 2nd order polynomial) from below (green curve in left figure [for 1+1 dimensions]) approximates the infinite volume result.





Numerical results, 1+1-dimensional GN model (1)

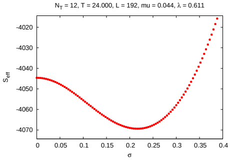
- Phase diagram with restriction to homogeneous condensate σ.
  (A Test of our method and implementation.)

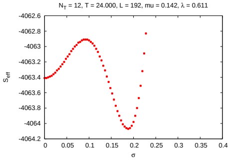


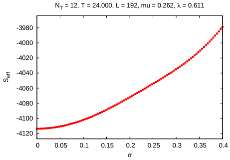
analytical results first obtained in [U. Wolff, Phys. Lett. B **157**, 303 (1985)]

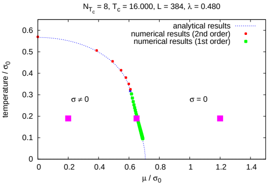
Numerical results, 1+1-dimensional GN model (2)

- $S_{\text{eff}}(\sigma)$ for homogeneous condensate σ.
  - Left: far inside the broken phase ($\mu/\sigma_0 = 0.20$, $T = T_c/3$).
  - Center: in the broken phase close to the 1st order phase boundary ($\mu/\sigma_0 = 0.65$, $T = T_c/3$).
  - Right: in the symmetric phase ($\mu/\sigma_0 = 1.20$, $T = T_c/3$).



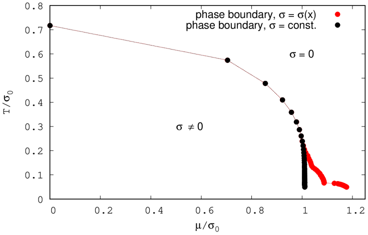


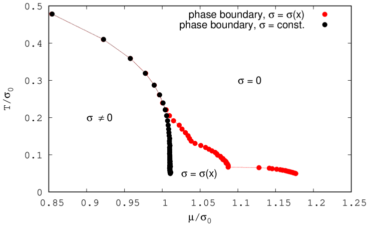




Numerical results, 2+1-dimensional GN model (1)

- Phase diagram:
  - **Black dots**: restriction to homogeneous condensate $\sigma = \text{const}$ (in agreement with available analytical results).
    [K. Urlichs, PhD thsis, University of Erlangen-Nuremberg (2007)]
  - **Red dots**: restriction to 1-dimensional modulations, $\sigma = \sigma(x)$.
    - Phase boundary via eigenvalues of the Hessian matrix ("stability analysis").
    - At finite volume, i.e. no extrapolation to infinite volume.

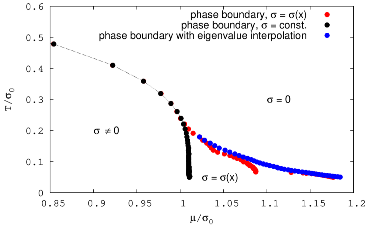


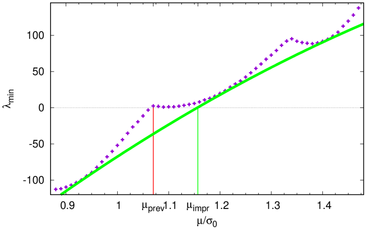


[M. Winstel, J. Stoll and M. Wagner, arXiv:1909.00064]

Numerical results, 2+1-dimensional GN model (2)

- Phase diagram:
  - **Black dots**: restriction to homogeneous condensate $\sigma = \mathrm{const.}$
  - **Red dots**: restriction to 1-dimensional modulations, $\sigma = \sigma(x)$.
    - Phase boundary via eigenvalues of the Hessian matrix ("stability analysis").
    - At finite volume, i.e. no extrapolation to infinite volume.
  - **Blue dots**: restriction to 1-dimensional modulations, $\sigma = \sigma(x)$.
    - Phase boundary via eigenvalues of the Hessian matrix ("stability analysis").
    - Extrapolated to infinite volume (see right figure; smallest eigenvalue of the Hessian matrix as a function of μ at fixed T).

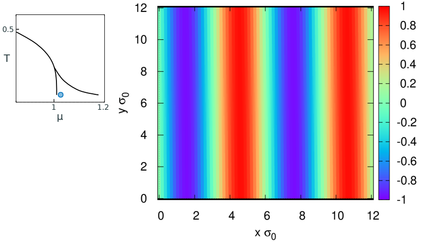


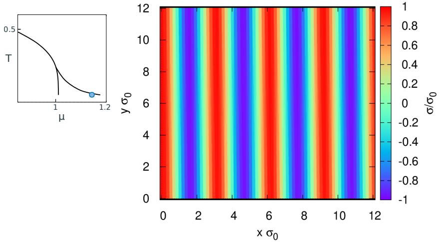


Numerical results, 2+1-dimensional GN model (3)

- Modulations of the condensate in the inhomogeneous phase at
  - $(\mu/\sigma_0, T/\sigma_0) = (1.025, 0.055)$ (left figure)
  - $(\mu/\sigma_0, T/\sigma_0) = (1.166, 0.055)$ (right figure)

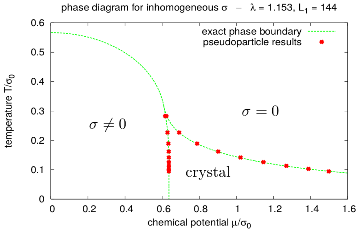
  (restriction to 1-dimensional modulations, $\sigma = \sigma(x)$).
- For increasing μ the wavelength decreases (as for the 1+1-dimensional GN model).

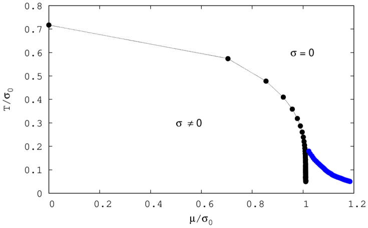


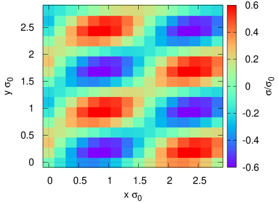


Numerical results, 2+1-dimensional GN model (4)

- Comparison of the phase diagram of the 1+1-dimensional (left figure) and the 2+1-dimensional (right figure) GN model.
  - Inhomogeneous phase in 2+1 dimensions smaller than for 1+1 dimensions.
  - Could become larger, when allowing 2-dimensional modulations, $\sigma = \sigma(x, y)$ ...?







Next steps

- Are there inhomogeneous phases with 2-dimensional modulations?
- Extend studies to 3+1 dimensions.
- Study the phase diagram of more realistic QCD-inspired models (chiral GN, Nambu-Jona-Lasinio (NJL), quark-meson model, ...) with particular focus on inhomogeneous phases.
  $\rightarrow$ Phase structure not only with respect to μ and T but also isospin and strangeness chemical potential μ_I, μ_S ...?
- Are there inhomogeneous phases at finite N_f?
  $\rightarrow$ Lattice field theory simulation of the path integral (talk by Julian Lenz on Friday).
  [L. Pannullo, J. Lenz, M. Wagner, B. Wellegehausen and A. Wipf, arXiv:1902.11066]
  [L. Pannullo, J. Lenz, M. Wagner, B. Wellegehausen and A. Wipf, arXiv:1909.11513]