Why the treatment of fermions on the lattice is difficult

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Introduction (1)

- *... In order to present your teaching skills, we kindly ask you to devote the first 10–15 minutes of your talk to explain "Why the treatment of fermions on the lattice is difficult" to a supposed student audience with knowledge of quantum mechanics and basics of field theory.*

Introduction (2)

- Motivation:
  - Lattice field theory is a method to solve (quantum) field theories numerically.
  - Most interesting/realistic quantum field theories contain fermions, e.g.
    * quantum electrodynamics contains electrons,
    * quantum chromodynamics (theory of quarks and gluons) contains quarks.
  - When treating these fermions by means of lattice field theory, certain difficulties arise:
    * Conceptual difficulties: when discretizing fermions in a straightforward way, unphysical states arise ("fermion doubling").
    * Practical difficulties: fermions require inversions of huge matrices (sophisticated algorithms and high performance computer systems mandatory).

Fermion doubling (1)

- To illustrate fermion doubling, consider a free spin-$1/2$ fermion in a single spatial dimension in relativistic quantum mechanics, e.g. an electron/positron in the absence of any external electromagnetic field.
- Space:
  - Continuum: $x \in \mathbb{R}$.
  - Lattice: $x = na$, $n \in \mathbb{Z}$, a is the sufficiently small lattice spacing (only space is discretized, time remains continuous in this example).

Fermion doubling (2)

- The corresponding wave equation is the Dirac equation (the relativistic spin-$1/2$ analog of the Schrödinger equation):
  - Continuum:

    $$i\gamma^0 \partial_t \psi(x,t) + i\gamma^1 \partial_x \psi(x,t) - m\psi(x,t) \quad = \quad 0.$$

  - Lattice:

    $$i\gamma^0 \partial_t \psi(x,t) + i\gamma^1 \frac{\psi(x+a,t) - \psi(x-a,t)}{2a} - m\psi(x,t) \quad = \quad 0.$$

  - $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric, i.e. $\eta_{\mu\nu} = \text{diag}(+1, -1, \ldots, -1)$.
  - m is the mass of the fermion.
  - In a single spatial dimension the wave function ψ has two components and γ^μ are 2×2 matrices.
  - In three spatial dimensions the wave function ψ has four components and γ^μ are 4×4 matrices.

Fermion doubling (3)

- Get rid of the derivative by means of the plane wave ansatz

$$\psi(x,t) \quad = \quad e^{i(Et-px)} u(p,E)$$

(u has also two/four components):

  - Continuum: $-\infty < p < +\infty$,

$$\Big(-\gamma^0 E + \gamma^1 p - m\Big) u(p,E) \quad = \quad 0.$$

  - Lattice: $-\pi/a < p \leq +\pi/a$, because $e^{-ipa} = e^{-i(p+2\pi/a)a}$,

$$\Big(-\gamma^0 E + i\gamma^1 \underbrace{\frac{e^{-ipa} - e^{+ipa}}{2a}}_{=-i\sin(pa)/a} - m\Big) u(p,E) \quad = \quad 0,$$

i.e.

$$\Big(-\gamma^0 E + \gamma^1 \frac{\sin(pa)}{a} - m\Big) u(p,E) \quad = \quad 0.$$

Fermion doubling (4)

- Get rid of the non-trivial matrix structure, i.e. decouple the two/four differential equations:
  - Continuum: ansatz $u(p,E) = (-\gamma^0 E + \gamma^1 p + m)v$ (v has also two/four components),

    $$\Big(E^2 - p^2 - m^2\Big)v \quad = \quad 0.$$

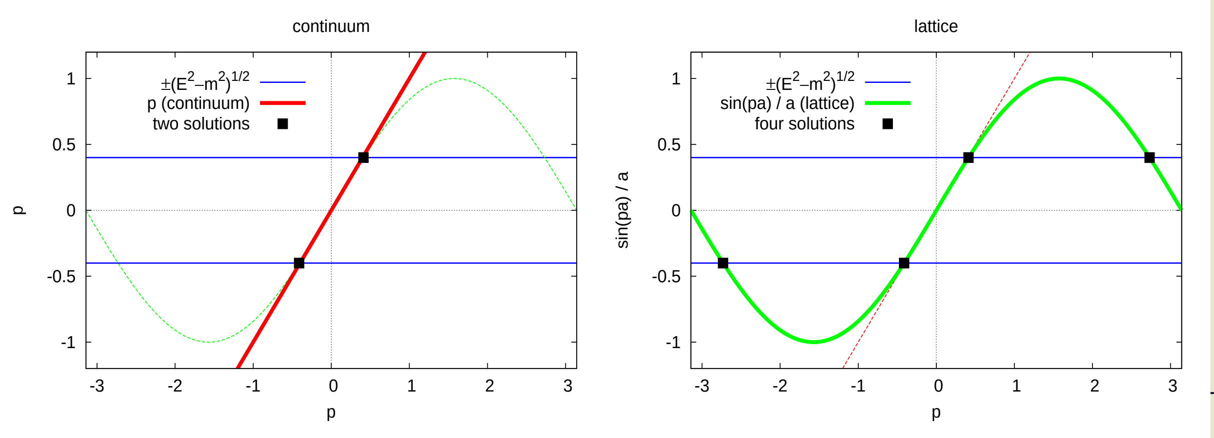
  - Lattice: ansatz $u(p,E) = (-\gamma^0 E + \gamma^1 \sin(pa)/a + m)v$ (v has also two/four components),

    $$\bigg(E^2 - \Big(\frac{\sin(pa)}{a}\Big)^2 - m^2\bigg)v \quad = \quad 0.$$

Fermion doubling (5)

- Solutions for given energy E:
  - Continuum: $p = \pm\sqrt{E^2 - m^2}$, $-\infty < p < +\infty$.
  - Lattice: $\sin(pa)/a = \pm\sqrt{E^2 - m^2}$, $-\pi/a < p \leq +\pi/a$.
  - Arbitrary v with the constraint $u(p, E) \neq 0$ or equivalently $\psi(x, t) \neq 0$ (one can show that there is a single possibility/two linearly independent possibilities to choose v both in the continuum and on the lattice).

$\longrightarrow$ Twice as many solutions/states on the lattice as in the continuum.



Fermion doubling (6)

- States are "duplicated" for every discretized spacetime dimension.
- For example when computing QED/QCD observables on a 3+1 dimensional spacetime lattice, for every physical state there are also fifteen unphysical "doubler states".
- There are ways to discretize fermions such that unphysical states do not appear:
  - Wilson fermions.
  - Staggered fermions.
  - Domain wall fermions.
  - Overlap fermions.
  - ...

  However all these discretizations have certain drawbacks: breaking of chiral symmetry, computationally extremely demanding, ...

Major computational costs (1)

- Computing the quark propagator amounts to inverting the Dirac operator; for interacting theories, e.g. QCD, where quark fields are coupled to gluonic fields A_μ, these inversions cannot be performed analytically:

$$\langle\bar{\psi}(x)\psi(y)\rangle \quad = \quad \bigg(\underbrace{\delta(x-y)\Big(\gamma_\mu(\partial_\mu^{(y)} - igA_\mu(y)) + m\Big)}_{\text{Dirac operator}}\bigg)^{-1}.$$

- On the lattice the Dirac operator is a large square matrix of dimension (# of lattice sites) $\times$ (# of spin components) $\times$ (# of color components).

- Example: QCD ($\rightarrow$ 3 color components) in four spacetime dimensions ($\rightarrow$ 4 spin components) on a 32^4 lattice
  $\rightarrow$ the Dirac operator is a square matrix of dimension $32^4 \times 4 \times 3 \approx 13 \times 10^6$
  $\rightarrow$ major computational costs to perform inversions of this matrix numerically.

Major computational costs (2)

- Performing the path integral over gluonic fields $\int DA_\mu$ requires the computation of the determinant of the Dirac operator
  $\longrightarrow$ even more challenging from a computational point of view.
- Consequence: sophisticated algorithms and high performance computer systems mandatory.