Inhomogeneous phases at high density in QCD-inspired models

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"Seminar of Research Training Group", FSU Jena

May 15, 2018

Outline

1. Introduction
2. GN model in 1+1 dimensions
3. Problems
   - Discretization of the fermionic determinant
   - Efficient computation of det(Q) and minimization of S_{eff}/N
   - Inhomogeneous phases and finite volume
4. Numerical results

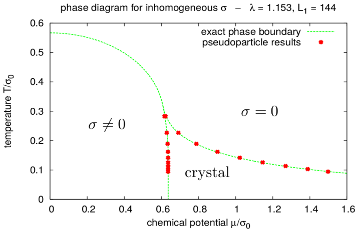
Introduction (1)

- **Long-term goal**: compute the phase diagram of QCD.
  - Extremely difficult ...
  - ... e.g. "sign problem" in lattice QCD for chemical potential $\mu \neq 0$, computations very challenging/impossible.
- **QCD-inspired models in the large-N limit**:

  QCD-inspired = symmetries similar as in QCD, e.g. chiral symmetry
  large-N limit = mean field (N: number of flavors)
  - Inhomogeneous phases at large μ and small temperature T.

    inhomogeneous phase = phase with a spatially non-constant order parameter
  - Cf. e.g.

    [O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **314**, 425 (2004) [hep-th/0402014]]
- **Are there inhomogeneous phases in QCD?**



Introduction (2)

- Goals of project A03 "**Inhomogeneous phases at high density**" of CRC-TR 211 "**Strong-interaction matter under extreme conditions**":
  - Study the phase diagram of various QCD-inspired models (Gross-Neveu (GN), chiral GN, Nambu-Jona-Lasinio (NJL), ...) with particular focus on inhomogeneous phases.
  - Methods:
    - Lattice field theory and related numerical methods (this talk).
    - Functional Renormalization Group (talk by A. Königstein).
  - Determine the spatial modulation of the condensates (= order parameters) without using specific ansätze (e.g. no restriction to a chiral density wave).
  - Are there inhomogeneous phases with 2- or 3-dimensional modulations?
  - Phase structure not only with respect to μ and T but also isospin and strangeness chemical potential μ_I, μ_S.
  - Are there inhomogeneous phases at finite N?

GN model in 1+1 dimensions (1)

- At the moment we study the GN model in 1+1 dimensions.
  - Phase diagram analytically known.
    [O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **314**, 425 (2004) [hep-th/0402014]]
  - Ideal to explore and test numerical/lattice field theory methods.
- Action:

$$S = \int d^2x \left(\sum_{j=1}^{N} \bar{\psi}_j \Big(\gamma_0(\partial_0 + \mu) + \gamma_1 \partial_1\Big) \psi_j - \frac{g^2}{2} \left(\sum_{j=1}^{N} \bar{\psi}_j \psi_j\right)^2\right).$$

- After introducing a scalar field σ (= condensate) and performing the integration over fermionic fields

$$S_{\text{eff}} = N\left(\frac{1}{2\lambda} \int d^2x\, \sigma^2 - \ln\Big(\det\big(\underbrace{\gamma_0(\partial_0 + \mu) + \gamma_1 \partial_1 + \sigma}_{=Q}\big)\Big)\right)$$

$$Z = \int D\sigma\, e^{-S_{\text{eff}}},$$

where $\lambda = Ng^2$.

GN model in 1+1 dimensions (2)

- $N \rightarrow \infty$: only one field configuration important in $\int D\sigma \, e^{-S_{\text{eff}}}$ (minimum of S_{eff}/N).
- For numerical treatment the degrees of freedom have to be reduced to a finite number
  $\rightarrow$ finite volume and discretization needed.
  - For example lattice field theory.
  - There are other possibilities to discretize, e.g. finite mode discretization, discretization by piecewise polynomial functions, etc.
    [M. Wagner, Phys. Rev. D **76**, 076002 (2007) [arXiv:0704.3023 [hep-lat]]]
- Challenges, problems:
  - Discretization of the fermionic determinant (various problems, e.g. fermion doubling problem, explicit breaking of chiral symmetry, unphysical zero modes, ...).
  - Efficient computation of $\det(Q)$ and minimization of S_{eff}/N (after discretization, Q is a large matrix).
  - Inhomogeneous phases and finite volume (size of the inhomogeneous structures versus size of the volume).

Discretization of the fermionic determinant (1)

- Various discretizations tested.
- Expansion in a set of basis functions, e.g. plane waves,

$$\psi(x,t) \rightarrow \sum_{m_t,m_x} c_{m_t,m_x} e^{i(p_{m_t} t + p_{m_x} x)} \quad , \quad \sigma(x) \rightarrow \sum_{m_x} d_{m_x} e^{i p_{m_x} x}$$

with $p_{m_t} = 2\pi(m_t - 1/2)/L_t$, $p_{m_x} = 2\pi m_x/L_x$, $d_{m_x} = (d_{-m_x})^*$.

[M. Wagner, Phys. Rev. D **76**, 076002 (2007) [arXiv:0704.3023 [hep-lat]]]

(−) Requires $\det(Q) = \det(Q^\dagger)$, not the case e.g. for $\mu_I \neq 0$ or $\mu_5 \neq 0$.
  - $\det(Q) \rightarrow \det(\langle f_n|Q|f_{n'}\rangle)$, where f_n are basis functions, e.g. $f_{m_t,m_x} = e^{i(p_{m_t} t + p_{m_x} x)}$.
  - Problem: $\text{span}\{f_n\} \neq \text{span}\{Q f_n\}$, which causes artificially small eigenvalues or zero modes in $\langle f_n|Q|f_{n'}\rangle$ not present in Q.
    $\rightarrow$ Wrong and weird results.
  - Increasing the number of basis functions does not cure the problem.
  - Solution: $\ln(\det(Q)) \rightarrow (1/2)\ln(\det(Q^\dagger Q))$ (requires $\det(Q) = \det(Q^\dagger)$).

(−) Number of spatial modes in $\psi(x,t)$ should be larger than number of modes in $\sigma(x)$.
  - Q depends on $\sigma(x)$; basis functions representing $\psi(x,t)$ must be able to resolve more detail for an accurate approximation of $\det(Q)$.

(+) No fermion doubling.

(+) Resulting condensates $\sigma(x)$ are continuous functions.
(When using lattice field theory, $\sigma(x)$ is represented by a set of points σ_x.)

Discretization of the fermionic determinant (2)

- Lattice discretization:
  - Naively discretized fermions.

$$\psi(x,t) \quad \rightarrow \quad \psi_{x,t} \quad , \quad \partial_x \psi(x,t) \quad \rightarrow \quad \frac{\psi_{x+a,t} - \psi_{x-a,t}}{2a} \quad , \quad \ldots$$

    ($x, t = 0, a, 2a, \ldots$; a: lattice spacing).

    ($-$) Fermion doubling.
  - Naively discretized with non-symmetric derivatives.

    ($-$) No fermion doubling, but other severe problems.
  - Staggered fermions.

    [P. de Forcrand and U. Wenger, PoS LATTICE **2006**, 152 (2006) [hep-lat/0610117]]

    ($-$) Fermion doubling still present.
  - ...
- Most promising seems to be a combination of two approaches:
  - Plane wave expansion in t direction.

    ($+$) Easy analytical simplifications possible, e.g. $\det(Q)$ factorizes.
  - Naive lattice discretization in x direction.

    ($+$) $\det(Q) = \det(Q^\dagger)$ not required, $\mu_I \neq 0$ and $\mu_S \neq 0$ might be possible.

    ($+$) Fermion doubling not a problem in the large-N limit ("$2 \times \infty = \infty$").

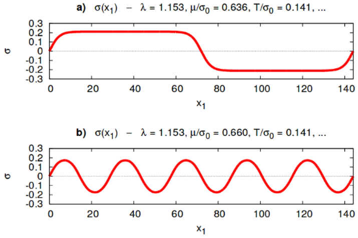
$$\psi(x,t) \quad \rightarrow \quad \psi_x(t) \quad = \quad \sum_m \psi_{x,m} e^{i p_m t} \quad , \quad \sigma(x) \quad \rightarrow \quad \sigma_x .$$

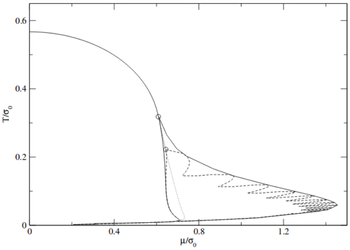
Efficient computation of det(Q) and ...

- Q is a large matrix, e.g. $\mathcal{O}(10^5) \times \mathcal{O}(10^5)$ entries.
- Efficient computation of det(Q) and minimization of $S_\text{eff}/N = \ldots - \ln(\det(Q))$ need
  - preparatory analytical simplifications, e.g. to factorize det(Q),
  - efficient algorithms and codes.
- Work in progress.
- Details are rather technical, beyond the scope of this presentation.

Inhomogeneous phases and finite volume (1)

- Periodic modulation of the inhomogeneous condensate, wave length λ depends on (μ, T) (left figure).
- Extent of the finite volume L typically fixed.
- If L is a multiple of λ, i.e. $L \approx n\lambda$, $n \in \mathbb{N}_+$
  $\rightarrow$ no particular problems with the finite volume, correct results.
- If $L \approx (n - 1/2)\lambda$, $n \in \mathbb{N}_+$
  $\rightarrow$ modulation of the inhomogeneous condensate does not fit into the finite volume, severely distorted results (right figure, oscillating dashed line).





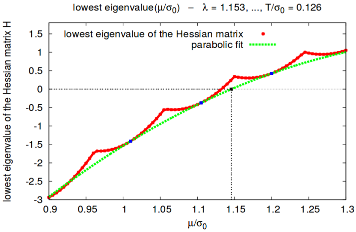
right figure from [P. de Forcrand and U. Wenger, PoS LATTICE **2006**, 152 (2006) [hep-lat/0610117]]

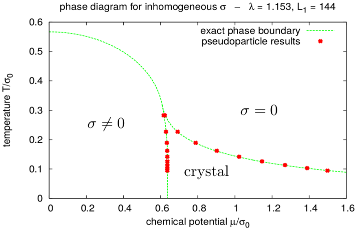
Inhomogeneous phases and finite volume (2)

- Infinite volume phase boundaries can be extracted from finite volume results.
- Phase boundary between inhomogeneous phase and restored phase:
  - Characterized by the appearance/disappearance of negative eigenvalues of the Hessian matrix

$$H_{xy} = \frac{\partial}{\partial \sigma_x} \frac{\partial}{\partial \sigma_y} S_{\text{eff}} \Big|_{\sigma=0} .$$

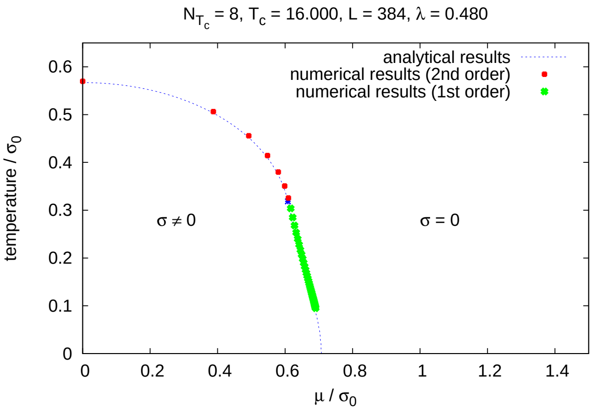
  - Lowest eigenvalue of H as a function of μ oscillates in a finite volume (red curve in left figure):
    - Minima: $L \approx n\lambda$, $n \in \mathbb{N}_+$, identical to the infinite volume result.
    - Maxima: $L \approx (n - 1/2)\lambda$, $n \in \mathbb{N}_+$, significantly different from the infinite volume result.
  - Fitting a smooth curve (e.g. a 2nd order polynomial) from below (green curve in left figure) corresponds to the infinite volume result.





Numerical results (1)

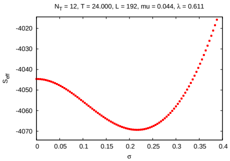
- Plane wave expansion in t direction, lattice discretization in x direction. Phase diagram with restriction to homogeneous condensate σ.

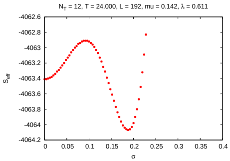


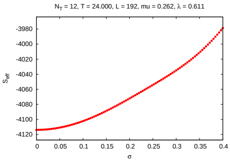
Numerical results (2)

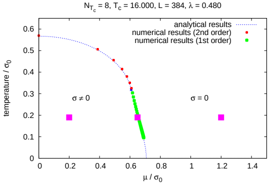
- Plane wave expansion in t direction, lattice discretization in x direction.

  $S_{\text{eff}}(\sigma)$ for homogeneous condensate σ.
  - Left: far inside the chirally broken phase ($\mu/\sigma_0 = 0.20$, $T = T_c/3$).
  - Center: in the chirally broken phase close to the 1st order phase boundary ($\mu/\sigma_0 = 0.65$, $T = T_c/3$).
  - Right: in the chirally restored phase ($\mu/\sigma_0 = 1.20$, $T = T_c/3$).





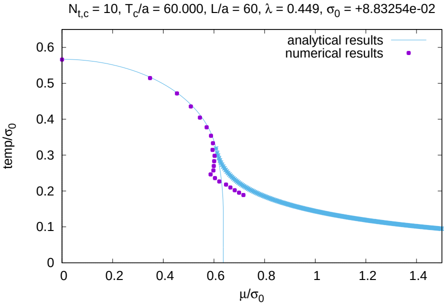




Numerical results (3)

- First tests for inhomogeneous condensate $\sigma = \sigma(x)$ successful:
  - Phase boundary via eigenvalues of the Hessian matrix.
  - Numerical results consistent with expectation.
  - Work in progress.

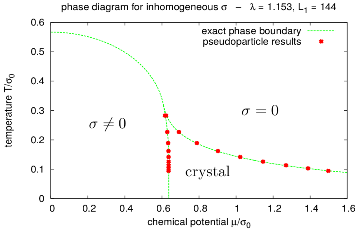
[M. Winstel, N. Zorbach, Bachelor of Science theses (ongoing), Goethe University Frankfurt am Main (2018)]

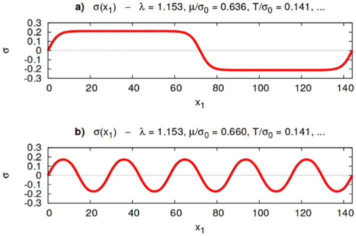


Numerical results (4)

- Expansion in piecewise polynomial functions both in t and in x direction. Phase diagram and condensate $\sigma(x)$.
  - **a)** close to the left boundary of the inhomogeneous phase.
  - **b)** inside the inhomogeneous phase.

[M. Wagner, Phys. Rev. D **76**, 076002 (2007) [arXiv:0704.3023 [hep-lat]]]





Next steps

- Study the phase diagram of various QCD-inspired models (Gross-Neveu (GN), chiral GN, Nambu-Jona-Lasinio (NJL), ...) with particular focus on inhomogeneous phases.
- Determine the spatial modulation of the condensates (= order parameters) without using specific ansätze (e.g. no restriction to a chiral density wave).
- Extend studies to 3+1 dimensions.
- Are there inhomogeneous phases with 2- or 3-dimensional modulations?
- Phase structure not only with respect to μ and T but also isospin and strangeness chemical potential μ_I, μ_S.
- Are there inhomogeneous phases at finite N?
  $\rightarrow$ Lattice field theory simulation of the path integral ...