Heavy mesons and tetraquarks from lattice QCD

seminar, Technische Universität Darmstadt

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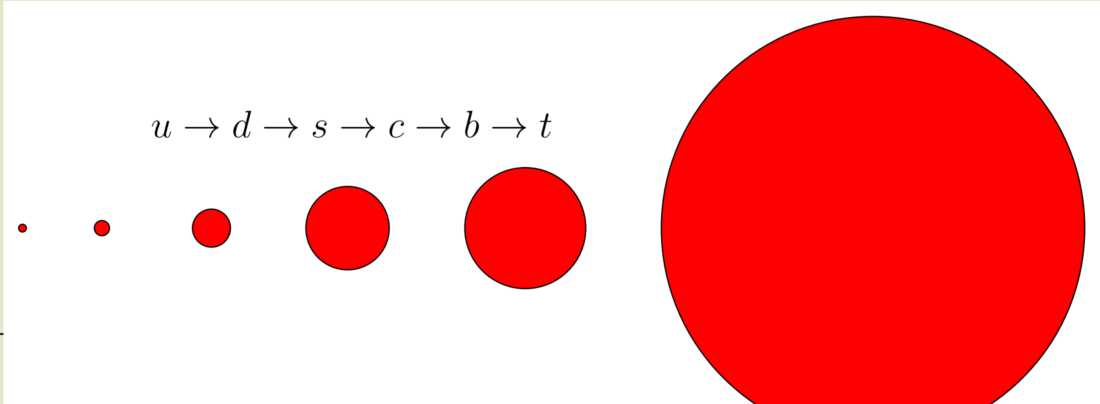
Outline

- **Part 1**:
  Introduction to QCD (quantum chromodynamics) and lattice QCD.
- **Part 2**:
  Heavy mesons and tetraquarks from lattice QCD
  (Two of our projects will be discussed).
  - **Part 2a**:
    D **mesons,** D_s **mesons and charmonium**
    ($\rightarrow$ precise lattice QCD results with all sources of error investigated, which can be compared to experiment in a meaningful way).
  - **Part 2b**:
    Heavy-heavy-light-light tetraquark candidates
    ($\rightarrow$ crude results based on approximations, but a lot of interesting physics, e.g. forces between hadrons, formation of tetraquarks).

Part 1: Introduction to QCD (quantum chromodynamics) and lattice QCD

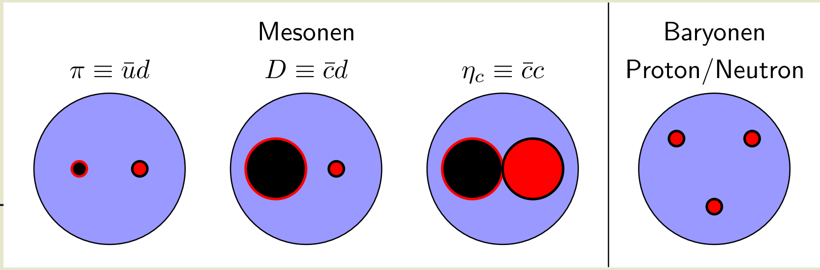
QCD: quarks and gluons

- **QCD**: theory of **quarks** and **gluons** ... explains the formation of hadrons (e.g. proton and neutron), their structure and their decays.
- **Quarks and antiquarks** (spin $1/2$):
  - 6 flavors ... **up**, **down**, **strange**, **charm**, **bottom**, **top** (different masses).
  - 3 colors ... **red**, **green**, **blue** (similar to electrical charge, but 3 different types).
- **Gluons** (spin 1):
  - Force carriers of QCD (similar to photons).
  - Carry (color) charge (in contrast to photons)
    $\rightarrow$ "unexpected phenomena", e.g. **confinement**.



QCD: confinement, hadrons

- There are no isolated quarks, they always appear in groups, typically quark-antiquarks pairs or groups of 3 quarks or antiquarks, so-called **hadrons** ($\rightarrow$ **confinement**).
- **Hadrons**:
  - **Mesons**: Integer spin, usually **quark-antiquark pairs** $q\bar{q}$ ... but also **2 quarks and 2 antiquarks** $qq\bar{q}\bar{q}$ **(Tetraquarks)** are possible.
    Examples: $\pi \equiv \bar{u}d$, $D \equiv \bar{c}d$, $\eta_s \equiv \bar{c}c$, ..., $ud\bar{b}\bar{b}$, ...
  - **Baryons**: Half-integer spin, usually 3 quarks qqq oder 3 antiquarks $\bar{q}\bar{q}\bar{q}$.
    Example: proton $\equiv uud$, neutron $\equiv udd$, ...
  - Several hundred mesons and baryons have been observed in experiments, they differ in
    - ∗ flavor content (6 possibilities for each quark/antiquark, u, d, s, c, b, t),
    - ∗ quantum numbers similar to the hydrogen atom (radial quantum number, total angular momentum J, parity P, ...).



QCD: definition

- Definition of QCD rather simple:

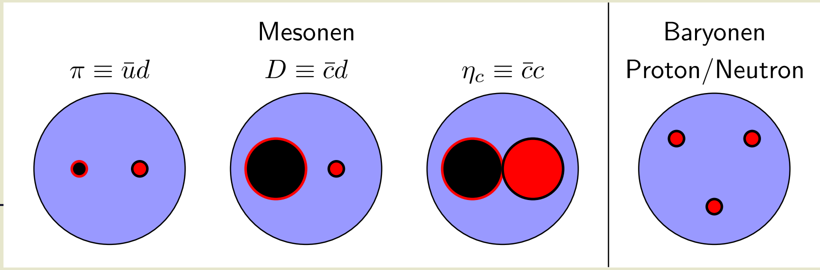
$$S = \int d^4x \Big(\sum_{f \in \{u,d,s,c,t,b\}} \bar{\psi}^{(f)} \Big(\gamma_\mu \Big(\partial_\mu - iA_\mu\Big) + m^{(f)}\Big)\psi^{(f)} + \frac{1}{2g^2}\mathrm{Tr}\Big(F_{\mu\nu}F_{\mu\nu}\Big)\Big)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

  - $\psi^{(f)}(\mathbf{r},t)$, $\bar{\psi}^{(f)}(\mathbf{r},t)$: **quark fields**.
  - $A_\mu(\mathbf{r},t)$: **gluon field**.

- No analytical solutions for e.g. meson or baryon masses available, because
  - field equations non-linear,
  - no small parameter (coupling constant), i.e. perturbation theory in general not applicable.

- Numerical method necessary $\rightarrow$ **lattice QCD**.



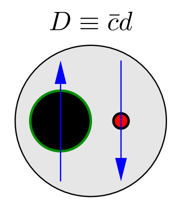
QCD: computation of hadron masses (1)

- Lattice QCD computation of a hadron mass in three steps:

Step (1): define a suitable hadron creation operator O

- A hadron creation operator is essentially a combination of quark field operators $\psi^{(f)}(\mathbf{r}) \equiv u(\mathbf{r}), d(\mathbf{r}), s(\mathbf{r}), c(\mathbf{r}), b(\mathbf{r}), t(\mathbf{r})$.
- The quark field operator $u(\mathbf{r})$ creates a u quark at position $\mathbf{r}$, $d(\mathbf{r})$ creates a d quark, ...
- A **suitable hadron creation operator** O generates in **crude approximation** the hadron of interest:
  - **Details are irrelevant, the final result for the hadron mass does not depend on these details.**
  - **Example:** D **meson** ... essentially a quark-antiquark pair $\bar{c}d$ with total angular momentum $J = 0$ and parity $P = -$; a possible D meson creation operator is

$$O \quad \equiv \quad \int d^3r \, \bar{c}(\mathbf{r})\gamma_5 d(\mathbf{r})$$



($\gamma_5 \rightarrow J^P = 0^-$, $\int d^3r \rightarrow \mathbf{p} = 0$).

QCD: computation of hadron masses (2)

Step (2): Compute the temporal correlation function $C(t)$ of the hadron creation operator O using lattice QCD

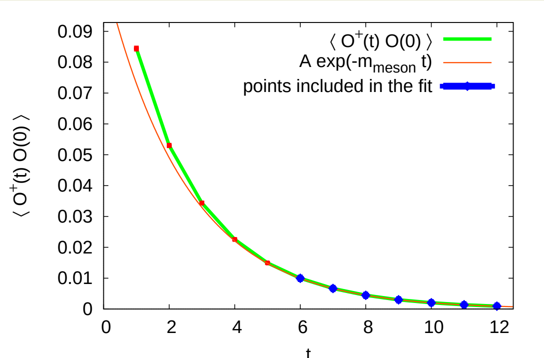
- **Correlation function**: $C(t) \equiv \langle\Omega|O^\dagger(t)O(0)|\Omega\rangle$ $\quad$ ($|\Omega\rangle$ = QCD ground state = vacuum).
- Lattice QCD is very technical:
  - Sophisticated codes have to be developed ...
  - ... which run on high performance computers several weeks or months ...
  - ... a few details on the next slide.

Step (3): extract the hadron mass from the exponential decay of the correlation function $C(t)$

- Using elementary quantum mechanics one can show

$$C(t) \quad = \quad \langle\Omega|O^\dagger(t)O(0)|\Omega\rangle \quad \stackrel{t\to\infty}{\propto} \quad e^{-m_D t}.$$

- Fit of $Ae^{-m_D t}$ to the lattice QCD results for $C(t)$ yields the D meson mass m_D.



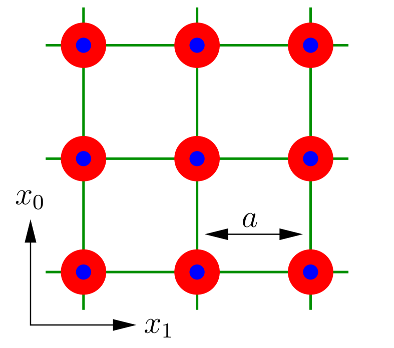
Lattice QCD

- **Goal**: numerical calculation of QCD observables, e.g. a temporal correlation function (and from that correlation function a hadron hadron mass).

- Starting point is the **path integral formulation**,

$$C(t) \quad = \quad \langle\Omega|O^\dagger(t)O(0)|\Omega\rangle \quad = \quad \prod_{x_\nu \in \text{Gitter}} \bigg(\prod_f d\psi^{(f)}(x_\nu)\, d\bar{\psi}^{(f)}(x_\nu)\bigg) dA_\mu(x_\nu)\ \ldots,$$

  - on each spacetime point x_ν (there are infinitely many) one has to solve an "ordinary integral" over the field variables $\psi^{(f)}(x_\nu)$ and $A_\mu(x_\nu)$,
  - i.e. an infinite-dimensional integral.



- Numerical realization:

  - Discretize spacetime by introducing a hypercubic lattice with sufficiently small lattice spacing $a \approx 0.05\,\text{fm} \ldots 0.10\,\text{fm}$.
  - Consider only a limited region of spacetime with extent $L \approx 2.0\,\text{fm} \ldots 4.0\,\text{fm}$.

$\rightarrow$ Path integral reduced to a finite-dimensional integral, but $\mathcal{O}(10^8)$ **integration variables**.
$\rightarrow$ Specifically developed stochastic algorithms are necessary.
$\rightarrow$ High performance computer systems are needed.

Lattice QCD: goals

- There are many typical goals of lattice QCD projects, e.g.:
  - Verification (or falsification) of QCD by comparing lattice QCD results and experimental results.
    $\rightarrow$ Part 2a: Precise computation of several meson masses, e.g. of D_s, η_c, J/Ψ, ...
  - Prediction of not yet experimentally observed mesons or baryons (e.g. valuable input for experiments).
    $\rightarrow$ Part 2b: Prediction of a $\bar{b}\bar{b}ud$ tetraquark with quantum numbers $I(J^P) = 0(1^+)$.
  - Investigation of the structure of mesons or baryons.
    $\rightarrow$ Part 2b: Does a meson have a quark-antiquark or rather a tetraquark structure ... if the latter is the case, is it a mesonic molecule or a diquark-antidiquark pair?
  - Resolve conflicts between experiments and model calculations.
    $\rightarrow$ Part 2a: Separation of $D_1(2430)$ and $D_1(2420)$ as a preparatory step to investigate $B^{(*)} \rightarrow D^{**}$.
  - Compute QCD observables, which are hard or impossible to measure experimentally.
    $\rightarrow$ Part 2b: Systems with two heavy $\bar{b}$ antiquarks, $\bar{b}\bar{b}ud$.

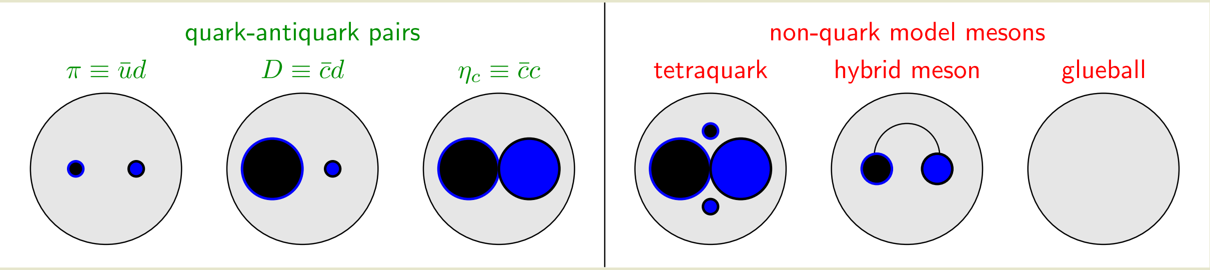
$(+)$ No assumptions. No approximations. Not a model. QCD results.

$(-)$ Projects take a rather long time ... typically several years.

Part 2: Heavy mesons and tetraquarks from lattice QCD

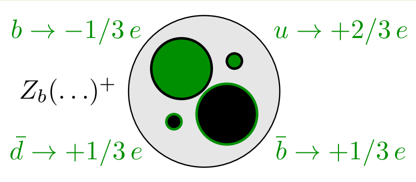
$\bar{q}q$ mesons and tetraquarks (1)

- **Meson**: system of quarks and gluons with integer total angular momentum $J = 0, 1, 2, \ldots$
- Most mesons seem to be **quark-antiquark pairs** $\bar{q}q$, e.q. $\pi \equiv \bar{u}d$, $D \equiv \bar{c}d$, $\eta_s \equiv \bar{c}c$ (quark-antiquark model calculations reproduce the majority of experimental results).
- Certain mesons are poorly understood (significant discrepancies between experimental results and quark model calculations), could have a more complicated structure, e.g.
  - **2 quarks and 2 antiquarks (tetraquark)**,
  - **a quark-antiquark pair and gluons (hybrid meson)**,
  - **only gluons (glueball)**.



$\bar{q}q$ mesons and tetraquarks (2)

- Indications for tetraquark structures:
  - Electrically charged mesons $Z_b(10610)^+$ and $Z_b(10650)^+$:
    - Mass suggests a $b\bar{b}$ pair ...
    - ... but $b\bar{b}$ is electrically neutral ...?
    - **Easy to understand, when assuming a tetraquark structure**: $Z_b(\ldots)^+ \equiv b\bar{b}u\bar{d}$ ($u \rightarrow +2/3\,e$, $\bar{d} \rightarrow -1/3\,e$).



  - Electrically charged Z_c states:
    - Similar to Z_b.
  - Mass ordering of light scalar mesons:
    - E.g. $m_\kappa > m_{a_0(980)}$ ...?

Several lattice QCD approches

Goal: Understand masses and structures of mesons using lattice QCD.

(1) $q\bar{q}$ **operators (quark-antiquark)** (part 2a)

- (+) High precision for simple states (uncertainty $\lesssim 0.5\%$).
- (−) Only for rather stable mesons, which have an ordinary quark-antiquark structure, e.g. D, D^*, D_s, D_s^*, η_c, J/ψ ...

(2) $q\bar{q}$- **and** $qq\bar{q}\bar{q}$ **operators (quark-antiquark + tetraquark)** (not in this talk)

- (+) Also for unstable mesons or mesons, which have a complicated structure (tetraquark candidates), e.g. $a_0(980)$, D_{s0}^*, $Z(\ldots)^+$, ...
- (+) Not only masses, but also information about the structure.
- (−) Technically very challenging.
- (−) Currently under development, no solid or precise results at the moment.

(3) $qq\bar{b}\bar{b}$ **operators, Born-Oppenheimer approximation** (part 2b)
(somewhere between (1) and (2))

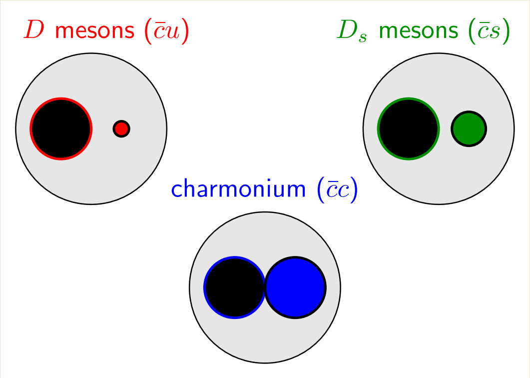
- (+) Allows to investigate specific tetraquark candidates at moderate computational costs, e.g. $Z_b(10610)^+$, $Z_b(10650)^+$.
- (−) Approximations involved.

Part 2a: D mesons, D_s mesons and charmonium

[M. Kalinowski, M.W. [ETM Collaboration], Phys. Rev. D **92**, 094508 (2015) [arXiv:1509.02396]]
[K. Cichy, M. Kalinowski, M.W. [ETM Collaboration], accepted by Phys. Rev. D (2016) [arXiv:1603.06467]]

D, D_s, charmonium (1)

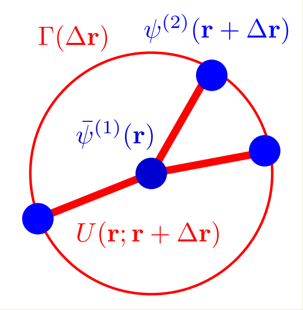
- **Goal**: precision lattice QCD computation of several D, D_s and charmonium masses:
  - D mesons: D, D^*, D_0^*, D_1, ...
  - D_s mesons: D_s, D_s^*, ...
  - Charmonium: η_c, J/ψ, ...



- Computation based on standard quark-antiquark creation operators:
  - D mesons $\rightarrow$ $\bar{c}u$, $\bar{c}d$, $\bar{u}c$ or $\bar{d}c$.
  - D_s mesons $\rightarrow$ $\bar{c}s$ or $\bar{s}c$.
  - Charmonium $\rightarrow$ $\bar{c}c$.

$\rightarrow$ Successful for rather stable mesons with quark-antiquark structure.
$\rightarrow$ Not suited e.g. for tetraquark candidates (D_{s0}^*, D_{s1} ...?), but still an important preparatory step.

D, D_s, charmonium (2)



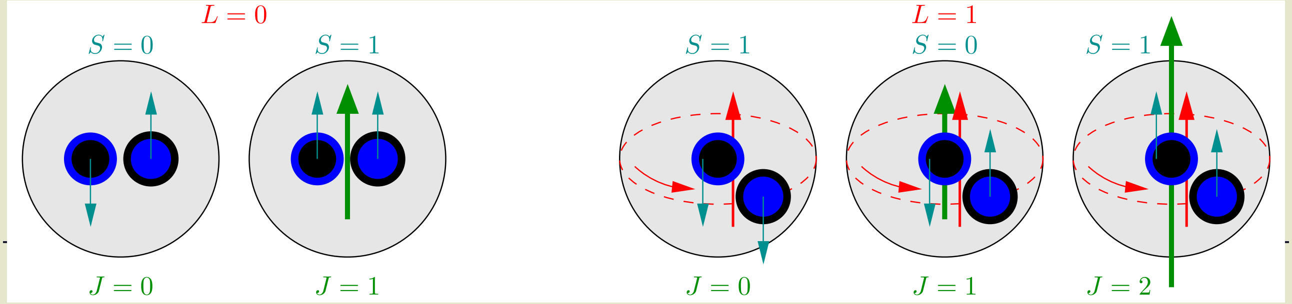
- $\mathcal{O}(100)$ different **meson creation operators**

$$O_{\Gamma,\bar{\psi}^{(1)}\psi^{(2)}} \equiv \int d^3r\, \bar{\psi}^{(1)}(\mathbf{r}) \underbrace{\int_{|\Delta\mathbf{r}|=R} d^3\Delta r\, U(\mathbf{r};\mathbf{r}+\Delta\mathbf{r})\Gamma(\Delta\mathbf{r})}_{=\text{ spherical harmonic } \times \text{ spin matrix}} \psi^{(2)}(\mathbf{r}+\Delta\mathbf{r}).$$

generate the following quantum numbers:

  - Quark flavor: $\bar{\psi}^{(1)}\psi^{(2)} = \bar{c}u$ for D, $\bar{\psi}^{(1)}\psi^{(2)} = \bar{c}s$ for D_s, $\bar{\psi}^{(1)}\psi^{(2)} = \bar{c}c$ for charmonium.
  - Total angular momentum $J = 0, 1, 2, \ldots$, parity $P = -, +$.
    Sperical harmonic $\rightarrow$ orbital angular momentum $L = 0, 1, 2, \ldots$
    Spin matrix $\rightarrow$ spin $S = 0, 1$.
  - $\int d^3x \rightarrow$ vanishing momentum $\mathbf{p} = 0$.

- Mass determination as in part 1 (compute correlation function, fit exponential function).



D, D_s, charmonium (3)

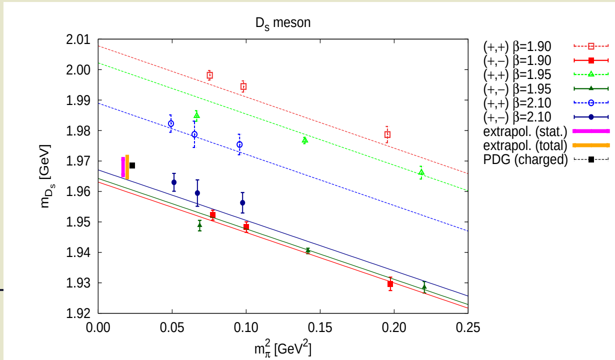
- Computations at unphysically heavy u/d quark masses and at finite lattice spacing, then combined linear extrapolation

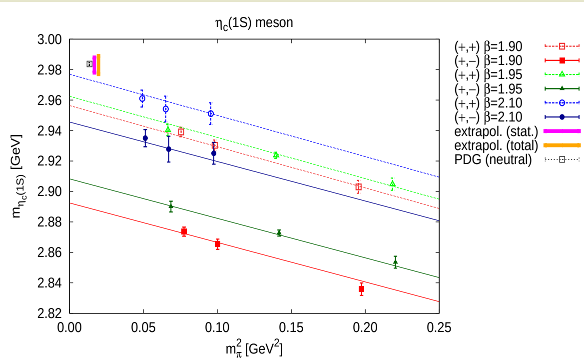
  - in the u/d quark mass $m_{u,d} \propto m_\pi^2$ to $m_\pi = 135\,\text{MeV}$ (horizontal axes) and
  - in the squared lattice spacing a^2 to the continuum, i.e. $a = 0$ (different colors)
  - using two different Wilson twisted mass lattice discretizations (same colors).

$\rightarrow$ Results in pink/yellow.

- Examples: D_s meson (left plot), η_c meson (right plot), both $J^P = 0^-$.

- **Perfect agreement between experimental results (black) and the quark mass and continum extrapolated lattice QCD results (pink/yellow) (errors on the per mille level).**





D, D_s, charmonium (4)

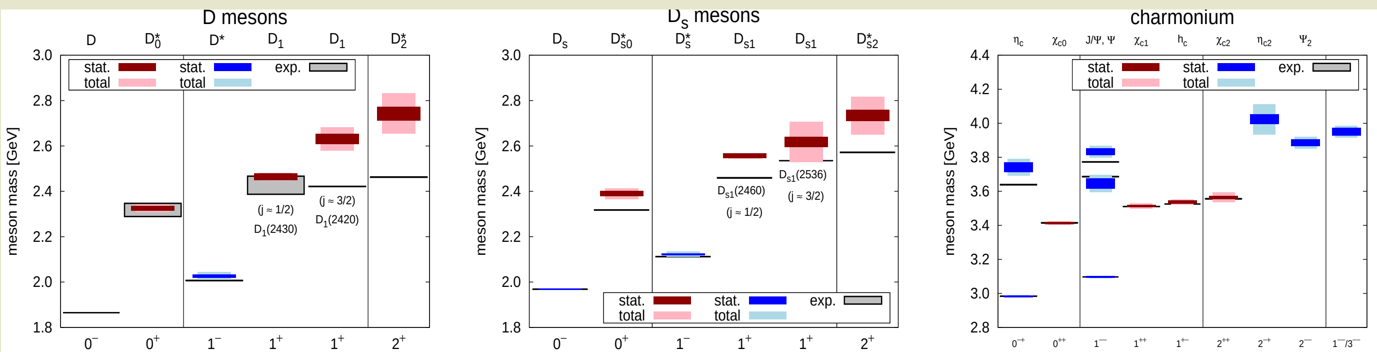
- **Summary of lattice QCD results** (blue $[P = -]$ and red $[P = +]$ boxes):
  - D meson masses (left):
    D, D_0^*, D^*, $D_1(2430)$, $D_1(2420)$, D_2^*.
  - D_s meson masses (center):
    D_s, D_{s0}^*, D_s^*, $D_{s1}(2460)$, $D_{s1}(2536)$, D_{s2}^*.
  - Charmonium masses (right):
    η_c, χ_{c0}, J/Ψ, χ_{c1}, h_c, χ_{c2}, η_{c2}, Ψ_2.

(+) Computations with 2+1+1 dynamical quark flavors (ETMC gauge link configurations).

(+) Extrapolated to physically light u/d quark masses.

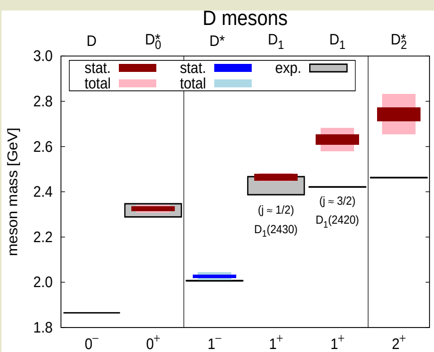
(+) Extrapolated to the continuum.

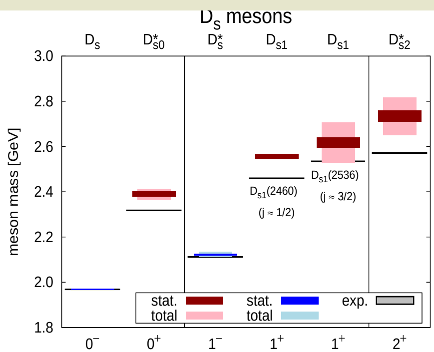
(−) Only quark-antiquark creation operators, no four-quark operators at the moment.

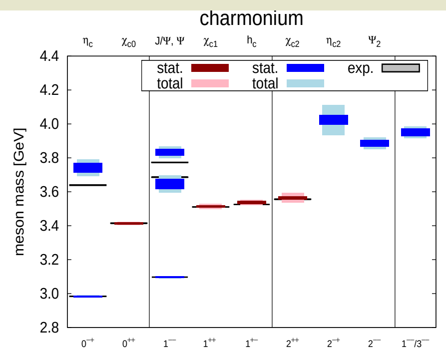


D, D_s, charmonium (5)

- **Summary of lattice QCD results** (blue $[P = -]$ and red $[P = +]$ boxes):
  - Comparison with existing experimental results (black boxes):
    - Agreement for the majority of states.
    - Tension/disagreement:
      - D_{s0}^*, $D_{s1}(2460)$: tetraquark candidates
        ... might require four-quark creation operators.
      - Radial excitations and higher orbital excitations (e.g. $\eta_c(2S)$, $J/\Psi(2S)$, D_2^*, D_{s2}^*)
        ... such excitations exhibit poor signals in lattice QCD (reflected by large errors), better statistics, i.e. longer computations required.







D, D_s, charmonium (6)

Summary of part 2a

- First principles QCD computation of several states, all sources of systematic error investigated and quantified, agreement with corresponding experimental results on the per mille level.

- Preparatory step for more advanced computations using also four-quark creation operators (investigation of tetraquark candidates and unstable mesons).

- Clear separation of the two $J^P = 1^+$ states
  $D_1(2430)$ (light spin $j \approx 1/2$)
  $D_1(2420)$ (light spin $j \approx 3/2$).

  – Important to study semileptonic decays $B^{(*)} \to D^{**} + l + \nu$.

  – Persistent conflict between experiment and theory ("$1/2$ versus $3/2$ puzzle").

Part 2b: Heavy-heavy-light-light tetraquark candidates

[P. Bicudo, M.W., Phys. Rev. D **87**, 114511 (2013) [arXiv:1209.6274]]
[P. Bicudo, K. Cichy, A. Peters, B. Wagenbach, M.W., Phys. Rev. D **92**, 014507 (2015) [arXiv:1505.00613]]
[P. Bicudo, K. Cichy, A. Peters, M.W., Phys. Rev. D **93**, 034501 (2016) [arXiv:1510.03441]]
[A. Peters, P. Bicudo, K. Cichy, M.W., arXiv:1602.07621]
[A. Peters, P. Bicudo, L. Leskovec, S. Meinel, M.W., arXiv: 1609.00181]
[P. Bicudo, J. Scheunert, M.W., arXiv:1609.00548]

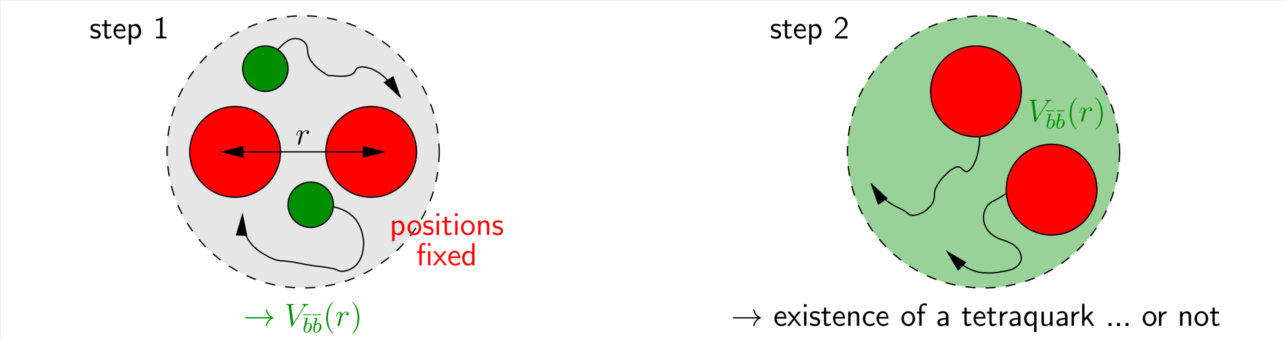
$\bar{b}\bar{b}qq$ tetraquarks (1)

- **Basic idea**: Investigate existence of heavy tetraquarks $\bar{b}\bar{b}qq$ in two steps.

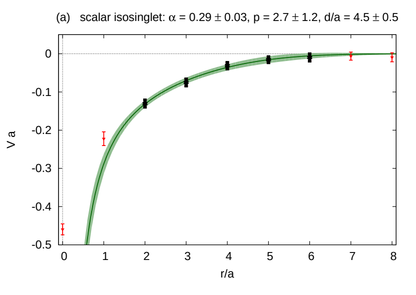
  (1) **Compute potentials of two static antiquarks ($\bar{b}\bar{b}$) in the presence of two lighter quarks ($qq \in \{ud, ss, cc\}$) using lattice QCD.**

  (2) **Check, whether these potentials are sufficiently attractive, to host a bound state by solving a corresponding Schrödinger equation.**
  (→ This would indicate a stable $\bar{b}\bar{b}qq$ tetraquark.)

((1) + (2) → Born-Oppenheimer approximation).



$\bar{b}\bar{b}qq$ tetraquarks (2)



Born-Oppenheimer approximation, step (1)

- Lattice QCD computation of $\bar{b}\bar{b}$ potentials $V_{\bar{b}\bar{b}}(r)$ (2 flavor ETMC gauge link configurations).

  (1) Use $\bar{b}\bar{b}qq$ creation operators

  $$O_{\bar{b}\bar{b}qq} \equiv (\mathcal{C}\Gamma)_{AB}(\mathcal{C}\tilde{\Gamma})_{CD}\Big(\bar{b}_C(-\mathbf{r}/2)q_A^{(1)}(-\mathbf{r}/2)\Big)\Big(\bar{b}_D(+\mathbf{r}/2)q_B^{(2)}(+\mathbf{r}/2)\Big).$$

  - Different light quark flavors $qq \in \{ud, ss, cc\}$.
  - Different light quark spin/parity.
  - Different heavy quark spin/parity (no effect on $V_{\bar{b}\bar{b}}(r)$).

  → Many different channels
  ... some attractive, some repulsive
  ... some correspond for large $\bar{b}\bar{b}$ separations to pairs of ground state mesons (B and/or B^*), some to excited mesons (one or two B_0^* and/or B_1^*).

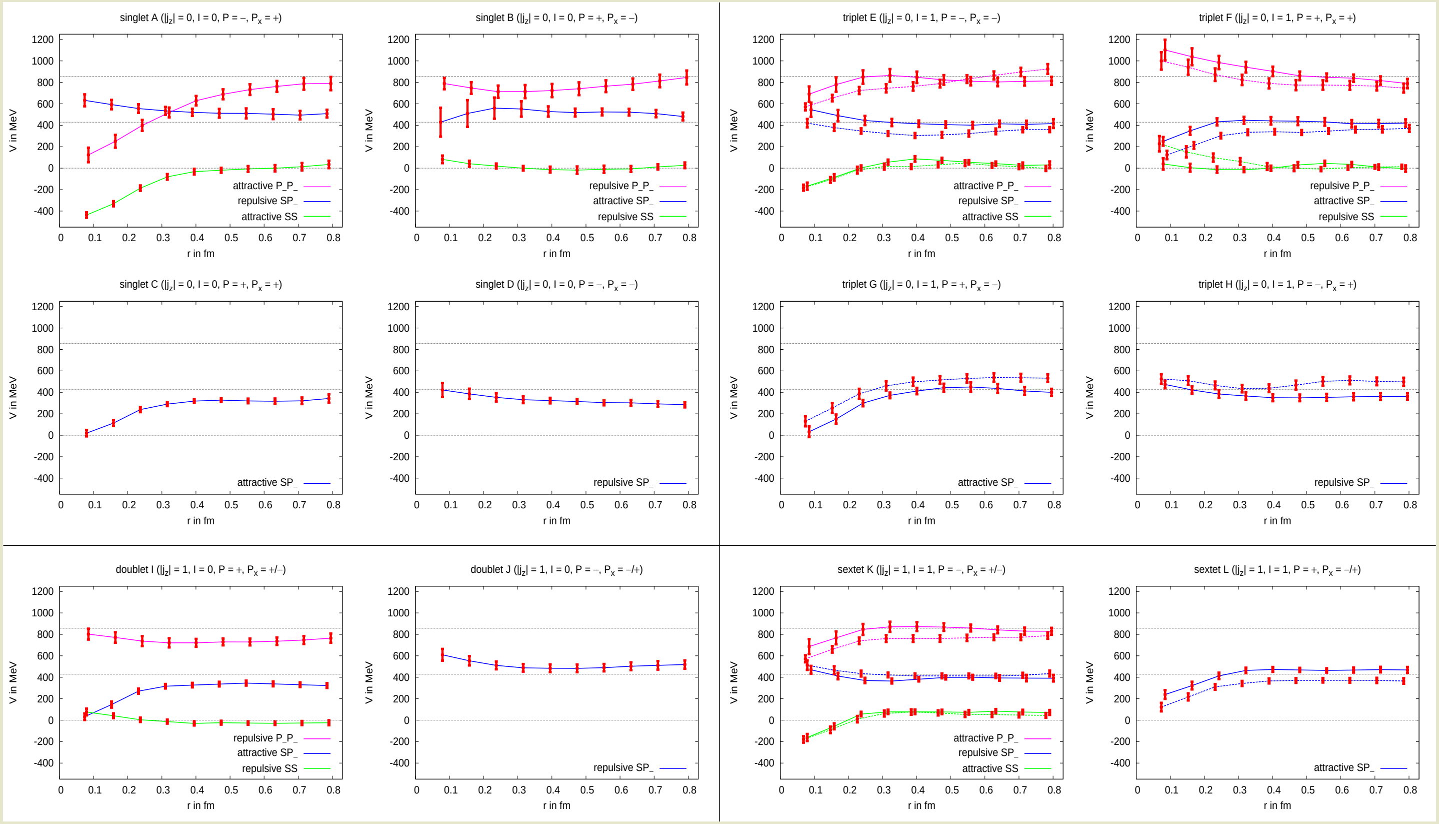
  (2) Compute temporal correlation functions.

  (3) Determine $V_{\bar{b}\bar{b}}(r)$ from the exponential decays of the correlation functions.

- **First principles QCD computation of forces between hadrons.**

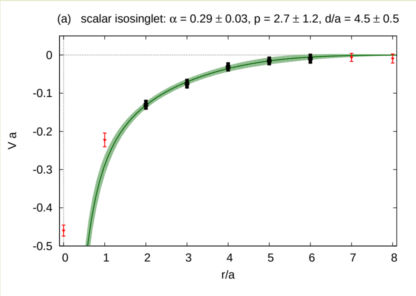
$\bar{b}\bar{b}qq$ tetraquarks (3)

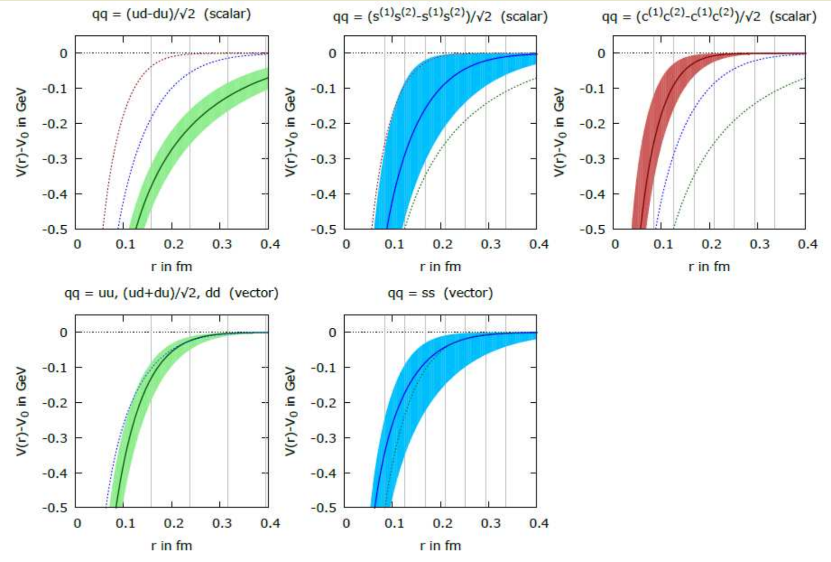
- $I = 0$ (left) and $I = 1$ (right); $|j_z| = 0$ (top) and $|j_z| = 1$ (bottom).



$\bar{b}\bar{b}qq$ tetraquarks (4)

- Two attractive channels corresponding to pairs of ground state mesons (B and/or B^*).
- Light quark mass dependence of these channels:
  $V_{\bar{b}\bar{b}}(r)$ wider and deeper for $qq = ud$ compared to $qq = ss$ compared to $qq = cc$.
  $\rightarrow$ **Good candidates to find tetraquarks are systems of two very heavy and two very light quarks, i.e. $\bar{b}\bar{b}ud$.**





$\bar{b}\bar{b}qq$ tetraquarks (5)

Born-Oppenheimer approximation, step (2)

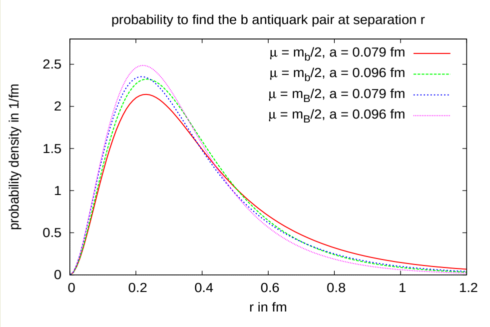
- Solve the Schrödinger equation for the relative coordinate $\mathbf{r}$ of the two $\bar{b}$ quarks,

$$\left(-\frac{1}{2\mu}\triangle + V_{\bar{b}\bar{b}}(r)\right) \underbrace{\psi(\mathbf{r})}_{=R(r)/r} \quad = \quad E\psi(\mathbf{r}) \quad , \quad \mu \quad = \quad m_b/2;$$

  possibly existing bound states, i.e. $E < 0$, indicate $\bar{b}\bar{b}qq$ tetraquarks.

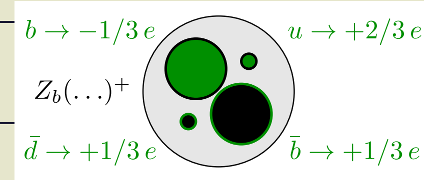
- A single bound state for one specific potential $V_{\bar{b}\bar{b}}(r)$ and light quarks $qq = ud$:
  - Binding energy $-E = 90^{+43}_{-36}\,\mathrm{MeV}$, i.e. confidence level $\approx 2\,\sigma$.
  - Quantum numbers of the $\bar{b}\bar{b}ud$ tetraquark: $I(J^P) = 0(1^+)$.
  - Average separation of $\bar{b}\bar{b}$: ≈ 0.25 fm.

$\rightarrow$ **Prediction of a tetraquark.**
(No experimental results for $\bar{b}\bar{b}ud$ yet.)



- No further bound states, in particular not for $qq = ss$ or $qq = cc$.

$\bar{b}\bar{b}qq$ tetraquarks (6)



Ongoing research

(1) **Including the spin of the $\bar{b}$ quarks:** (Static quarks are spinless.)

- Consider many channels at the same time via a coupled channel Schrödinger equation.
- Result: $I(J^P) = 0(1^+)$ tetraquark persists, binding energy $-E = 59^{+30}_{-38}$ MeV.

(2) **Structure of the tetraquark:** (Mesonic molecule or diquark-antidiquark?)

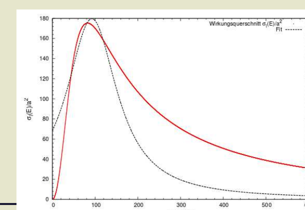
- Several hadron creation operators and correlation matrices required ...

(3) $\bar{b}b\bar{q}q$ **instead of** $\bar{b}\bar{b}qq$**:** (i.e. $Z_b(10610)^+$, $Z_b(10650)^+$)

- Theoretically harder, because light quarks can form a far separated light meson, e.g. a pion ... result would then be about "bottomonium + pion", not about a tetraquark.
- Crude result: $I(J^P) = 1(1^+)$ $\bar{b}b\bar{d}u$ tetraquark, binding energy $-E = (58 \pm 71)$ MeV.

(4) **Tetraquark resonances:**

- Resonance parameters (phase shifts, resonance masses and widths) can be determined numerically using quantum mechanics/scattering theory.
- Indication of a resonance for $I(J^P) = 0(1^-)$.



Recent research, future plans, potential synergies

(≈60%) **Hadron spectrum, structure, decays (in particular exotic mesons):**

- Tetraquark candidates with four finite mass quarks: $a_0(980)$, κ, D^*_{s0}, $D_{s1}(2460)$, ...
- $\bar{b}\bar{b}qq$ and $\bar{b}b\bar{q}q$ tetraquarks with static quarks (HQET) and with NRQCD. (part 2b)
- Hybrid static potentials and hybrid mesons.
- Standard quark-antiquark mesons (orbital, radial and parity excitations): D, D_s, charmonium, B, B_s, bottomonium ... $B \rightarrow D^{**}$ ... (part 2a)

(FAIR, Emmy Noether research group, DFG grant)

(≈20%) **Lattice QCD simulations at fixed or frozen topology.**

(Emmy Noether research group)

(≈15%) **Static potential:**

- Matching of lattice QCD and perturbative QCD, determination of $\Lambda_{\overline{\mathrm{MS}}}$.
- Non-perturbative definition and interpretation of color-octet potentials.
- Non string-like trial states, efficient computation of off-axis separations.

(≈ 5%) **Phase structure of QCD-inspired models for $\mu > 0$, inhomogeneous condensates.**