The pseudoparticle approach in SU(2) Yang-Mills theory

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Outline

Part I: the pseudoparticle approach, a model for SU(2) Yang-Mills theory

- Basic principle.
- Numerical results: static quark antiquark potential, topological susceptibility, critical temperature.

Part II: properties of confining gauge field configurations

- Pseudoparticles of different size and profile.
- Instantons, antiinstantons and akyrons.

Part III: fermions in the pseudoparticle approach (work in progress)

- Problems with fermionic fields in the pseudoparticle approach.
- The Gross-Neveu model as testing ground.

Summary and outlook

Part I: the pseudoparticle approach, a model for SU(2) Yang-Mills theory

Basic principle (1)

- Pseudoparticle approach (PP approach; F. Lenz, M.W., 2005):
  - A numerical technique to approximate Euclidean path integrals (in this talk: mainly SU(2) Yang-Mills theory):

$$\left\langle \mathcal{O} \right\rangle = \frac{1}{Z} \int DA \, \mathcal{O}[A] e^{-S[A]}$$

$$S[A] = \frac{1}{4g^2} \int d^4x \, F^a_{\mu\nu} F^a_{\mu\nu} \quad , \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu .$$

  - A tool to analyze the importance of certain classes of gauge field configurations with respect to confinement.
  - Related work:
    - Ensembles of regular gauge instantons and merons (F. Lenz, J. W. Negele, M. Thies, 2003).
    - Ensembles of calorons with non-trivial holonomy (P. Gerhold, E.-M. Ilgenfritz, M. Müller-Preussker, 2006).

Basic principle (2)

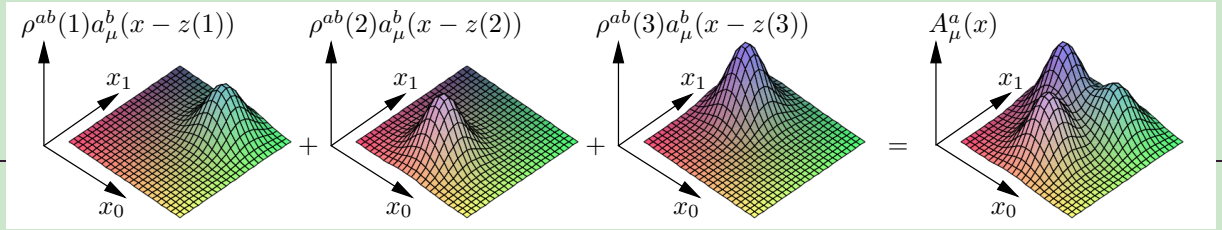
- PP: any gauge field configuration a_μ^a, which is localized in space and in time.
- Consider only those gauge field configurations, which can be written as a sum of a fixed number (≈ 400) of PPs:

$$A_\mu^a(x) \quad = \quad \sum_j \rho^{ab}(j) a_\mu^b(x - z(j))$$

(j: PP index; $\rho^{ab}(j)$: degrees of freedom of the j-th PP, i.e. amplitude and color orientation; $z(j)$: position of the j-th PP).

- Define the functional integration as an integration over the PP degrees of freedom:

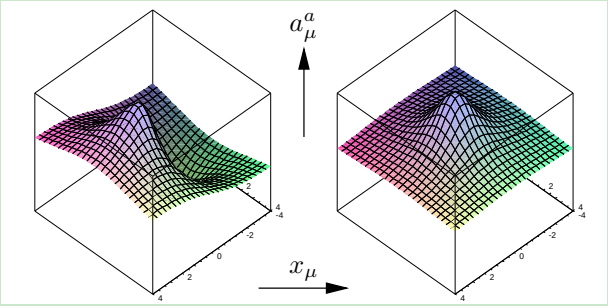
$$\int DA \ldots \quad \rightarrow \quad \int \left(\prod_j d\rho^{ab}(j) \right) \ldots$$



Building blocks of PP ensembles

- Building blocks of PP ensembles: "instantons", "antiinstantons", akyrons (λ: PP size).

$$
\begin{aligned}
a^a_{\mu,\mathrm{instanton}}(x) &= \eta^a_{\mu\nu}\frac{x_\nu}{x^2+\lambda^2} \\
a^a_{\mu,\mathrm{antiinstanton}}(x) &= \bar{\eta}^a_{\mu\nu}\frac{x_\nu}{x^2+\lambda^2} \\
a^a_{\mu,\mathrm{akyron}}(x) &= \delta^{a1}\frac{x_\mu}{x^2+\lambda^2}.
\end{aligned}
$$



- Instantons, antiinstantons and akyrons form a basis of all gauge field configurations in the "continuum limit".

- Degrees of freedom: amplitudes $\mathcal{A}(i)$, color orientations $\mathcal{C}^{ab}(i)$, positions $z(i)$.

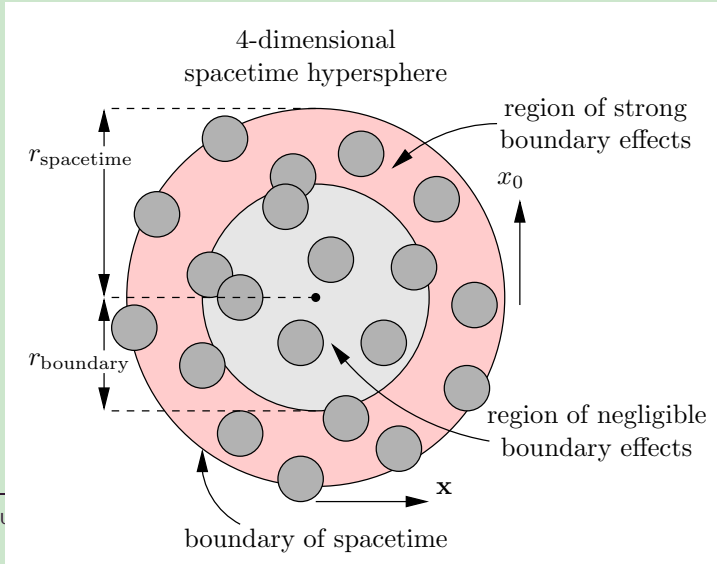
$$
\begin{aligned}
A^a_\mu(x) &= \mathcal{A}(i)\mathcal{C}^{ab}(i)a^a_{\mu,\mathrm{instanton}}(x-z(i)) \\
A^a_\mu(x) &= \mathcal{A}(i)\mathcal{C}^{ab}(i)a^a_{\mu,\mathrm{antiinstanton}}(x-z(i)) \\
A^a_\mu(x) &= \mathcal{A}(i)\mathcal{C}^{ab}(i)a^a_{\mu,\mathrm{akyron}}(x-z(i)).
\end{aligned}
$$

PP ensembles (1)

- PP ensemble: a fixed number of PPs inside a spacetime hypersphere.
- Gauge field:

$$A_\mu^a(x) = \sum_i \mathcal{A}(i)\mathcal{C}^{ab}(i)a^b_{\mu,\text{instanton}}(x - z(i)) + \sum_j \mathcal{A}(j)\mathcal{C}^{ab}(j)a^b_{\mu,\text{antiinstanton}}(x - z(j)) + \sum_k \mathcal{A}(k)\mathcal{C}^{ab}(k)a^b_{\mu,\text{akyron}}(x - z(k)).$$

- Choose color orientations $\mathcal{C}^{ab}(i)$ and positions $z(i)$ randomly.
- **A_μ^a is no classical solution (not even close to a classical solution)!**
- Long range interactions between PPs.



PP ensembles (2)

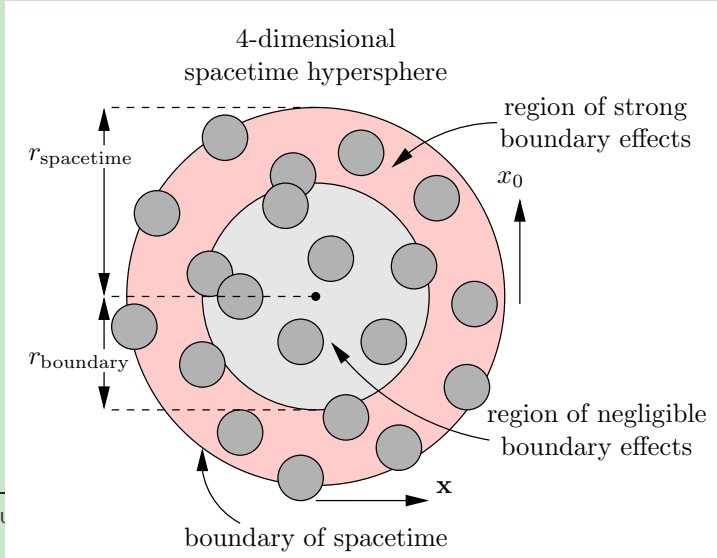
- Approximation of the path integral:

$$\left\langle \mathcal{O} \right\rangle \;=\; \frac{1}{Z} \int \left(\prod_i d\mathcal{A}(i) \right) \mathcal{O}(\mathcal{A}(i)) e^{-S(\mathcal{A}(i))}$$

  (integration over PP amplitudes).

- Solve this multidimensional integral via Monte-Carlo simulations.

- Exclude boundary effects: observables have to be "measured" sufficiently far away from the boundary.



Quark antiquark potential (1)

- Common tool to determine the potential of a static quark antiquark pair: Wilson loops (z: closed spacetime curve),

$$W_z[A] \quad = \quad \frac{1}{2}\mathrm{Tr}\left(P\left\{ \exp\left(i \oint dz_\mu \, A_\mu(z) \right) \right\} \right).$$

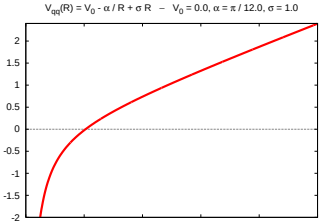
- Rectangular Wilson loop (R, T: spatial and temporal extension): $W_{(R,T)}$.
- Wilson loops $\leftrightarrow$ quark antiquark potential (R: quark antiquark separation):

$$V_{\mathrm{q\bar{q}}}(R) \quad = \quad -\lim_{T\to\infty} \frac{1}{T} \ln\left\langle W_{(R,T)} \right\rangle.$$

- Assumption: the potential for large quark antiquark separations can be parameterized according to

$$V_{\mathrm{q\bar{q}}}(R) \quad = \quad V_0 - \frac{\alpha}{R} + \sigma R.$$

$V_{\mathrm{q\bar{q}}}$ plotted against R



Quark antiquark potential (2)

Method 1: Determine the string tension σ and the Coulomb coefficient α

- "Guess" the functional dependence of ensemble averages of Wilson loops:

$$-\ln\left\langle W_{(R,T)}\right\rangle \;=\; V_0\Big(R+T\Big) - \alpha\left(\frac{R}{T}+\frac{T}{R}\right) + \beta + \sigma RT.$$

- Determine the string tension σ and the Coulomb coefficient α by fitting the "Wilson loop ansatz" to Monte-Carlo data for $-\ln\langle W_{(R,T)}\rangle$.
- Several approaches:
  - Area perimeter fits.
  - Creutz ratios.
  - Generalized Creutz ratios.
  - ...

Quark antiquark potential (3)

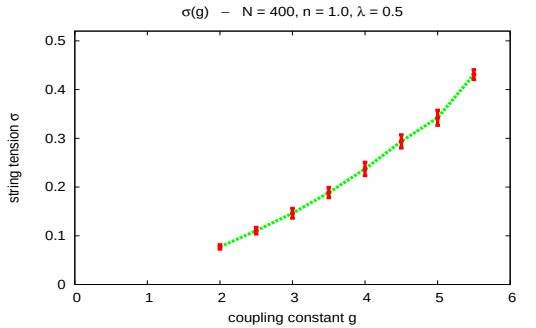
Method 1: Determine the string tension σ and the Coulomb coefficient α

- Parameterization of the quark antiquark potential:

$$V_{\mathrm{q}\bar{\mathrm{q}}}(R) \quad = \quad V_0 - \frac{\alpha}{R} + \sigma R.$$

- Results for PP ensembles containing ≈ 400 PPs:
  - **String tension $\sigma > 0$**
    $\longrightarrow$ linear potential for large quark antiquark separations, confinement.
  - **σ is an increasing function of the coupling constant g**
    $\longrightarrow$ adjust the physical scale by choosing appropriate values for g.
  - **Coulomb coefficient $\alpha > 0$**
    $\longrightarrow$ attractive "Coulomb-like" correction (as predicted by the bosonic string picture and by lattice calculations).

σ plotted against g



Quark antiquark potential (4)

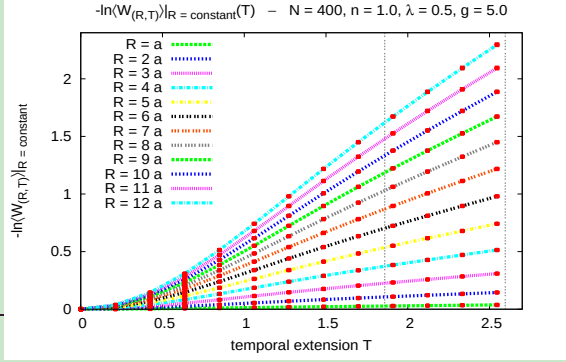
Method 2: Calculate the quark antiquark potential directly

- For large T:

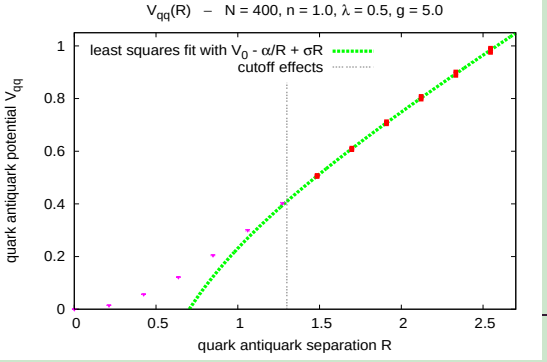
$$V_{\mathrm{q\bar{q}}}(R)T \quad \approx \quad -\ln\left\langle W_{(R,T)}\right\rangle.$$

- From the slope of $-\ln\langle W_{(R,T)}\rangle|_{R=\mathrm{constant}}$ we can read off $V_{\mathrm{q\bar{q}}}(R)$.
- Results are in agreement with our previous results.

$-\ln\langle W_{(R,T)}\rangle|_{R=\mathrm{constant}}$ plotted against T



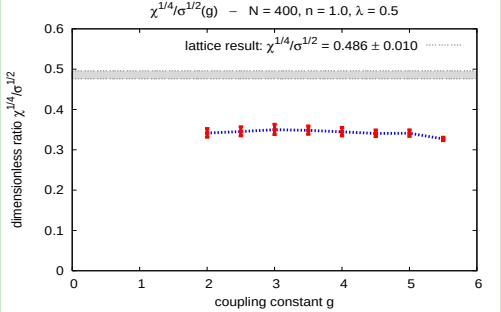
$V_{\mathrm{q\bar{q}}}$ plotted against R



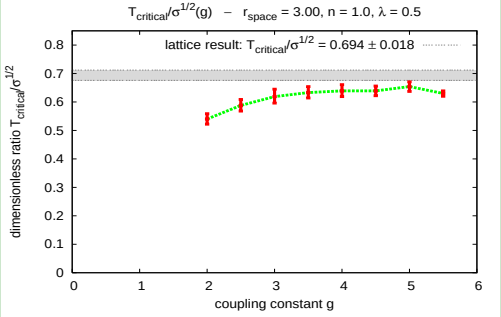
Quantitative results

- For quantitative results, including the string tension, we need other dimensionful quantities:
  - Topological susceptibility $\chi = \langle Q_V^2 \rangle / V$.
  - Critical temperature of the confinement deconfinement phase transition T_{critical}.
- Dimensionless quantities (physically meaningful): $\chi^{1/4}/\sigma^{1/2}$, $T_{\text{critical}}/\sigma^{1/2}$.
- Consider different $g = 2.0 \ldots 5.5$ (diameter of the spacetime hypersphere $0.8\,\text{fm} \ldots 1.9\,\text{fm}$).
- **Results are in qualitative agreement with results from lattice calculations**.
- **Consistent scaling behavior of σ, χ and T_{critical}.**

$\chi^{1/4}/\sigma^{1/2}$ plotted against g



$T_{\text{critical}}/\sigma^{1/2}$ plotted against g



Part II: properties of confining gauge field configurations

Properties of confining gauge field ...

- **What are essential properties of confining gauge field configurations?**
- **Which gauge field configurations are responsible for confinement?**
- Apply the PP approach with different types of PPs to study the effect of different classes of gauge field configurations on confinement:
  - PPs of different size.
  - PPs with a limited range of interaction (PPs with Gaussian profile).
  - PPs without topological charge (akyrons).

PPs of different size (1)

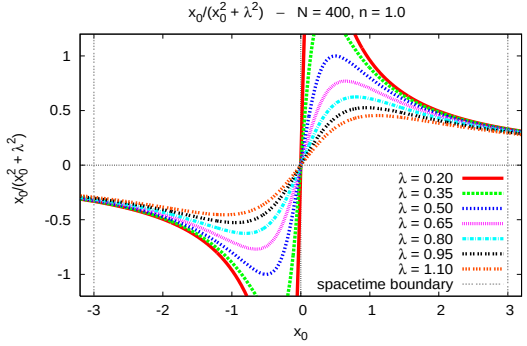
- Consider ensembles with PPs of different size λ:

$$a^a_{\mu,\mathrm{instanton}}(x) = \eta^a_{\mu\nu}\frac{x_\nu}{x^2+\lambda^2} \quad , \quad a^a_{\mu,\mathrm{antiinstanton}}(x) = \bar{\eta}^a_{\mu\nu}\frac{x_\nu}{x^2+\lambda^2} \quad ,$$
$$a^a_{\mu,\mathrm{akyron}}(x) = \delta^{a1}\frac{x_\mu}{x^2+\lambda^2}.$$

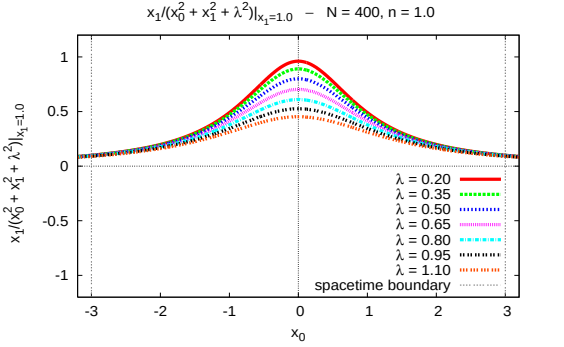
- **λ strongly affects the shape of a PP near its center but has essentially no effect on the $1/|x|$ long range behavior.**

- Typical PP profiles:

$x_0/(x_0^2+\lambda^2)$ plotted against x_0



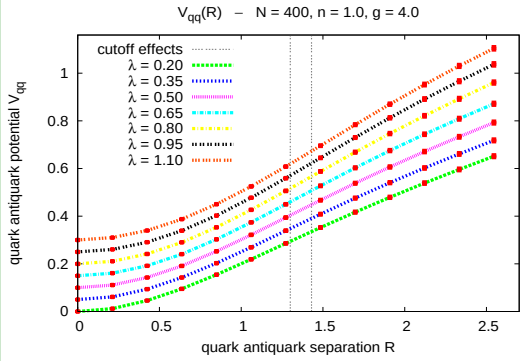
$x_1/(x_0^2+x_1^2+\lambda^2)|_{x_1=1.0}$ plotted against x_0



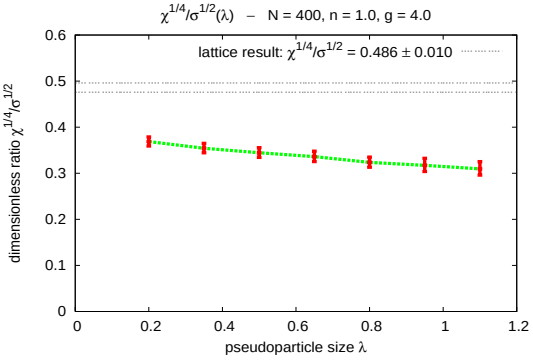
PPs of different size (2)

- $\lambda = 0.2, \ldots, 1.1.$
- The $\mathrm{q\bar{q}}$ potential and the dimensionless ratio $\chi^{1/4}/\sigma^{1/2}$ are essentially unaffected by the PP size λ.

$V_{\mathrm{q\bar{q}}}$ plotted against R



$\chi^{1/4}/\sigma^{1/2}$ plotted against λ

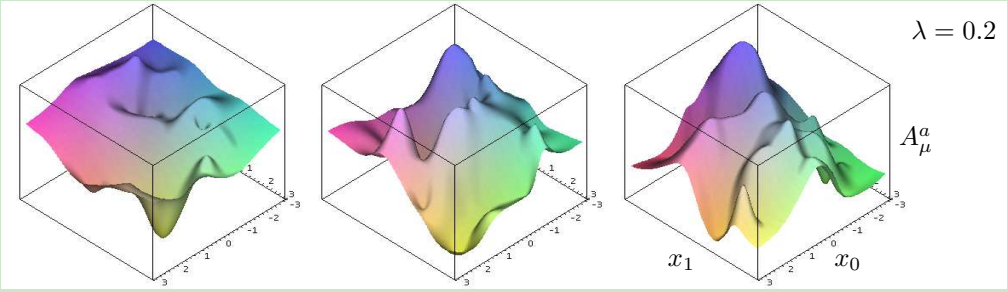


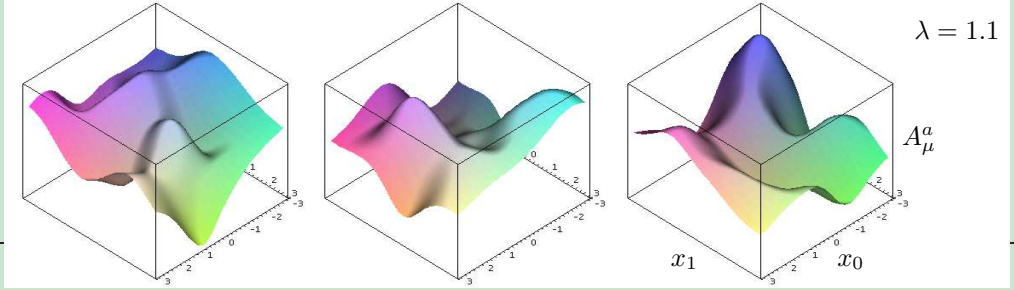
→ **Confinement is a consequence of the $1/|x|$ long range behavior of the PPs, which is unaffected by the size parameter λ.**

PPs of different size (3)

Typical gauge field configurations (PP size $\lambda = 0.2 \leftrightarrow \lambda = 1.1$)

- The global structure of typical gauge field configurations is the same.
- **For $\lambda = 0.2$ there are additional local UV fluctuations. These UV fluctuations have no effect on confinement and the string tension.**





PPs with Gaussian profile (1)

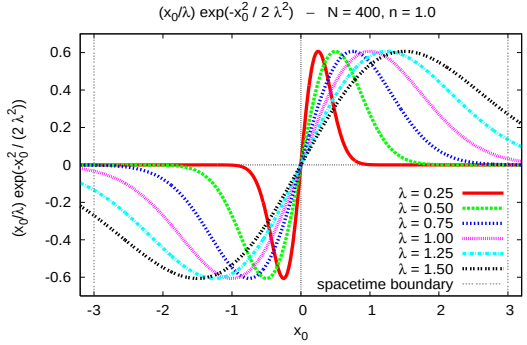
- Consider ensembles with Gaussian localized PPs of different size λ:

$$a^a_{\mu,\mathrm{instanton}}(x) = \eta^a_{\mu\nu} x_\nu e^{-x^2/2\lambda^2} \quad , \quad a^a_{\mu,\mathrm{antiinstanton}}(x) = \bar{\eta}^a_{\mu\nu} x_\nu e^{-x^2/2\lambda^2} \quad ,$$
$$a^a_{\mu,\mathrm{akyron}}(x) = \delta^{a1} x_\mu e^{-x^2/2\lambda^2} .$$

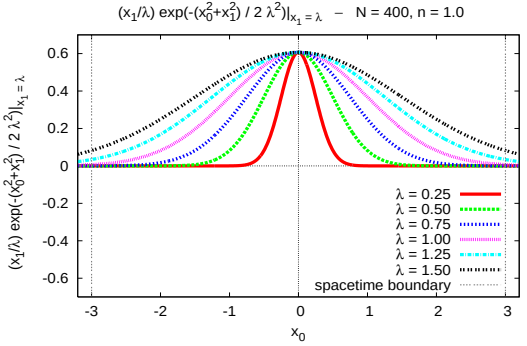
- **Gaussian localized PPs have a limited range of interaction, which is proportional to their size λ.**

- Typical PP profiles:

$(x_0/\lambda)\exp(-x_0^2/2\lambda^2)$ plotted against x_0

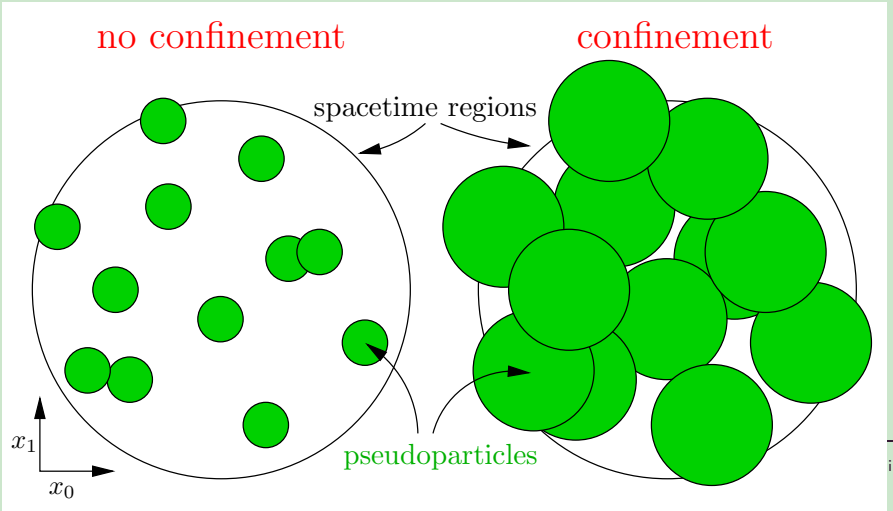


$(x_1/\lambda)\exp(-(x_0^2+x_1^2)/2\lambda^2)|_{x_1=\lambda}$ plotted against x_0

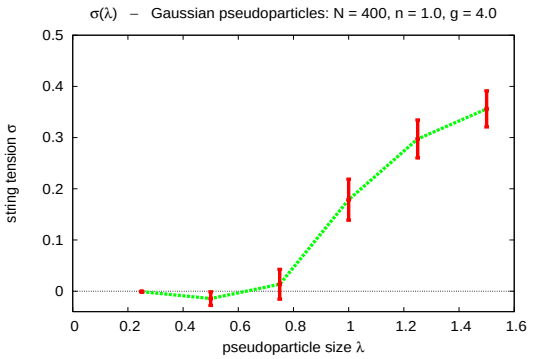


PPs with Gaussian profile (2)

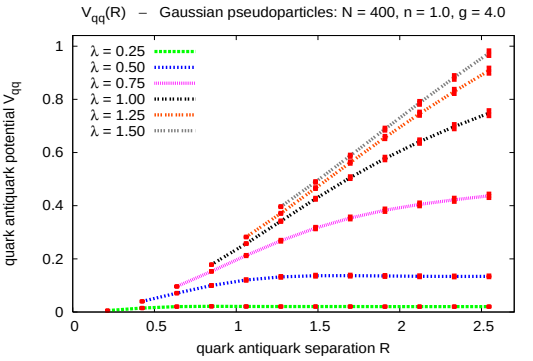
- Short range PPs ($\lambda \leq 0.50$)
  $\longrightarrow$ little overlap between neighboring PPs.
  $\longrightarrow$ no confinement.
- Long range PPs ($\lambda \geq 1.00$)
  $\longrightarrow$ significant overlap between neighboring PPs.
  $\longrightarrow$ confinement.
- **PP percolation $\leftrightarrow$ confinement.**



σ plotted against λ



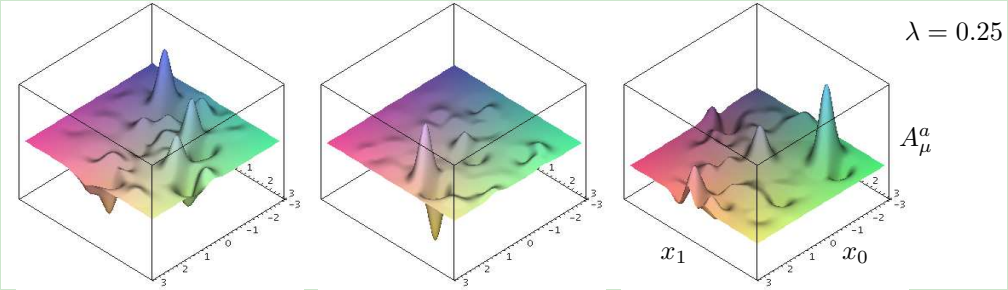
$V_{q\bar{q}}$ plotted against R

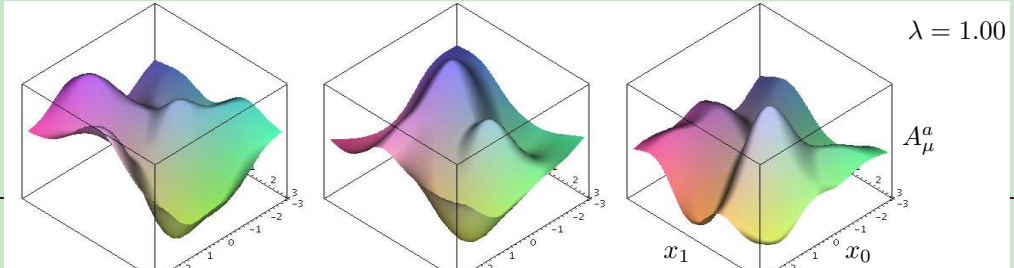


PPs with Gaussian profile (3)

Typical gauge field configurations ($\lambda = 0.25 \leftrightarrow \lambda = 1.00$)

- $\lambda = 0.25$: local UV fluctuations $\rightarrow$ no confinement.
- $\lambda = 1.00$: global excitations $\rightarrow$ confinement.
- **Gauge field configurations responsible for confinement contain extended structures and large area excitations.**

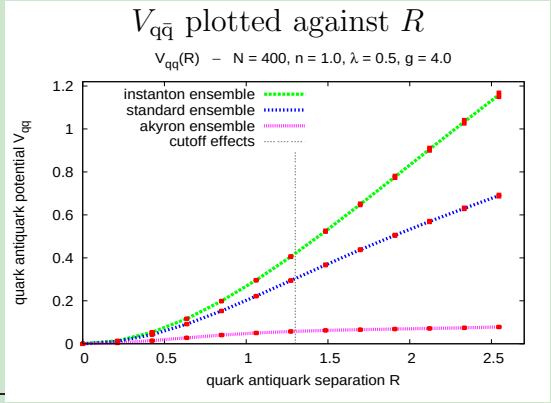




Instantons, antiinstantons and akyrons

- Consider the following ensembles:
  - Akyron ensemble: 400 akyrons (**topological charge density** $q = 0$).
  - Standard ensemble: 150 instantons, 150 antiinstantons, 100 akyrons.
  - Instanton ensemble: 200 instantons, 200 antiinstantons.
- No confinement in the akyron ensemble
  $\longrightarrow$ akyrons alone are not suited to reproduce Yang-Mills physics.
  $\longrightarrow$ **supports the expectation that confinement and topological charge are closely related.**
- Standard ensemble $\leftrightarrow$ instanton ensemble
  $\longrightarrow$ $(\chi^{1/4}/\sigma^{1/2})_{\text{standard}} = 0.35$
  $(\chi^{1/4}/\sigma^{1/2})_{\text{instanton}} = 0.26$

  $(\chi^{1/4}/\sigma^{1/2})_{\text{lattice}} = 0.49.$
  $\longrightarrow$ **using akyrons is beneficial with respect to quantitative results.**



Part III: fermionic fields in the pseudoparticle approach (first steps)

Basic principle (fermionic fields) (1)

- **How can fermions be included in the PP approach?**
- Action and partition function of **any theory with quadratic fermion interaction** (no restriction to SU(2) Yang-Mills theory anymore):

$$S[\psi,\bar{\psi},\phi] = \int dx \left(\bar{\psi}Q(\phi)\psi + \mathcal{L}(\phi)\right)$$

$$Z = \int D\psi\, D\bar{\psi}\, D\phi\, e^{-S[\psi,\bar{\psi},\phi]}$$

(Q: Dirac operator; ϕ: any type and number of bosonic fields, e.g. the non-Abelian gauge field in QCD).

Basic principle (fermionic fields) (2)

- Consider only those fermionic field configurations, which can be represented by a linear superposition a fixed number of localized building blocks:

$$\psi(x) \quad = \quad \sum_j \underbrace{\eta_j G_j(x)}_{j\text{-th PP}}$$

(η_j: Grassmann valued spinors; G_j: functions, which are localized in space as well as in time, i.e. PPs).

- Define the functional integration over all fermionic field configurations as an integration over the Grassmann valued spinors:

$$\int D\psi \, D\bar{\psi} \ldots \quad = \quad \int \left(\prod_j d\eta_j \, d\bar{\eta}_j \right) \ldots$$

Basic principle (fermionic fields) (3)

- Integrate out the fermions:

$$S_{\text{effective}}[\phi] = \int d^{d+1}x\, \mathcal{L}(\phi) - \ln\left(\det\left(\langle G_j|Q|G_{j'}\rangle\right)\right)$$

$$Z \propto \int D\phi\, e^{-S_{\text{effective}}[\phi]}$$

($\langle G_j|Q|G_{j'}\rangle$ is a finite matrix; **"Q-regularization"**).

- If $\det(Q)$ is real and positive, $\det(Q) = \sqrt{\det(Q^\dagger Q)}$. This suggests another PP regularization:

$$S_{\text{effective}}[\phi] = \int d^{d+1}x\, \mathcal{L}(\phi) - \frac{1}{2}\ln\left(\det\left(\langle G_j|Q^\dagger Q|G_{j'}\rangle\right)\right)$$

(**"$Q^\dagger Q$-regularization"**).

- **The "$Q^\dagger Q$-regularization" has significant advantages over the naive "Q-regularization".**

Q versus $Q^\dagger Q$ (1)

- For the sake of simplicity: consider all PPs G_j to be orthonormal, i.e. $\langle G_j | G_{j'} \rangle = \delta_{jj'}$ (this is not a restriction!).
- The problem of the Q-regularization:
  - **Applying the Dirac operator Q to one of the PPs $G_{j'}$ in general yields a function, which is (partially) outside the PP function space $\mathrm{span}\{G_n\}$, i.e.**

    $$QG_{j'}(x) \quad = \quad \sum_k a_{j'k} G_k(x) + h_{j'} H_{j'}(x)$$

    ($H_{j'}$ normalized, $H_{j'} \perp \mathrm{span}\{G_n\}$).
  - If $|\sum_k a_{j'k} G_k| \gg |h_{j'}| \quad \rightarrow \quad$ no problem.
  - If $|\sum_k a_{j'k} G_k| \lesssim |h_{j'}| \quad \rightarrow \quad$ when computing the fermionic matrix $\langle G_j | Q | G_{j'} \rangle$, a significant part of $QG_{j'}$ is simply ignored, just because $H_{j'}$ is perpendicular to the PP function space $\mathrm{span}\{G_n\}$.

Q versus $Q^\dagger Q$ (2)

- The advantage of the $Q^\dagger Q$-regularization:
  - **Both the left hand sides $\langle G_j|Q^\dagger$ and the right hand sides $Q|G_{j'}\rangle$ of the matrix elements $\langle G_j|Q^\dagger Q|G_{j'}\rangle$ might be outside the PP function space $\mathrm{span}\{G_n\}$, but they form the same function space, $\mathrm{span}\{QG_n\}$, in which their overlap is computed.**
- For "better arguments" cf. M. Wagner, arXiv:0704.3023 [hep-lat].

Testing ground: Gross-Neveu model (1)

- Action and partition function of the $1+1$-dimensional Gross-Neveu model:

$$S = \int d^2x \left(\sum_{n=1}^{N} \bar{\psi}^{(n)} \Big(\gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 \Big) \psi^{(n)} - \frac{g^2}{2} \left(\sum_{n=1}^{N} \bar{\psi}^{(n)} \psi^{(n)} \right)^2 \right)$$

$$Z = \int \left(\prod_{n=1}^{N} D\psi^{(n)} \, D\bar{\psi}^{(n)} \right) e^{-S}$$

(N: number of flavors; μ: chemical potential; g: coupling constant).

Testing ground: Gross-Neveu model (2)

- Introduce a real scalar field σ and integrate out the fermions:

$$S_{\text{effective}} = N\left(\frac{1}{2\lambda}\int d^2x\,\sigma^2 - \ln\left(\det\left(\gamma_0(\partial_0+\mu)+\gamma_1\partial_1+\sigma\right)\right)\right)$$

$$Z \propto \int D\sigma\, e^{-S_{\text{effective}}}$$

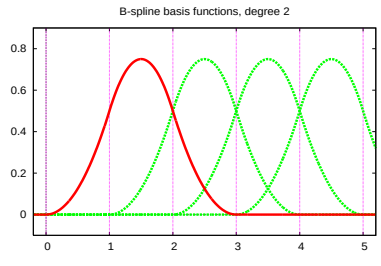
  $(\lambda = Ng^2)$.

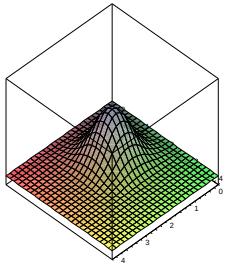
- Large-N limit:
  - $N \to \infty$, $\lambda = Ng^2 = \text{constant}$.
  - There is no need to compute the σ-path integral anymore.
  - It is sufficient to minimize $S_{\text{effective}}$ with respect to σ.
  - $\sigma = -g^2 \sum_{n=1}^{N} \bar{\psi}^{(n)}\psi^{(n)}$ (chiral condensate).

Fermionic PPs

- Fermionic PPs in this talk: a large number of uniformly distributed "hat functions" (B-spline basis functions of degree 2).
  - "Sensible set of field configurations" (any not too heavily oscillating field configuration can be approximated)
    $\longrightarrow$ we can expect to reproduce correct Gross-Neveu results.
  - Piecewise polynomial functions
    $\longrightarrow$ certain integrals can be calculated analytically.

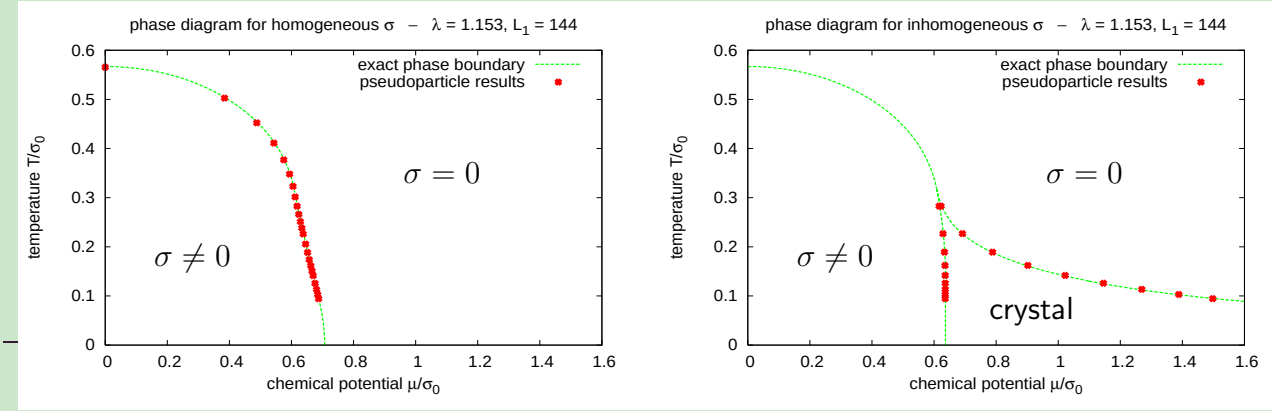
PPs G_j as functions of x





Phase diagram

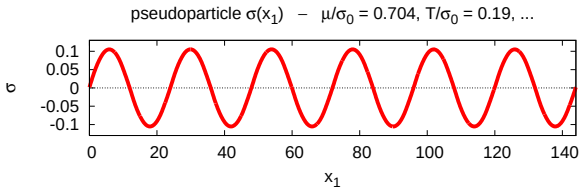
- Q-**regularization: completely wrong and useless results.**
  - No improvement, when using a larger number of PPs.
  - No improvement, when using a different type of PPs.
- $Q^\dagger Q$-**regularization: excellent agreement with analytical results.**
  - Homogeneous chiral condensate: analytical results by U. Wolff, 1985.
  - Inhomogeneous chiral condensate: analytical results by O. Schnetz, M. Thies, K. Urlichs, 2004.

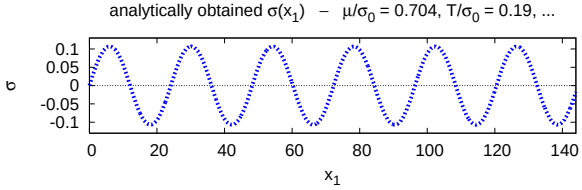


Chiral condensate

- $Q^\dagger Q$**-regularization: excellent agreement with analytical results.**
  - The sin-like behavior "inside the crystal phase" changes to a kink-antikink structure, when approaching the left phase boundary.

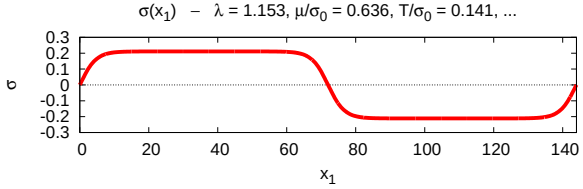
PP result

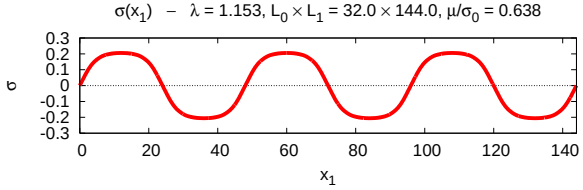


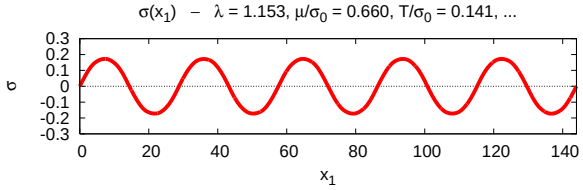


analytical result

close to the left phase boundary







inside the crystal phase

Summary and Outlook

Summary and conclusions (1)

- The PP approach with ≈ 400 instantons, antiinstantons and akyrons is able to reproduce many essential features of SU(2) Yang-Mills theory:
  - **Linear quark antiquark potential, i.e. confinement.**
  - **Consistent scaling behavior of σ, χ and T_{critical}.**
  - Dimensionless quantities $\chi^{1/4}/\sigma^{1/2}$, $T_{\text{critical}}/\sigma^{1/2}$ are in qualitative agreement with results from lattice calculations.
- Essential properties of confining gauge field configurations:
  - Long range PPs necessary for confinement (PP percolation)
    $\longrightarrow$ **confinement $\leftrightarrow$ extended structures and large area excitations.**
  - Instantons and antiinstantons (PPs with non-vanishing topological charge density) necessary for confinement
    $\longrightarrow$ **confinement $\leftrightarrow$ topological charge.**
  - Short range behavior of PPs irrelevant for confinement
    $\longrightarrow$ **confinement not affected by UV fluctuations.**

Summary and conclusions (2)

- Inclusion of fermions in the PP approach:
  - Always apply the $Q^\dagger Q$-regularization and not the naive Q-regularization.
  - The application of the PP approach to compute the phase diagram of the $1{+}1$-dimensional Gross-Neveu model in the large-N-limit has been a successful test.
  - Next steps:
    * Apply the PP approach to QCD.
    * Try to identify a small number of physically relevant fermionic PPs (PPs, which are able to approximate the low lying eigenmodes of the Dirac operator?).

Outlook

- Current research:
  - Improve the static quark antiquark potential:
    - Use improved operators (smeared Wilson loops) to extract the static quark antiquark potential (o.k.).
    - Casimir scaling for higher quark representations (o.k.).
    - String breaking for the adjoint representation (???).
  - Consider SU(2) Yang-Mills theory with dynamical fermions:
    - Chiral symmetry breaking by computing the low lying eigenmodes of the Dirac operator in the quenched approximation (Banks-Casher relation) (???).
- Goal: obtain a model with a small number of degrees of freedom, which exhibits chiral symmetry breaking and a confinement deconfinement phase transition at the same time.
- Compute further observables: pion masses, ...