The pseudoparticle approach

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Outline

Part I: the pseudoparticle approach, a model for SU(2) Yang-Mills theory

- Basic principle.
- Numerical results: static quark antiquark potential, topological susceptibility, critical temperature, gluelump masses.

Part II: properties of confining gauge field configurations

- Pseudoparticles of different profile.
- Instantons, antiinstantons and akyrons.

Part III: fermions in the pseudoparticle approach

- Problems with fermionic fields in the pseudoparticle approach.
- The Gross-Neveu model as testing ground.

Summary and outlook

Part I: the pseudoparticle approach, a model for SU(2) Yang-Mills theory

Basic principle of the PP approach (1)

- A numerical technique to approximate Euclidean path integrals (in this talk: mainly SU(2) Yang-Mills theory):

$$\left\langle \mathcal{O} \right\rangle = \frac{1}{Z} \int DA\, \mathcal{O}[A] e^{-S[A]}$$

$$S[A] = \frac{1}{4g^2} \int d^4x\, F^a_{\mu\nu} F^a_{\mu\nu} \quad , \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu .$$

- A tool to analyze the importance of certain classes of gauge field configurations with respect to confinement.
- Related work:
  - Ensembles of regular gauge instantons and merons (F. Lenz, J. W. Negele, M. Thies, 2003).
  - Ensembles of calorons with non-trivial holonomy (P. Gerhold, E.-M. Ilgenfritz, M. Müller-Preussker, 2006).
  - Ensembles of dyons (D. Diakonov, V. Petrov, 2007).

Basic principle of the PP approach (2)

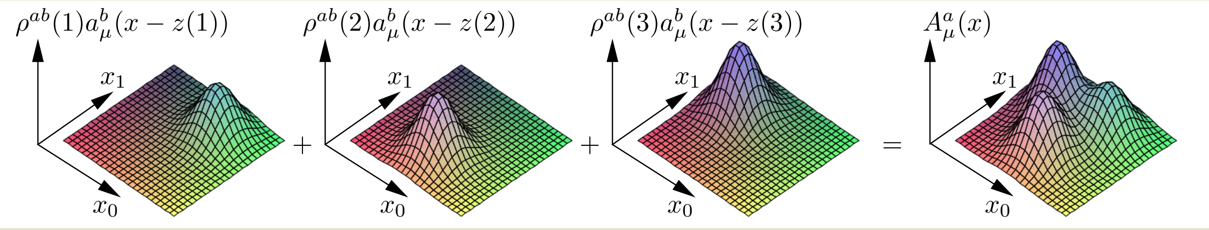
- PP: any gauge field configuration a_μ^a, which is localized in space and in time.
- Consider only those gauge field configurations, which can be written as a sum of a fixed number (≈ 400) of PPs:

$$A_\mu^a(x) = \sum_j \rho^{ab}(j) a_\mu^b(x - z(j))$$

(j: PP index; $\rho^{ab}(j)$: degrees of freedom of the j-th PP, i.e. amplitude and color orientation; $z(j)$: position of the j-th PP).

- Define the functional integration as an integration over the PP degrees of freedom:

$$\int DA \ldots \quad \rightarrow \quad \int \left(\prod_j d\rho^{ab}(j) \right) \ldots$$



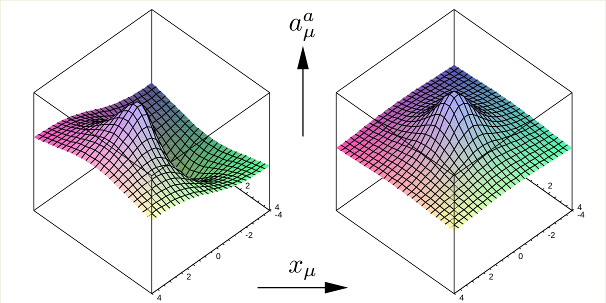
Building blocks of PP ensembles

- Standard choice of building blocks: "instantons", "antiinstantons", akyrons (λ: PP size).

$$a^a_{\mu,\text{instanton}}(x) = \eta^a_{\mu\nu} \frac{x_\nu}{x^2 + \lambda^2}$$

$$a^a_{\mu,\text{antiinstanton}}(x) = \bar{\eta}^a_{\mu\nu} \frac{x_\nu}{x^2 + \lambda^2}$$

$$a^a_{\mu,\text{akyron}}(x) = \delta^{a1} \frac{x_\mu}{x^2 + \lambda^2}.$$



- Instantons, antiinstantons and akyrons form a basis of all gauge field configurations in the "continuum limit".

- Degrees of freedom: amplitudes $\mathcal{A}(j) \in \mathbb{R}$, color orientations $\mathcal{C}^{ab}(j) \in \mathrm{SO}(3)$, positions $z(j) \in \mathbb{R}^4$.

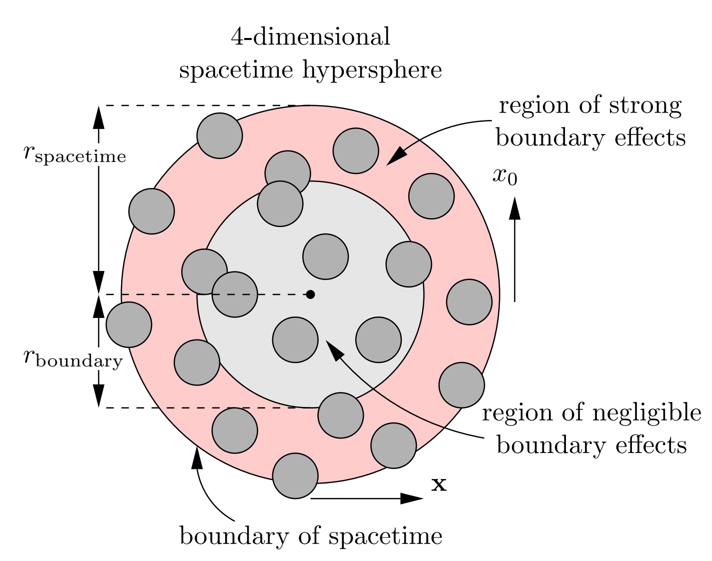
$$A^a_\mu(x) = \sum_j \mathcal{A}(j) \mathcal{C}^{ab}(j) a^a_{\mu,\ldots}(x - z(j)).$$

PP ensembles (1)

- PP ensemble: a fixed number of PPs inside a spacetime hypersphere.
- Gauge field:

$$A_\mu^a(x) = \sum_j \mathcal{A}(j)\mathcal{C}^{ab}(j)a^b_{\mu,\text{instanton}}(x - z(j)) + \sum_k \mathcal{A}(k)\mathcal{C}^{ab}(k)a^b_{\mu,\text{antiinstanton}}(x - z(k)) + \sum_l \mathcal{A}(l)\mathcal{C}^{ab}(l)a^b_{\mu,\text{akyron}}(x - z(l)).$$

- Choose color orientations $\mathcal{C}^{ab}(j)$ and positions $z(j)$ randomly.
- **A_μ^a is no classical solution (not even close to a classical solution)!**
- Long range interactions between PPs.



PP ensembles (2)

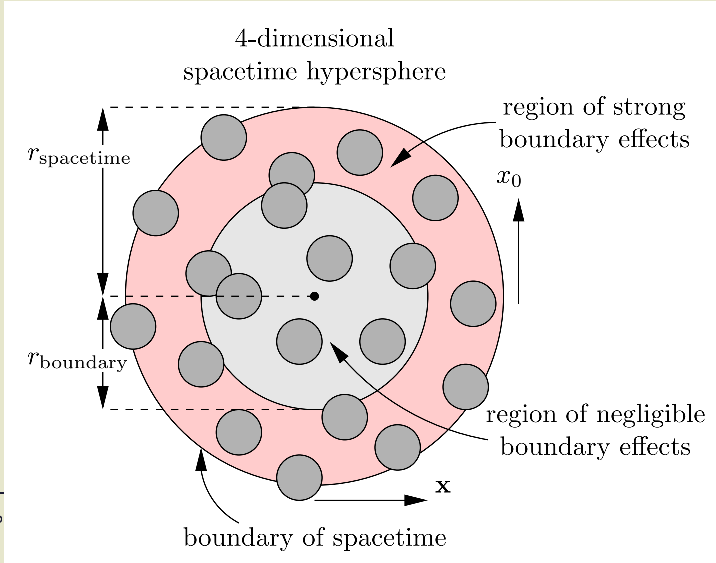
- Approximation of the path integral:

$$\left\langle \mathcal{O} \right\rangle \;=\; \frac{1}{Z} \int \left(\prod_j d\mathcal{A}(j)\right) \mathcal{O}(\mathcal{A}(j)) e^{-S(\mathcal{A}(j))}$$

  (integration over PP amplitudes).

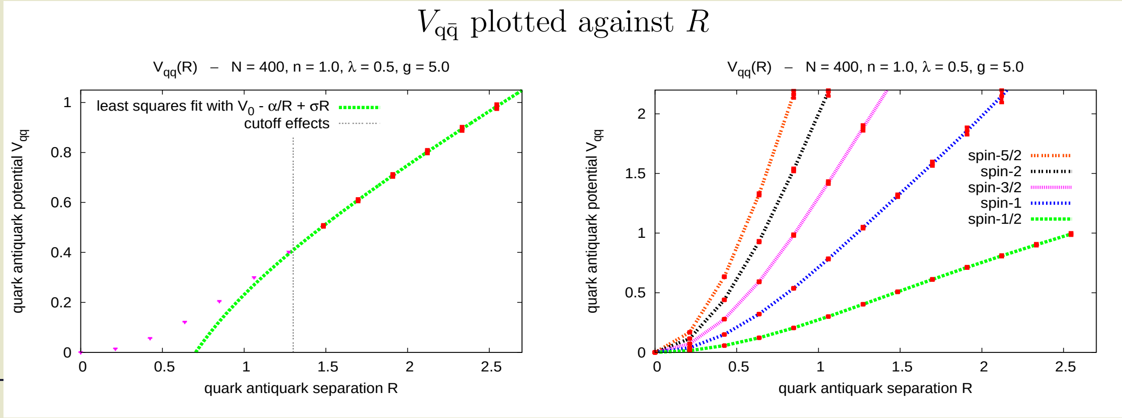
- Solve this multidimensional integral via Monte-Carlo simulations.

- Exclude boundary effects: observables have to be "measured" sufficiently far away from the boundary.



Numerical results (1)

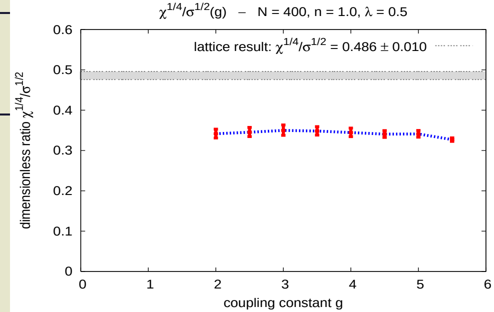
- Static quark antiquark potential:
  - Linear for large separations, i.e. confinement.
  - Fit with $V(R) = V_0 - \alpha/R + \sigma R$:
    - String tension $\sigma > 0$ (in the following σ is used to set the scale).
    - Attractive $1/R$-correction, which is of the right order of magnitude compared to lattice results and the bosonic string picture.
  - Potentials for different quark representations exhibit Casimir scaling.



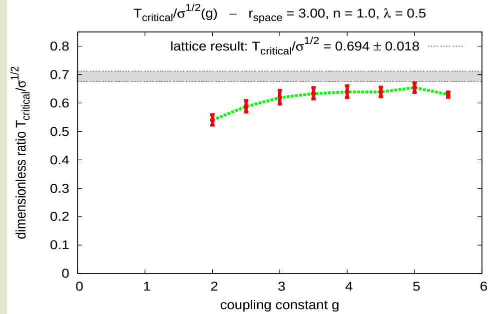
Numerical results (2)

- Further dimensionful quantities:
  - Topological susceptibility $\chi = \langle Q_V^2 \rangle / V$.
  - Critical temperature of the confinement deconfinement phase transition T_{critical}.
  - Mass of magnetic and electric gluelumps (adjoint representation) m_B and m_E (work done by Ch. Szasz).
- Dimensionless quantities (physically meaningful): $\chi^{1/4}/\sigma^{1/2}$, $T_{\mathrm{critical}}/\sigma^{1/2}$, $R_{\mathrm{SB}}^B \sigma^{1/2}$, $R_{\mathrm{SB}}^E \sigma^{1/2}$.
- Considering different g amounts to considering different spacetime volumes.
- **Qualitative agreement with lattice results.**
- **Consistent scaling behavior.**

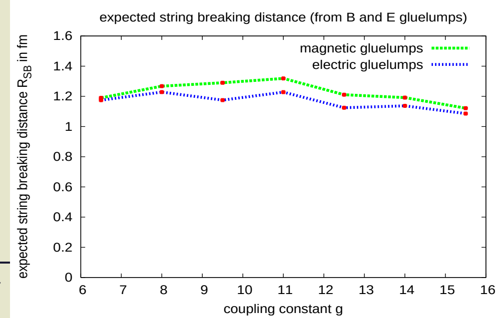
$\chi^{1/4}/\sigma^{1/2}$ plotted against g



$T_{\mathrm{critical}}/\sigma^{1/2}$ plotted against g



R_{SB}^B and R_{SB}^E in fm plotted ...



Part II: properties of confining gauge field configurations

Properties of confining gauge field ...

- **What are essential properties of confining gauge field configurations?**
- **Which gauge field configurations are responsible for confinement?**
- Apply the PP approach with different types of PPs to study the effect of different classes of gauge field configurations on confinement:
  - PPs with a limited range of interaction (PPs with Gaussian profile).
  - PPs without topological charge (akyrons).
  - ...

PPs with Gaussian profile (1)

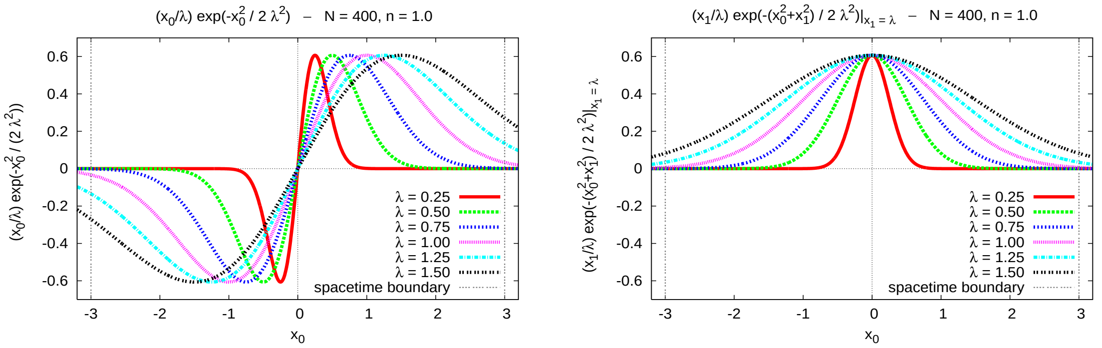
- Consider ensembles with Gaussian localized PPs of different size λ:

$$a^a_{\mu,\mathrm{instanton}}(x) = \eta^a_{\mu\nu} x_\nu e^{-x^2/2\lambda^2} \quad , \quad a^a_{\mu,\mathrm{antiinstanton}}(x) = \bar{\eta}^a_{\mu\nu} x_\nu e^{-x^2/2\lambda^2} \quad ,$$
$$a^a_{\mu,\mathrm{akyron}}(x) = \delta^{a1} x_\mu e^{-x^2/2\lambda^2} .$$

- **Gaussian localized PPs have a limited range of interaction, which is proportional to their size λ.**

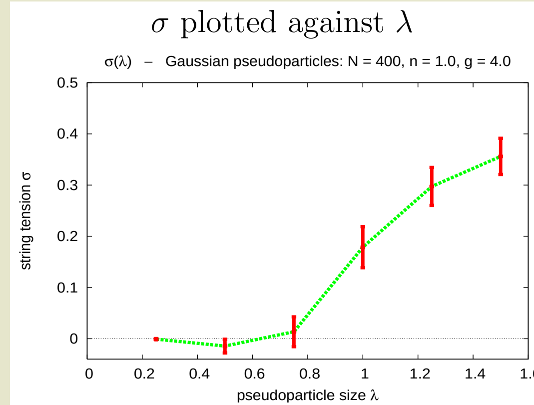
- Typical PP profiles:

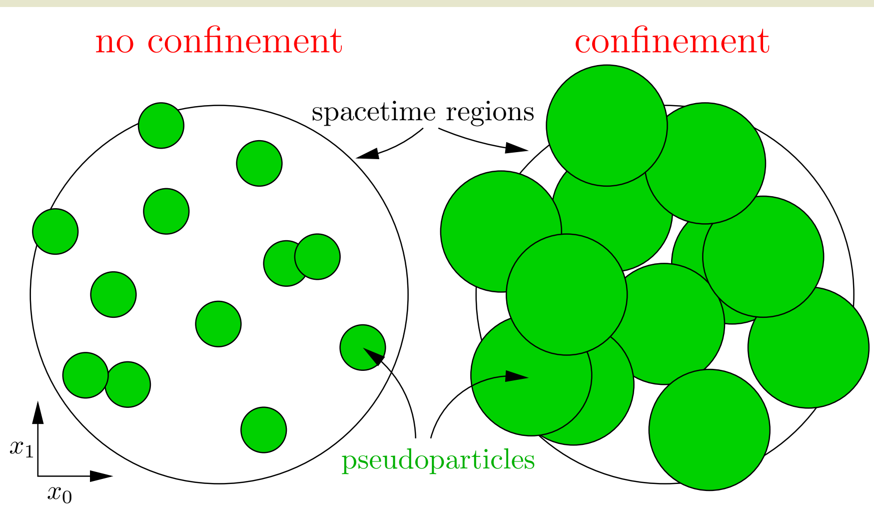
$(x_0/\lambda)\exp(-x_0^2/2\lambda^2)$ plotted against x_0 — $(x_1/\lambda)\exp(-(x_0^2+x_1^2)/2\lambda^2)|_{x_1=\lambda}$ plotted against x_0

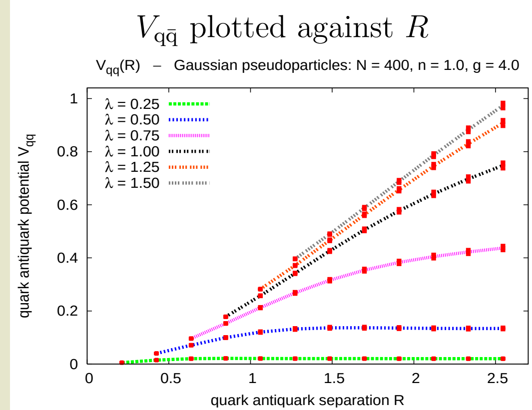


PPs with Gaussian profile (2)

- Short range PPs ($\lambda \leq 0.50$)
  $\longrightarrow$ little overlap between neighboring PPs.
  $\longrightarrow$ no confinement.
- Long range PPs ($\lambda \geq 1.00$)
  $\longrightarrow$ significant overlap between neighboring PPs.
  $\longrightarrow$ confinement.
- **PP percolation $\leftrightarrow$ confinement.**



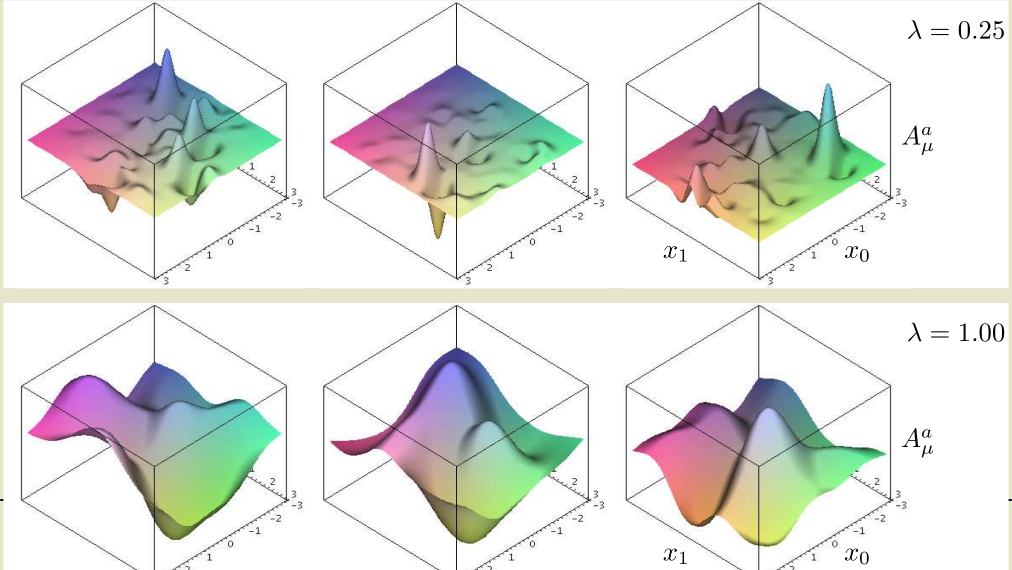




PPs with Gaussian profile (3)

Typical gauge field configurations ($\lambda = 0.25 \leftrightarrow \lambda = 1.00$)

- $\lambda = 0.25$: local UV fluctuations $\rightarrow$ no confinement.
- $\lambda = 1.00$: global excitations $\rightarrow$ confinement.
- **Gauge field configurations responsible for confinement contain extended structures and large area excitations.**



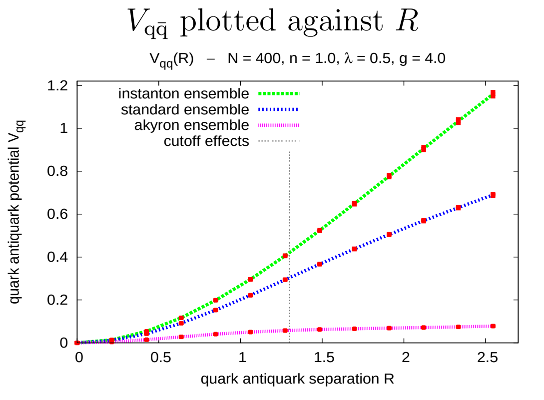
Instantons, antiinstantons and akyrons

- Consider the following ensembles:
  - Akyron ensemble: 400 akyrons (**topological charge density $q = 0$**).
  - Standard ensemble: 150 instantons, 150 antiinstantons, 100 akyrons.
  - Instanton ensemble: 200 instantons, 200 antiinstantons.

- No confinement in the akyron ensemble
  $\longrightarrow$ akyrons alone are not suited to reproduce Yang-Mills physics.
  $\longrightarrow$ **supports the expectation that confinement and topological charge are closely related.**

- Standard ensemble $\leftrightarrow$ instanton ensemble
  $\longrightarrow (\chi^{1/4}/\sigma^{1/2})_{\mathrm{standard}} = 0.35$
  $(\chi^{1/4}/\sigma^{1/2})_{\mathrm{instanton}} = 0.26$

  $(\chi^{1/4}/\sigma^{1/2})_{\mathrm{lattice}} = 0.49.$
  $\longrightarrow$ **using akyrons is beneficial with respect to quantitative results.**



Part III: fermionic fields in the pseudoparticle approach

Basic principle (fermionic fields) (1)

- **How can fermions be included in the PP approach?**
- Action and partition function of **any theory with quadratic fermion interaction** (no restriction to SU(2) Yang-Mills theory anymore):

$$S[\psi, \bar{\psi}, \phi] \;=\; \int dx \, \Big(\bar{\psi} Q(\phi) \psi + \mathcal{L}(\phi)\Big)$$

$$Z \;=\; \int D\psi \, D\bar{\psi} \, D\phi \, e^{-S[\psi, \bar{\psi}, \phi]}$$

(Q: Dirac operator; ϕ: any type and number of bosonic fields, e.g. the non-Abelian gauge field in QCD).

Basic principle (fermionic fields) (2)

- Consider only those fermionic field configurations, which can be represented by a linear superposition a fixed number of localized building blocks:

$$\psi(x) \quad = \quad \sum_j \underbrace{\eta_j G_j(x)}_{j\text{-th PP}}$$

(η_j: Grassmann valued spinors; G_j: functions, which are localized in space as well as in time, i.e. PPs).

- Define the functional integration over all fermionic field configurations as an integration over the Grassmann valued spinors:

$$\int D\psi \, D\bar{\psi} \ldots \quad = \quad \int \left(\prod_j d\eta_j \, d\bar{\eta}_j\right) \ldots$$

Basic principle (fermionic fields) (3)

- Integrate out the fermions:

$$S_{\text{effective}}[\phi] = \int d^{d+1}x\,\mathcal{L}(\phi) - \ln\Big(\det\Big(\langle G_j|Q|G_{j'}\rangle\Big)\Big)$$

$$Z \propto \int D\phi\, e^{-S_{\text{effective}}[\phi]}$$

  ($\langle G_j|Q|G_{j'}\rangle$ is a finite matrix; **"Q-regularization"**).

- If $\det(Q)$ is real and positive, $\det(Q) = \sqrt{\det(Q^\dagger Q)}$. This suggests another PP regularization:

$$S_{\text{effective}}[\phi] = \int d^{d+1}x\,\mathcal{L}(\phi) - \frac{1}{2}\ln\Big(\det\Big(\langle G_j|Q^\dagger Q|G_{j'}\rangle\Big)\Big)$$

  (**"$Q^\dagger Q$-regularization"**).

- **The "$Q^\dagger Q$-regularization" has significant advantages over the naive "Q-regularization".**

Q versus $Q^\dagger Q$ (1)

- For the sake of simplicity: consider all PPs G_j to be orthonormal, i.e. $\langle G_j | G_{j'} \rangle = \delta_{jj'}$ (this is not a restriction!).
- The problem of the Q-regularization:
  - **Applying the Dirac operator Q to one of the PPs $G_{j'}$ in general yields a function, which is (partially) outside the PP function space $\mathrm{span}\{G_n\}$, i.e.**

$$QG_{j'}(x) \quad = \quad \sum_k a_{j'k} G_k(x) + h_{j'} H_{j'}(x)$$

    ($H_{j'}$ normalized, $H_{j'} \perp \mathrm{span}\{G_n\}$).
  - If $|\sum_k a_{j'k} G_k| \gg |h_{j'}| \quad \rightarrow \quad$ no problem.
  - If $|\sum_k a_{j'k} G_k| \lesssim |h_{j'}| \quad \rightarrow \quad$ when computing the fermionic matrix $\langle G_j | Q | G_{j'} \rangle$, a significant part of $QG_{j'}$ is simply ignored, just because $H_{j'}$ is perpendicular to the PP function space $\mathrm{span}\{G_n\}$.

Q versus $Q^\dagger Q$ (2)

- The advantage of the $Q^\dagger Q$-regularization:
  - **Both the left hand sides $\langle G_j | Q^\dagger$ and the right hand sides $Q | G_{j'} \rangle$ of the matrix elements $\langle G_j | Q^\dagger Q | G_{j'} \rangle$ might be outside the PP function space $\mathrm{span}\{G_n\}$, but they form the same function space, $\mathrm{span}\{Q G_n\}$, in which their overlap is computed.**
- For "better arguments" cf. M. Wagner, Phys. Rev. D **76**, 076002 (2007) [arXiv:0704.3023 [hep-lat]].

Testing ground: Gross-Neveu model (1)

- Action and partition function of the $1+1$-dimensional Gross-Neveu model:

$$S = \int d^2x \left(\sum_{n=1}^{N} \bar{\psi}^{(n)} \Big(\gamma_0(\partial_0 + \mu) + \gamma_1 \partial_1\Big) \psi^{(n)} - \frac{g^2}{2} \left(\sum_{n=1}^{N} \bar{\psi}^{(n)} \psi^{(n)}\right)^2\right)$$

$$Z = \int \left(\prod_{n=1}^{N} D\psi^{(n)} \, D\bar{\psi}^{(n)}\right) e^{-S}$$

(N: number of flavors; μ: chemical potential; g: coupling constant).

Testing ground: Gross-Neveu model (2)

- Introduce a real scalar field σ and integrate out the fermions:

$$S_{\mathrm{effective}} = N\left(\frac{1}{2\lambda}\int d^2x\,\sigma^2 - \ln\Big(\det\Big(\gamma_0(\partial_0+\mu)+\gamma_1\partial_1+\sigma\Big)\Big)\right)$$

$$Z \propto \int D\sigma\, e^{-S_{\mathrm{effective}}}$$

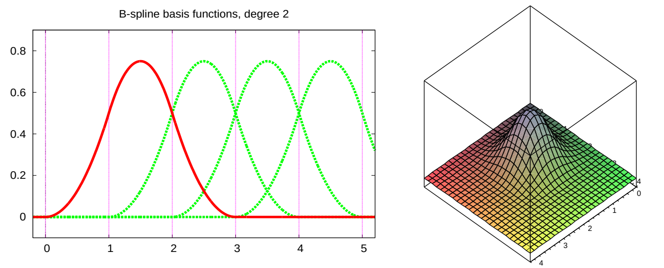
($\lambda = Ng^2$).

- Large-N limit:
  - $N \to \infty$, $\lambda = Ng^2 = \mathrm{constant}$.
  - There is no need to compute the σ-path integral anymore.
  - It is sufficient to minimize $S_{\mathrm{effective}}$ with respect to σ.
  - $\sigma = -g^2 \sum_{n=1}^{N} \bar{\psi}^{(n)}\psi^{(n)}$ (chiral condensate).

Fermionic PPs

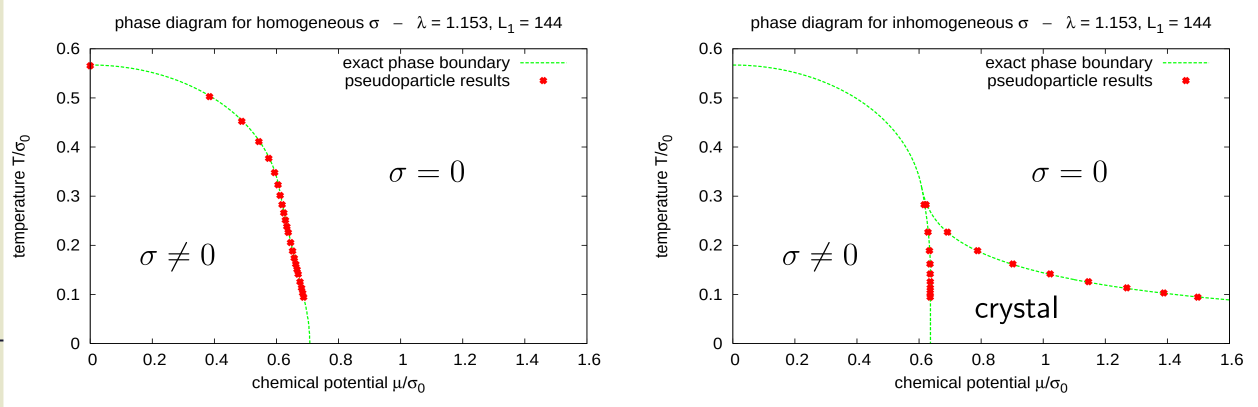
- Fermionic PPs (in this talk): a large number of uniformly distributed "hat functions" (B-spline basis functions of degree 2).
  - "Sensible set of field configurations" (any not too heavily oscillating field configuration can be approximated)
    $\longrightarrow$ we can expect to reproduce correct Gross-Neveu results.
  - Piecewise polynomial functions
    $\longrightarrow$ certain integrals can be calculated analytically.

PPs G_j as functions of x



Phase diagram

- **Q-regularization: completely wrong and useless results.**
  - No improvement, when using a larger number of PPs.
  - No improvement, when using a different type of PPs.
- **$Q^\dagger Q$-regularization: excellent agreement with analytical results.**
  - Homogeneous chiral condensate: analytical results by U. Wolff, 1985.
  - Inhomogeneous chiral condensate: analytical results by O. Schnetz, M. Thies, K. Urlichs, 2004.



Summary and Outlook

Summary and conclusions (1)

- The PP approach with ≈ 400 instantons, antiinstantons and akyrons is able to reproduce many essential features of SU(2) Yang-Mills theory:
  - **Linear quark antiquark potential, i.e. confinement.**
  - Casimir scaling for different quark representations.
  - **Consistent scaling behavior of σ, χ, $T_{\textrm{critical}}$, m_B and m_E.**
  - Dimensionless quantities are in qualitative agreement with lattice results.
- Essential properties of confining gauge field configurations:
  - Long range PPs necessary for confinement (PP percolation)
    $\rightarrow$ **confinement** $\leftrightarrow$ **extended structures and large area excitations.**
  - Instantons and antiinstantons (PPs with non-vanishing topological charge density) necessary for confinement
    $\rightarrow$ **confinement** $\leftrightarrow$ **topological charge.**

Summary and conclusions (2)

- Inclusion of fermions in the PP approach:
  - **Always apply the $Q^\dagger Q$-regularization and not the naive Q-regularization.**
  - The application of the PP approach to compute the phase diagram of the $1+1$-dimensional Gross-Neveu model in the large-N-limit has been a successful test.
  - Next steps:
    * Apply the PP approach to QCD.
    * Try to identify a small number of physically relevant fermionic PPs (PPs, which are able to approximate the low lying eigenmodes of the Dirac operator?).

Outlook

- Current research:
  - Improve the static quark antiquark potential:
    - Use improved operators (smeared Wilson loops) to extract the static quark antiquark potential (o.k.).
    - String breaking for the adjoint representation (???).
  - Consider SU(2) Yang-Mills theory with dynamical fermions:
    - Chiral symmetry breaking by computing the low lying eigenmodes of the Dirac operator in the quenched approximation (Banks-Casher relation) (???).
- Goal: obtain a model with a small number of degrees of freedom, which exhibits chiral symmetry breaking and a confinement deconfinement phase transition at the same time.
- Compute further observables: pion masses, ...