Extracting hadron masses from fixed topology simulations

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[A. Dromard and M. Wagner, PoS LATTICE 2013, 339 (2013) [arXiv:1309.2483 [hep-lat]]]
 [C. Czaban and M. Wagner, PoS LATTICE 2013, 465 (2013) [arXiv:1310.5258 [hep-lat]]]
 [A. Dromard and M. Wagner, arXiv:1404.0247 [hep-lat]]

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Introduction, motivation (1)

- **Topological objects:** non-trivial structures in field configurations, which cannot be removed by continuous deformations, while keeping the action finite (their position can be changed and they can be deformed).
- Analogy: a knot in a rope.
- **Topological charge:** "number of topological objects" in a field configuration.



Introduction, motivation (2)

- Examples of field theories, where topological objects/charge exist(s):
 - 1D QM on a circle.
 - U(1) gauge theory and the Schwinger model in 2D.
 - **SU(2)**/SU(3) Yang-Mills theory and QCD in 4D.



Introduction, motivation (3)

- Typical lattice simulation algorithms might have difficulties changing the topological charge/sector (→ topology freezing).
- Reasons:
 - Field configurations are updated in a nearly continuous way.
 - Topological sectors are separated by large action barriers (in the continuum by infinite barriers).
- Topology changes are strongly suppressed, when
 - using overlap sea quarks,

[S. Aoki *et al.* [JLQCD Collaboration], Phys. Rev. D **78**, 014508 (2008) [arXiv:0803.3197 [hep-lat]]] [S. Aoki *et al.*, PTEP **2012**, 01A106 (2012)]

- the lattice spacing is small ($a \lesssim 0.05$ fm, i.e. close to the continuum). [M. Lüscher and S. Schaefer, JHEP **1107**, 036 (2011) [arXiv:1105.4749 [hep-lat]]]

[S. Schaefer, PoS LATTICE 2012, 001 (2012) [arXiv:1211.5069 [hep-lat]]]

Introduction, motivation (4)

• The simulation of a path integral requires averaging over field configurations from all topological sectors,

$$Z_V \equiv \int DA \, D\psi \, D\bar{\psi} \, e^{-S_E[A,\bar{\psi},\psi]} \quad , \quad \dots$$

• When topology is fixed,

$$Z_{Q,V} \equiv \int DA \, D\psi \, D\bar{\psi} \, \delta_{Q,Q[A]} e^{-S_E[A,\bar{\psi},\psi]} \quad , \quad \dots,$$

results exhibit systematic errors; in particular two-point correlation functions are not proportional $e^{-M_H t}$ for large temporal separations t.

• These errors are proportional to 1/V (V: spacetime volume); their behavior can be calculated as a power series in 1/V; using the results one can determine physical quantities, e.g. hadron masses, from correlation functions from fixed topology simulations.

Introduction, motivation (5)

- There are also cases, where one might fix topology on purpose.
- Example: using a mixed action setup of high quality overlap quarks and computationally inexpensive Wilson (tm) quarks at light quark masses ...
 - ... at light quark masses and $Q \neq 0$ the valence overlap Dirac matrix has near zero modes (Atiyah-Singer index theorem) ...
 - ... which are not present (and, therefore, compensated) in the sea Wilson (tm) Dirac matrix ...
 - $-\ldots$ the consequence is an ill-behaved continuum limit.

[K. Cichy, G. Herdoiza and K. Jansen, Nucl. Phys. B **847**, 179 (2011) [arXiv:1012.4412 [hep-lat]]] [K. Cichy *et al.*, Nucl. Phys. B **869**, 131 (2013) [arXiv:1211.1605 [hep-lat]]]

• A possible solution to this problem might be to fix topology to Q = 0, where also the valence overlap Dirac matrix has no near zero modes.

Introduction, motivation (6)

- Topology can be fixed by
 - either sorting the generated field configurations according to their topological charge
 - or by employing topology fixing actions.
 - [H. Fukaya *et al.*, Phys. Rev. D **73**, 014503 (2006) [hep-lat/0510116]]
 - [W. Bietenholz et al., JHEP 0603, 017 (2006) [hep-lat/0511016]]
 - [F. Bruckmann et al., Eur. Phys. J. A 43, 303 (2010) [arXiv:0905.2849 [hep-lat]]]

Part 1

The behavior of two-point correlation functions at fixed topology

Literature

• Idea:

- Topology fixing causes systematic errors, which are finite volume corrections.
- Expand two-point correlation functions at fixed topology $C_{Q,V}(t)$ (which are used to determine hadron masses) as a power series in 1/V.
- Seminal paper: expansion of $C_{Q,V}(t)$ up to $\mathcal{O}(1/V)$ and in part up to $\mathcal{O}(1/V^2)$.
 - [R. Brower, S. Chandrasekharan, J. W. Negele and U. J. Wiese, Phys. Lett. B 560, 64 (2003) [hep-lat/0302005]]
- General discussion of n-point functions at fixed topology including also higher orders in 1/V.

[S. Aoki, H. Fukaya, S. Hashimoto and T. Onogi, Phys. Rev. D 76, 054508 (2007) [arXiv:0707.0396 [hep-lat]]]

• Our contribution: expansion of $C_{Q,V}(t)$ up to $\mathcal{O}(1/V^3)$.

[A. Dromard and M. Wagner, arXiv:1404.0247 [hep-lat]]

$$Z_{Q,V}$$
 (Z at fixed Q and finite V) (1)

• In the following: expansion of

$$Z_{Q,V} \equiv \int DA \, D\psi \, D\bar{\psi} \, \delta_{Q,Q[A]} e^{-S_E[A,\bar{\psi},\psi]},$$

the partition function at fixed topological charge Q and spacetime volume V, in powers of 1/V (QCD at fixed topology is not a quantum theory).

• $Z_{Q,V}$ is the Fourier transform of $Z_{\theta,V}$, the partition function at vacuum angle θ ($S_{E,\theta}[A, \bar{\psi}, \psi] \equiv S_E[A, \bar{\psi}, \psi] + i\theta Q[A]$; $\theta = 0 \leftrightarrow$ ordinary QCD):

$$\begin{aligned} Z_{Q,V} &= \int DA \, D\psi \, D\bar{\psi} \left(\frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{i(Q-Q[A])\theta}\right) e^{-S_E[A,\bar{\psi},\psi]} &= \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{iQ\theta} \underbrace{\int DA \, D\psi \, D\bar{\psi} \, e^{-S_{E,\theta}[A,\bar{\psi},\psi]}}_{\equiv \mathcal{Z}_{\theta,V}} &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{iQ\theta} \mathcal{Z}_{\theta,V}. \end{aligned}$$

(QCD at $\theta \neq 0$ is a quantum theory, e.g. a Hamiltonian and states exist, ...).

$Z_{Q,V}$ (Z at fixed Q and finite V) (2)

- One can show $E_n(+\theta, V_s) = E_n(-\theta, V_s)$; it implies $(d/d\theta)E_n(\theta, V_s)|_{\theta=0} = 0$.
- One can show $\chi_t = e_0^{(2)}(\theta)|_{\theta=0}$ (the second derivative of the vacuum energy density at $\theta = 0$),

$$\chi_{t} \equiv \lim_{V \to \infty} \frac{\langle Q^{2} \rangle}{V} = -\lim_{V \to \infty} \frac{1}{\mathcal{Z}_{\theta,V} V} \frac{d^{2}}{d\theta^{2}} \mathcal{Z}_{\theta,V} \Big|_{\theta=0} =$$

$$= -\lim_{V \to \infty} \frac{1}{\mathcal{Z}_{\theta,V} V} \frac{d^{2}}{d\theta^{2}} \sum_{n} e^{-E_{n}(\theta,V_{s})T} \Big|_{\theta=0} =$$

$$= \lim_{V \to \infty} \frac{1}{\mathcal{Z}_{\theta,V} V_{s}} \sum_{n} \frac{d^{2} E_{n}(\theta,V_{s})}{d\theta^{2}} e^{-E_{n}(\theta,V_{s})T} \Big|_{\theta=0} =$$

$$= \lim_{V_{s} \to \infty} \frac{E_{0}^{(2)}(\theta,V_{s})}{V_{s}} \Big|_{\theta=0} = e_{0}^{(2)}(\theta) \Big|_{\theta=0}$$

(ordinary finite size effects neglected, e.g. $E_0(\theta, V_s) = e_0(\theta)V_s$).

• This calculation explains the appearance of χ_t in the following equations.

$$Z_{Q,V}$$
 (Z at fixed Q and finite V) (3)

• $\mathcal{Z}_{\theta,V}$ is dominated by the vacuum at large temporal extension T,

$$Z_{Q,V} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{iQ\theta} \mathcal{Z}_{\theta,V} =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \, e^{iQ\theta} e^{-E_0(\theta,V_s)T} \Big(1 + \mathcal{O}(e^{-\Delta E(\theta)T}) \Big).$$
(1)

• The vacuum energy $E_0(\theta, V_s)$ can be written as a power series in θ ,

$$E_0(\theta, V_s)T = e_0(\theta)V = \left(\sum_{n=0}^{\infty} \frac{\mathcal{E}_{2n}}{(2n)!} \theta^{2n}\right)V,$$

where
$$\mathcal{E}_n \equiv e_0^{(n)}(\theta) \Big|_{\theta=0}$$
, in particular $\mathcal{E}_2 = \chi_t$.

• The integral in (1) can be solved (calculating order by order in 1/V) using standard techniques: $\int_{-\pi}^{+\pi} d\theta \to \int_{-\infty}^{+\infty} d\theta$ (exponentially suppressed errors) ... residue theorem ... Gaussian integrals ("saddle point approximation") ...

$Z_{Q,V}$ (Z at fixed Q and finite V) (4)

• The calculation is lengthy ... details not suited for a talk ...

these equations is a simple model, quantum mechanize on a clinic letch in the low case and with a potential. We also explain and demonstrate in densal, here physical meanse (com- sponding is unified topology) can be extracted from front longing simulations (for related exploratory studies in the Solvinger model and the $O(2)$ and $O(2)$ models are Signs model of [13, 14, 15, 16]. This might provide height singlest and picklosure for exploring the same	according to $\chi_{0} = \lim_{U_{1}\to\infty} \frac{\chi_{0}^{(U)}(0, U_{2})}{U_{1}}\Big _{U_{1}=0} = \chi_{0}^{(U)}(0)\Big _{U_{1}=0}$ (3.7) (throughout this paper X ^(b) denotes the <i>u</i> -th ordershifts of the quantity X with respect to H. We construct we analyze constrained the actions of the 1.6 bits attacket is the first set of actional	Finally the saidle point method requires us deform the contour of knogradios to pass through the saidle point, which is pins a constant with of the main and by the purely insignary δ_{ij} . This insolves the same coordinate $s = (b - a_i)/(\beta^2 b_i)/(1^{-1/2})$ parameterizing the shifted contert of integration yielding	• $k = 1, n = k;$ $\left\ \frac{f^{\mu}(\theta_{1}) r}{\theta_{1}(\theta_{1}) r} r^{\mu} \right\ _{H^{1}(\Omega)} = \frac{1}{Z_{0}^{\mu}} \frac{f_{0}}{\delta f_{0}^{\mu}} + \frac{1}{Z_{0}^{\mu}} \left(\frac{f_{0}}{\delta d_{0}^{\mu}} - \frac{f_{0}^{\mu}}{\delta d_{0}^{\mu}} \right) \theta_{0}^{\mu} + O\left(\frac{1}{(f_{0})^{\mu}} \right). (2.30)$	• $k = 3, v_1 = v_2 = 3, v_3 = 4;$ $3 \times \left\ \frac{(f^{(0)}(\theta_1)(f^{(0)}(\theta_2)Y)}{(2f)^{(0)}(\theta_2)Y)} s^{(0)} \right\ = O\left(\frac{1}{(f_2Y)^2} \theta_2^2 \right).$ (2.30)
strategy to QCD. Partie of dis-verk have been presented at a recent conference [17]. 2. The martition function Z_{OU} at fived tonology and finite	with find topolog. These are expected to be suggeneed exponentially with increasing spatial volume U_c for action 4.3 for a discussion). In colore worth we assume V_c to be sufficiently large such that $E_0(\theta_c, V_c) = c_0(\theta_c V_c)$ where $c_0(\theta)$ is the energy density of the HOURD.	$Z_{Q,T} = \frac{d}{2\pi (f^{(2)}[\delta_1(\eta^*)] \otimes 2} \int_{-\infty}^{\infty} ds \exp\left(-\frac{1}{2}t^2 - \sum_{n=0}^{\infty} \frac{f^{(2)}[\delta_1(\eta^*)]}{s[f_1(\eta^*)]\delta_1(\eta^*) \otimes 2}s^n\right).$ (2.35) After defining	• $s = s_1 = 0$ $\left\ \frac{f^{(0)}(0, 0')}{(\delta_n' f^{(0)}(0, 0')^2} s^0 \right\ = \frac{1}{(\delta_n' V)^2} \frac{\delta_n}{\delta \delta_n'} + O\left(\frac{1}{(\delta_n' V)^2} s^0\right). (2.31)$	• $x = u_c n_c = n_q = n_q = u_c$ $\left\ \frac{(I^{(4)}(0, V)^2)}{(4!)^{1/2} (m_c(0, V)^2)} x^{1/2} \right\ = \frac{1}{(\delta_{21} V)^2} \frac{36 \chi \xi_{1}^2}{56 \chi \xi_{21}^2} + O\left(\frac{1}{(\delta_{21} V)^2}\right). (2.31)$
spacetime volume	At sufficiently large T the partition function is dominated by the vacuum, i.e. $Z_{e_{V}} = e^{-i_{0}(\theta_{V}, \tau)} \left(1 + O(e^{-\Delta \theta_{V}(\tau)})\right),$ (2.8)	$ h(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds e^{-s/2}h(s)$ (2.16) a more compact notation for the result (2.16) is	• $k = 1, v = 8$ $\left\ -\frac{f^{(0)}(k, W)}{r^{(0)}} r^{0} \right\ = \frac{1}{r^{(0)}} \frac{f_{0}}{r^{(0)}} + O\left(-\frac{1}{r^{(0)}}\right), (2.22)$	• $k = 3, m_1 = m_2 = 4, m_2 = 6$ $\sim - \frac{ (f^{(0)}(0, V)^2 f(f^{(0)}(0, V) - n) }{2} = n f \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ (5.27)
In this section we calculate the dependence of the particular $Q_{n,k}$ particular function in most topological charge Q on the spacetime volume Y_{n} denoted as $Z_{Q,V_{n}}$ up to $O(1/V^{2})$.	where $\Delta E(\theta) = E(A,V_c) - E(\theta,V_c)$. The exponentially suppressed correction will be entitled in the following (cf. section 4.2 for a discussion). To ease notation, we define	$Z_{Q,V} = \frac{e^{-\beta \theta_{i} V}}{(2\pi f^{12}(\theta_{i})V^{1/2})} \left\ \exp \left(- \sum_{n=0}^{\infty} \frac{f^{(n)}(\theta_{i})V}{n!(f^{(n)}(\theta_{i}))V'f^{(2)}} x^{n} \right) \right\ .$ (2.37)	• k = 1, n = 10. (3.97) 1 (2.17) 38k ₂ − (3.17) 1	$3 \times \left\ \frac{-(d)^2 \theta(f_1^{-1}(\theta_1^{-1})^{-1})^{-1}}{(d)^2 \theta(f_1^{-1})^{-1}}^{-1}\right\ = O\left(\frac{1}{(d_2^{-1})^{-1}}\right).$ (2.32) • $k = 4, m_1 = m_2 = m_1 = 4$.
2.1 Calculation of the $1/V$ expansion of $Z_{Q,N}$ The Euclidean QCD partition function at non-vanishing θ angle and finite spacetime volume V is idented as	$f(\theta) = f(\theta, Q, V) = \alpha_0(\theta) - \frac{\theta_0 \theta}{V}$. (2.9) Using also (2.8) the partition function at fixed topology (2.5) can be written according to	where G case also be written as	$\left\ \frac{\phi_{1}}{(0)}\psi_{1}^{(0)}(y_{1})\psi_{1}^{(0)}\right\ = O\left(\frac{1}{(0, V)^{2}}\right). (2.23)$ • $k = 2, v_{1} = v_{2} = 3$	$\left\ \frac{(f^{(0)}(\theta_{0})V)^{4}}{(4)^{\mu}(f^{(0)}(\theta_{0})V)^{\mu}}e^{\mu t}\right\ = O\left(\frac{1}{(t_{0}^{1})^{(\mu)}}\right). (2.33)$
$Z_{WV} = \int DA D\psi D\bar{\psi} e^{-\chi_{W}(\bar{\psi},\bar{\psi})} = \sum_{n} e^{-\delta \psi N N T}$ (2.1) true and	$Z_{QP} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta e^{-P\theta \theta}$, (2.10)	$G = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\ \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(\theta_n)t'}{n! f^{((n)}(\theta_n)t')^{n/2}} s^n \right)^n \right\ . (2.39)$	$\left\ \frac{(f^{(0)}(y_i)V)^3}{(2f)^3(\theta_i)V)^3}s^0\right\ = \frac{1}{Z_0V}\frac{4Z_0^2}{12Z_0^2}\theta_i^2 + O\left(\frac{1}{(Z_0V)^2}\right). (2.34)$	Inserting these expressions into (2.19) leads to
programs $\bar{S}_{EB}(\bar{A}, \bar{\psi}, v) = S_Z(\bar{A}, \bar{\psi}, v) + i\delta Q(\bar{A}),$ (2.2) where T is the periodic time extension, V_c the spatial volume, $V = TV_c, E_c(\bar{A}, V_c)$ is the	where the integral (2.10) can be approximately product of the status plott method. To thus each we expend $1/(6V)$ acrossed its consistence n_{1} and replace $(L_{1}^{-1}, M) \in \Gamma_{1}^{-1}$, which introduces another exponentially suppressed error (of section 4.2 for a discussion).	We now insert $f(\theta)V$ and δ_{i} (eqs. (2.13) and (2.14)) and perform the integration over s- order by order in $1/2_{i}V$ (note that $\delta_{i} \sim 1/2_{i}V$). To this end we use the relations	• $k = 2, v_0 = 3, v_0 = 5;$ $ (U^{(2)}(\theta_i V)(J^{(2)}(\theta_i V)) _{0} _{1} = o(-1-a);$ (3.10)	$G = 1 + \frac{1}{\xi_{1}V} \left(- \frac{\xi_{2}}{\xi_{2}} + \left(- \frac{\xi_{0}}{4\xi_{2}} + \frac{1}{4\xi_{1}^{2}}\right)^{q} \right) + \frac{1}{(\xi_{1}V)^{q}} \left(- \frac{\xi_{0}}{4\xi_{2}} + \frac{3M_{1}}{3M_{2}^{2}} \right) \\ + \frac{1}{-1} \left(- \frac{\xi_{0}}{\xi_{0}} + \frac{7\xi_{0}\xi_{0}}{3M_{1}^{2}} + \frac{3M_{1}^{2}}{2M_{1}^{2}} \right) + \mathcal{O} \left(- \frac{1}{\xi_{0}} - \frac{1}{\xi_{0}^{2}} + \frac{M_{1}^{2}}{M_{1}^{2}} \right) $ (2.30)
energy eigenvalue of the r-th eigenvalue of the linearitonian and S_{c} the Dicklena QCD action whitese θ -term. Similarly, the Bucklean QCD particles function at fixed topological charge Q and finite spacetime tolume V is defined as	$Z_0 = \frac{1}{2m} \int_{-\infty}^{\infty} d\theta \exp \left(-f(\theta_i)V - \frac{f^{(0)}(\theta_i)V}{2}(\theta - \theta_i)^2 - \sum_{n=0}^{\infty} \frac{f^{(0)}(\theta_i)V}{n!}(\theta - \theta_i)^n\right).$ (2.11) $\theta = \min \log \log (\log \theta - \log \theta) + \log (\log \theta - \log \theta)$	$I^{(2n)}(0_n)V = \sum_{k=0}^{\infty} \frac{d_{2k}V}{(2k-2n)!} \theta^{\mu-2n}, n = 1, 2,,$	$2 \times = \frac{1}{320} \sqrt{2} (\theta_1 /2^{-1} + \frac{1}{3}) (\theta_2 /2^{-1} + \frac{1}{3}) (\theta_2 /2^{-1} + \frac{1}{3}) (2.20)$ • $k = 2, m = m = 4.$	$ \mathcal{L}_{1}V ^{2} \langle -36\mathcal{L}_{2}^{2} - 26\mathcal{L}_{2}^{2} - 3672\mathcal{L}_{2}^{2} \rangle = \langle \mathcal{L}_{1}V ^{2} - \mathcal{L}_{2}V ^{2} + f$ and, after inserting the expansion of δ_{1} (2.14), yields
$Z_{Q,V} = \int DA D\psi \delta_{Q,Q,R} e^{-S_{R}/k \hat{L}_{R}}$ (2.3)	is can be determined as a power ison in $(2\pi)^{-1}$. Due to $\omega_{n}(-\alpha_{1}) + \alpha_{1}(-\alpha_{2})$, the expansion of the meansumemergy density around $\theta = 0$ is $\omega_{n}(\theta) = \sum_{n=1}^{\infty} \frac{d_{2n}\theta^{2n}}{(2\alpha_{1})^{n}} = \langle \sigma_{n} = \sigma_{n}^{(2)}(\theta) \rangle$ (9.19)	$f^{(2n+1)}(\theta_i)V = \sum_{i=0}^{n} \frac{C_{ij}(v)}{(2-2n+1)!} \theta_i^{(2-2n+1)} = v = 2, 3,,$ $ _{ij}^{(2n+1)} = 0 = (_{ij}^{(2n+1)} - _{ij}^{(2n+1)} - _{ij}^{(2n+1)} - _{ij}^{(2n+1)} + v = 0, 3,,$	$\left\ \frac{(f^{(1)} \theta_{i} V)^{2}}{(dt)^{i}f^{(2)} \theta_{i} V)^{2}}s^{2}\right\ = \frac{1}{(f_{i}V)^{2}}\frac{3\beta_{i}f_{i}^{2}}{192d_{i}^{2}} + O\left(\frac{1}{(f_{i}V)^{2}}\theta_{i}^{2}\right). (2.36)$	$\hat{G} = 1 - \frac{1}{\hat{L}_{4} U} \frac{\hat{e}_{4}}{(\hat{e}_{4} V)^{2}} \left(- \frac{\hat{e}_{4}}{(\hat{e}_{4} V)^{2}} + \frac{3 \hat{e}_{4} \hat{e}_{1}}{3 \hat{e}_{4} \hat{e}_{1}^{2}} \right)$ $1 - \left(\hat{e}_{4} - \frac{1}{2} \hat{e}_{4} \hat{e}_{1} - \frac{1}{2} \hat{e}_{4} \hat{e}_{1} - \frac{3 \hat{e}_{4} \hat{e}_{1}}{3 \hat{e}_{4} \hat{e}_{1}^{2}} \right)$
Using $\delta_{Q,Q(4)} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} db e^{i(Q-Q)A/\theta}$ (2.4)	$u_{2}(r) = \sum_{k=0}^{n} (2k)^{n-1} = u_{2}(r) + u_{k-0}(r) + u_{k-0}($	p = 1 - 0 (2.19) The terms in (2.18) are	• $k = 2, v_1 = 4, v_2 = 6;$ $v_1 = \left[(f^{(1)}(0, Y) f^{(0)}(0, Y), v_2 \right] = \frac{1}{-1} \frac{7L}{2L} \frac{1}{4} + c_1 \left(-\frac{1}{-1} \right) $ (2.27)	$+ \frac{1}{(k_1^{-1})^{\prime\prime}} \left(-\frac{34k_1^{\prime\prime}}{34k_1^{\prime\prime}} + \frac{1}{24k_2^{\prime\prime}} - \frac{3772k_1^{\prime\prime}}{3772k_1^{\prime\prime}} + \left(\frac{1}{14k_1^{\prime\prime}} - \frac{1}{3k_2^{\prime\prime}} \right) Q^{\prime\prime} \right) \qquad (2.30)$ $+ \mathcal{O} \left(\frac{1}{(k_1^{-1})^{\prime\prime\prime}} - \frac{1}{(k_1^{-1})^{\prime\prime}} Q^{\prime} \right).$
it is easy to see that $Z_{Q,1}$ and $Z_{R,2}$ are related by a Fourier transform,	$f(\theta)V = \sum_{k=0}^{\infty} \frac{\mathcal{L}_{ab}\theta^{ab}}{(2k)!}V - iQ\theta.$ (2.13)	 for k = 1 propertional to 1/(L₁)[*])^{n/(n-1)}, for k = 2 reconstituted to 1/(L₁)[*])^{(n)=n(1/(n-1))}. 	• $k = 2, v_1 = 4, v_2 = 8$	The remaining terms in (2.17) expressed in powers of V are
$S_{QN} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i\phi} Z_{QN}, \qquad (2.5)$	It is straightforward to solve the defining equation for θ_{ii} $d/d\theta f(\theta V _{\theta=0}, = 0$, with respect to θ_{ii}	- for $k=3$ propertional to $1/\langle \mathcal{E}_1 V \rangle^{(n_1+n_2)(n_2/2-3)},\ldots$	$2 \times \left\ \frac{ f^{(1)}(\theta_{i} W) f^{(0)}(\theta_{i} W) }{2} e^{i\theta} \right\ = O\left(\frac{1}{2\pi i m^{2}}\right),$ (2.28)	$f(\theta_s)V = I_0V + \frac{1}{I_0V}\frac{1}{2}Q^2 + \frac{1}{ f_1V ^2}\frac{I_1}{24I_2}Q^4 + O\left(\frac{1}{ f_1V ^3}\right)$ (2.36)
Using this together with $(2, 1) = 2_{n_1}(-s_1, s_1)$ (11), which implies $(s_1) = (s_1, s_2, s_3, s_4) = 0$. Using this together with $(2, 1)$ and $(2, 2)$ one can express the topological susceptibility, defined as: $y_{12} = \sum_{k=0}^{n_1} \frac{(Q^2)}{2}$. (2.6)	$\theta_{i} = i \left(\frac{1}{E_{k}^{i} V} Q + \frac{1}{(E_{k}^{i})^{i}} \frac{dx_{i}^{2}}{dx_{i}^{2}} \frac{Q^{2}}{dx_{i}^{2}} + Q \left(\frac{1}{(E_{k}^{i})^{i}} \right)^{i} \right)$ (2.14) ¹ Thresphere the rest energy is a dependent of the set of $(U_{i}^{i})^{i} + U_{i}^{i} + \frac{1}{(E_{i}^{i})^{i}} \frac{1}{(E_{i}^{i})}$	Moreover, $u_1, u_2, u_3, u_4, u_4, u_5, u_6, u_8$, have 54 be even, determine the corresponding sorm in (2.19) vanishes, the set 2(2.10). Tanily even (0.4) as and v_1 contractions in loading sorder in d_1 are power of $\theta_1 \sim 1/\theta_2$ Y. Therefere, up to $O(1/ \theta_2 V ^2)$ is is sufficient to consider the following terms:	• $k = 2, v_1 = v_2 = \psi$ $\left \frac{(f^{(0)} 0, 1)^{(j)}}{(dt^{(j)} f^{(j)} 0, 1)^{(j)}} z^{(j)} \right = O\left(\frac{1}{(l_1^j)^{(j)}}\right),$ (2.29)	and $f^{(2)}(b_i V = Z_iV \left(1 - \frac{1}{(Z_iV)^2}\frac{t_i}{2Z_i}Q^i + O\left(\frac{1}{(Z_iV)^2}Q^i\right)\right).$ (2.37)
8	4	5	*	7
Combining (2.17), (2.36), (2.26) and (2.37) yields the final result for $Z_{\rm QV},$	O denotes a métable hadron creation operator, for enample for the charged pion π^+ a contrast, charge is	with My survailed and has defined parity. Then $\alpha(\theta)$ can be written as a power series, arranged $\beta=0$ according to	Using (2.35) and (2.36) yields an explicit expression up to $\mathcal{O}(1/(\mathcal{E}_k t))^2)_i$	point correlation functions at fixed topology,
Containing (21.17), (1.26), (2.26) and (2.17) yields the final mean for $Z_{2(1)}$, $Z_{2(1)} = \frac{1}{\sqrt{22(2)}} \left(\exp\left(-L_{2(2)}^{-1} Z_{2(2)}^{-1} Z_{2(2)}$	Obtained a similar hadron mode sequence, for energies of the charged plane γ^2 accounts, there $-2\pi \frac{1}{m_p^2}\int_{-\pi}^{\pi}f^2\sigma_{\mu}^2(r)\sigma_{\mu}(r)$ (23) (6.5 g (FH) for a invaluation in the inter hadron sequences on the numerical of the methan quences $(-2\pi)^{-1}\sigma_{\mu}^2(r)$ and $\sigma_{\mu}^2(r)$ and $\sigma_{\mu}^2(r)$ and $\sigma_{\mu}^2(r)$ and methan quences $(-2\pi)^{-1}\sigma_{\mu}^2(r)$ and $\sigma_{\mu}^2(r)$	which y available and has defined party. Then $\alpha(0)$ can be written as a power avia, $\alpha(0) = \sum_{i=1}^{N} \frac{(2i)^{\alpha}(2i)^{\alpha}}{(2i)^{\alpha}} = \alpha(0)\exp\left(\ln\left(\sum_{i=1}^{N} \frac{(2i)^{\alpha}(2i)^{\alpha}}{(2i)^{\alpha}(2i)^{\alpha}}\right)\right),$ (3.9) $\max_{i=1}^{N} \frac{(2i)^{\alpha}(2i)^{\alpha}}{(2i)^{\alpha}} = \alpha(0)\exp\left(\ln\left(\sum_{i=1}^{N} \frac{(2i)^{\alpha}(2i)^{\alpha}}{(2i)^{\alpha}(2i)^{\alpha}}\right)\right).$	$\begin{aligned} &(\log\left\{1.3\right\}) \sin\left\{2.31\right\} \sin\left\{4.31\right\} \sin\left\{4.31\right\} \sin\left\{4.31\right\} \sin\left\{4.31\right\} \sin\left\{3.31\right\} $	point consistent functions at their topology, $C_{ijj}(t) = \frac{1}{\sqrt{1 + e_{ij}^{2}/k^{2}}}$ $cos_{ij}\left(-\frac{1}{\sqrt{1 + e_{ij}^{2}/k^{2}}} - \frac{1}{\sqrt{1 + e_{ij}^{2}$
Contains (2.17), (1.30), (1.30) and (1.31) yields the final matrix $k_{-k_{0}}$, $\chi_{0} = \frac{1}{\sqrt{2\pi}} \frac{1}{2\pi} \left(q \left(-\frac{1}{4\pi k_{0}^{2}} \frac{1}{2\pi} - \frac{1}{4\pi} \frac{f_{0}}{2\pi} q^{2} \right)^{-1} \frac{f_{0}}{4\pi} \frac{1}{2\pi} \frac{1}{2\pi} \frac{f_{0}}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} \frac{f_{0}}{2\pi} \frac{1}{2\pi} \frac$	Observe a stabilization entries operator. So enough to the chapter latter ' accesses that '' $(1 + \frac{1}{\sqrt{2}} \int_{0}^{1} f_{1}^{2} f_{2}^{2} (r) r) dr$ (1.5) If $f_{2} = \int_{0}^{1} f_{1}^{2} f_{2}^{2} (r) dr$ (and the same stabilized on the stabi	which yscalards at the definit ratio, $\operatorname{Term} (\phi)$ are for which as a power sets, $\begin{aligned} & \varphi_{i}^{*} = \operatorname{scalard}(g_{i}) & \varphi_{i}^{*}(g_{i}) & \varphi_{i}$	$\begin{array}{l} \log \left(B \right) \approx d\left(B \right) \sin dx \approx split regardler, \eta \approx 10^{-1}(S /T)^{-1}, \\ C_{21} \left(S _{21} - \frac{2}{22D_{21}} \int_{T}^{1} \left((U - SE - \frac{1}{2A_{21}} \int_{T}^{1} \left(SE - \frac{1}{2A_{21}} \int$	point combines functions at less legacities $G_{21}(z) = \frac{1}{\sqrt{1 + c_{2}/c_{2}}} \frac{1}{c_{2}/c_{2}} \frac{1}{c_{2}/c_{2}/c_{2}} \frac{1}{c_{2}/c_{2}/c_{2}/c_{2}} - \frac{1}{c_{2}/c_{2}$
$ \begin{aligned} & \text{Continue} (2.17), (2.30), (2.30) & \text{of} (1.37), \text{pick to the and set } k \in \mathcal{L}_{0,1}, \\ & \mathcal{L}_{0,1} = \frac{1}{\sqrt{2\pi^2 T_1^{-1}}} \left(q - \left(-4\pi k_1 N^2 - 1 \right) \frac{f_1}{2\pi} (q^2 - \frac{f_1}{2\pi} f_2 N^2 - \frac{f_1}{2\pi} (q^2 - \frac{f_1}{2\pi} f_1 N^2 $	O denote a sinkle halos metric quertas. En remark for the depth dia γ^* screenes de la serie $-\frac{1}{\sqrt{1-\beta}}\int_{0}^{-\beta}d\rho^{(\beta)}(r)(r)$ (4) of $-\frac{1}{\sqrt{1-\beta}}\int_{0}^{-\beta}d\rho^{(\beta)}(r)(r)$ (5) For a first particular bis du la surgementar per part de construites trates des una surgement de la serie	which consider and be defined only. The off one for white as a power sets $ \begin{split} & = (k - 1) \exp \left(- \frac{k - 1}{k - 1} \int_{-\infty}^{\infty} \frac{d(k - 1)}{k - 1} \int_{-\infty}^{\infty} $	$\begin{aligned} &\log \left(2.3\right) & \approx 0.237\right) (4.66) + a + q \sin \theta + q \sin (a + a + 0.5) (d_1)^2 f_1 \\ & C_{10}(1) S_{10} + \frac{q + 2}{2M_{10}} f_1(\alpha + (-J_0) - \frac{1}{2M_{10}} \frac{1}{2} $	point combine functions at least topology. $\begin{split} & G_{21}(t) = \frac{\sqrt{10}}{\sqrt{10}} \frac{1}{\sqrt{10}} \frac{1}{\sqrt{10}} \frac{1}{\sqrt{10}} \left(\frac{1}{\sqrt{10}} \frac{1}{10$
Contains (2.77), (2.30), (2.30) and (2.37), relate the Fourier and For $E_{0,1}$, $Z_{0,2} = \frac{1}{\sqrt{2\pi^2}} \left(\sqrt{c_1} - \left(-L_{0,1} + C_{0,1} + C$	Observe a similar holder metric spectra. For earged to the chapter plane τ accesses $M = \frac{1}{2} \left(\frac{1}{\sqrt{n}} \int_{0}^{1} \frac{\partial (x) (x)(x)}{\partial (x)} \right)$ (2.9) In the plane metric of the spectra plane of the plane metric of the spectra plane of the spectra metric of the sp	statisty secondaria set be defined unity. Then only one be written as a power setue $ a(r_{0}) = \frac{a(r_{0}) - a(r_{0}) $	Deg (33) and (339) both an sight equation $q_0 = 0.75(2/17),$ $C_{00}(0) \mathcal{Z}_{00} + \frac{\partial \mathcal{Z}_{00}}{\partial q_0} \left(\cos \left(-\partial \mathcal{U} - \frac{1}{2\sqrt{1/2}} \mathcal{U} - \frac{1}{(2\sqrt{1/2})} \mathcal{Z}_{00}^{(2)} \right) - \left(\frac{1}{(-\sqrt{1/2})} \mathcal{U}_{00}^{(2)} \right) \mathcal{U}_{00}^{(2)} + \frac{1}{(2\sqrt{1/2})} \mathcal{U}_{00}^{(2)} \right) - \frac{1}{(2\sqrt{1/2})} \left(- \frac{1}{(2\sqrt{1/2})} \mathcal{U}_{00}^{(2)} - \frac{1}{(2\sqrt{1/2})} \mathcal{U}_{00}^{(2)} \right) - \frac{1}{(2\sqrt{1/2})} \mathcal{U}_{00}^{(2)} - \frac{1}{(21/$	point conclusion functions as their topology: $\begin{split} & \mathcal{Q}_{0}(t) = \frac{-q_{1}}{\sqrt{1+q_{1}}\sqrt{2}} \\ & \qquad \qquad$
Contains (2.17), (1.30), (3.20) and (3.17) relates the final matrix k_{cyc} , $k_{cyc} - \frac{1}{\sqrt{2}\sqrt{2}\sqrt{2}} \left(\sqrt{c_c} - \frac{1}{\sqrt{2}\sqrt{2}} \sqrt{c_c} \sqrt{c_c} - \frac{1}{\sqrt{2}\sqrt{2}} \sqrt{c_c} - \frac{1}{\sqrt{2}\sqrt{2}} \sqrt{c_c} \sqrt{c_c} - \frac{1}{\sqrt{2}\sqrt{2}} \sqrt{c_c} - \frac{1}{\sqrt{2}\sqrt{2}} \sqrt{c_c} - \frac{1}{\sqrt{2}\sqrt{2}} \sqrt{c_c} \sqrt{c_c} - \frac{1}{\sqrt{2}\sqrt{2}} \sqrt{c_c} - 1$	Observe a sinkly holes more specific. For early the the chapter plane r accesses $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} d\rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (26) In the first plane $\rho_{\mu} = \frac{1}{\sqrt{2}} \int \sigma_{\mu} d\rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (36) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} d\rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (37) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (38) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (36) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (37) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (38) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (38) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (38) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu} \partial \rho_{\mu}$ (38) $\mu = \frac{1}{\sqrt{2}} \int \sigma_{\mu} \partial \rho_{\mu} $	addy available at the Addy andy Then off) are the values as a set of the strength of the	$\begin{split} & \log \left(23 \right) \approx d\left(23 \right) \operatorname{Melk} an again equation q_0 = 0.7 f_0 ^2 / f_1 \\ & = \frac{d^2 g}{d^2 g} \left(q_0 \left(-2\delta - \frac{1}{2\sqrt{2}} \frac{1}{2} \frac{d^2}{d^2} - \frac{1}{(2\sqrt{2})} \frac{d^2 g}{d^2} \right) \right) = \\ & - \frac{1}{\sqrt{2\delta d^2}} \frac{d^2 + \delta - \delta }{(2\sqrt{2} - \sqrt{2})} \left(\frac{1}{d^2} - \frac{1}{(2\sqrt{2})} \frac{d^2 - \delta }{d^2} \right) \right) = \\ & - \frac{d^2 - \delta }{d^2 d^2} \frac{d^2 + \delta - \delta }{(2\sqrt{2} - \sqrt{2})} \frac{d^2 - \delta }{d^2} \frac{d^2 - \delta }{d^2} \\ & - \frac{1}{(2\sqrt{2} - \sqrt{2})} \frac{d^2 - \delta }{d^2 d^2} \frac{d^2 - \delta }{d^2 - \sqrt{2}} \frac{d^2 - \delta }{d^2} \frac{d^2 - \delta }{d^2} \\ & - \frac{1}{(2\sqrt{2} - \sqrt{2})} \frac{d^2 - \delta }{d^2 d^2} \frac{d^2 - \delta }{d^2} d^2 $	point conclusion functions as their topology: $\begin{split} & \mathcal{Q}_{0}(t) = \frac{-\sqrt{1+q_{0}}}{\sqrt{1+q_{0}}} \\ & = \frac{1}{\sqrt{1+q_{0}}} \left(\frac{-\lambda_{0}(t) - \frac{1}{\sqrt{1+q_{0}}}}{\sqrt{1+q_{0}}} - \frac{1}{\sqrt{2}}\right)_{0}^{2} t^{2} \\ & = \frac{1}{\sqrt{1+q_{0}}} \left(\frac{\lambda_{0}}{\sqrt{1+q_{0}}} - \frac{1}{\sqrt{2}}\right)_{0}^{2} t^{2} \\ & = \frac{1}{\sqrt{1+q_{0}}} \left(\frac{\lambda_{0}}{\sqrt{1+q_{0}}} - \frac{1}{\sqrt{1+q_{0}}} - \frac{1}{\sqrt{1+q_{0}}}\right)_{0}^{2} t^{2} \\ & = \frac{1}{\sqrt{1+q_{0}}} \left(\frac{\lambda_{0}}{\sqrt{1+q_{0}}} - \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} \right)_{0}^{2} t^{2} \\ & = \frac{1}{\sqrt{1+q_{0}}} \left(\frac{\lambda_{0}}{\sqrt{1+q_{0}}} - \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} \right)_{0}^{2} t^{2} \\ & = \frac{1}{\sqrt{1+q_{0}}} \left(\frac{\lambda_{0}}{\sqrt{1+q_{0}}} + \frac{1}{\sqrt{1+q_{0}}} + \frac{1}$
$ \begin{aligned} & \text{Contract}(\Omega^{1}, 1, \mathbb{H}), (\Omega^{10}) \neq (1, \mathbb{H}) $	Obtain a table halo metric queries, for angel k the charge plant $r_{\rm s}$ are entropy of $\mu_{\rm s}$ ($\mu_{\rm s}$ $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$) $\mu_{\rm s}$ ($\mu_{\rm s}$) \mu_{\rm s}) $\mu_{\rm s}$) \mu_{\rm s}) $\mu_{\rm s}$) $\mu_{\rm s}$) \mu_{\rm s}) her har matrix	which smallest and the Adds ratio. The ord/in the twins as a power sets and $-\frac{1}{2}\left(\frac{1}{2}\sum_{k=1}^{N-1}\sum_{k=1}^{N-1}\sum_{m=1}^{N-1}\sum_{k=1}^{N-1}\sum_{m=1}^{N-1}$	$\begin{split} & \log \left(2.8 \right) & \approx 1.2 \mathrm{M} \right) \mathrm{d} \mathrm{d} \mathrm{d} \mathrm{M} \right) \mathrm{d} \mathrm{d} $	point carefultion functions at least hyperby: $\begin{split} & \mathcal{L}_{0}(z) = \frac{1}{\sqrt{1+e_{0}}(z)^{2}} & \\ & \mathcal{L}_{0}^{2}(z) = \frac{1}{1+e_$
Calcing (27), 130, (23) at (13) yields the fast stands it $L_{a,c}$, $\chi_{a} = \frac{1}{\sqrt{2\pi}} \left(m \left(-k_{a} (\lambda_{a}^{-1}) - \frac{1}{\sqrt{2\pi}} \frac{\lambda_{a}^{-1}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}^{-1})^{-1}} \frac{\lambda_{a}^{-1}}}{(\lambda_{a}$	<text><text><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></text></text>	which would not be that parts. The only near bar into a 1 years on the strength $(\begin{array}{c} \displaystyle \sum_{m=0}^{m} \sum_{m=0}^{m}$	Deg (1.8) and (2.8) (Add as a split regardless up to $(0.5)/(J_1/h_1)$ $C_{221}(0.5) = -\frac{c_{221}}{C_{221}} \int (c_{221}(-J_2/h_1)^2 - J_2/h_1^2 - J_2$	point conclusion functions at their topological structure of the structur
Contains (2.17), (1.30), (1.30) and (1.31)) relate the dual state $L_{s,0}$, $\begin{split} \chi_{0} &= \frac{1}{\sqrt{2\pi}N} \left(m\left(-4\pi \lambda_{0}^{2}\lambda_{0}^{2}-1\frac{\lambda_{0}^{2}}{4\lambda_{0}^{2}}\right)^{-1}\frac{f_{0}^{2}}{4\lambda_{0}^{2}}\left(-\frac{\lambda_{0}^{2}}{2\lambda_{0}^{2}}\right)^{-1}\right) \\ &= \frac{1}{\sqrt{2\pi}N} \left(m\left(-4\pi\lambda_{0}^{2}\lambda_{0}^{2}-1\frac{\lambda_{0}^{2}}{4\lambda_{0}^{2}}\right)^{-1}\left(-\frac{\lambda_{0}^{2}}{2\lambda_{0}^{2}}\right)^{-1}\frac{1}{4\lambda_{0}^{2}}\left(-\frac{\lambda_{0}^{2}}{4\lambda_{0}^{2}}\right)^{-1}\left(-\frac{\lambda_{0}^{2}}{2\lambda_{0}^{2}}\right)^{-1}\left(-\frac{\lambda_{0}^{2}}{2\lambda_{0}^{2}}\right)^{-1}\left(-\frac{\lambda_{0}^{2}}{2\lambda_{0}^{2}}\right)^{-1}\frac{1}{4\lambda_{0}^{2}}\left(-\frac{\lambda_{0}^{2}}{4\lambda_{0}^{2}}\right)^{-1}\left(-\frac{\lambda_{0}^{$	<text><text><equation-block><text><text><text><text><text><equation-block><equation-block><text></text></equation-block></equation-block></text></text></text></text></text></equation-block></text></text>	<text><text><text><text><text><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></text></text></text></text></text>	$\begin{split} & \log \left(23 \right) & \approx d \left(23 \right) & \sin $	point contribution functions at their topologic: $\begin{split} & \varphi_{n}(t) = \frac{1}{\sqrt{2} + \sqrt{2} \sqrt{2}}, \\ & \varphi_{n}(t) = \frac{1}{\sqrt{2} + \sqrt{2} + \sqrt{2} \sqrt{2}}, \\ & \varphi_{n}(t) = \frac{1}{\sqrt{2} + \sqrt{2} +$
$ \begin{aligned} & \text{Contrary (17), (13), (13) and (13) yields the limit and list $\mathcal{L}_{q,c}$, \\ & \mathcal{L}_{q,c} = \frac{1}{\sqrt{2\pi r_{q}^{2} - 1}} \left(\frac{(-1/2) \sqrt{2\pi r_{q}^{2} - 1}}{2\pi r_{q}^{2} - 1} \sqrt{2\pi r_{q}^{2} - 1} \frac{1}{2\pi r_{q}^{2} - 1} \sqrt{2\pi r_{q}^{2} - 1} \frac{1}{2\pi r_{$	<text><text><equation-block><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></equation-block></text></text>	<text><text><text><text><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></text></text></text></text>	$\begin{split} & \log \left(23 \right) & \approx 12.39 , (\operatorname{Meh} \ u \ \operatorname{spin} \ s$	point available functions at least lengths: $\begin{aligned} & \mathcal{L}_{0}(\boldsymbol{u}) = \frac{\partial \mathcal{U}}{\partial \boldsymbol{u}} & \frac{\partial \mathcal{U}}{\partial \boldsymbol{u}} $
Catalan (17), 130, (130) well (37) yields the line atom for $\mathcal{L}_{q,c}$: $\begin{aligned} & \mathcal{L}_{q,c} = \frac{2\pi g^2 m}{2\pi g^2} \left(m \left(-k_0 \mathcal{M}_{q,c}^{-1} - \frac{k_0^2 m}{2m_0^2} - \frac{k_0^2 m}{2m$	<text><text><text><equation-block><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></equation-block></text></text></text>	<text><text><text><text><text><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></text></text></text></text></text>	$\begin{split} & \log \left(2.8 \right) & \approx 1.2 \text{ M}_1 \left(4.8 \text{ Les}_1 + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right$	per transfer large t
Catana (17), 130, (13) with (13) yields the data matter k_{w}): $ \begin{aligned} & $	<text><text><text><equation-block><text><text><text><text><text><text><equation-block><equation-block><equation-block><text></text></equation-block></equation-block></equation-block></text></text></text></text></text></text></equation-block></text></text></text>	<text><text><text><text><text><text><text><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></text></text></text></text></text></text></text>	$ \begin{array}{l} \log \left(23 \right) \mbox{ of } 23 \right) \mbox{ of } 23 \\ \mbox{ of } \left(-\frac{1}{23 \sqrt{2} \sqrt{2}} \left(-\frac{1}{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} $	<text><text><text><text><text><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></text></text></text></text></text>
Cataca (17), 13(1), (13) with (13) which the matter the catacata (13) with	<text><text><text><equation-block><text><text><text><text><text><text><equation-block><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></equation-block></text></text></text></text></text></text></equation-block></text></text></text>	<text><text><text><equation-block><text><text><text><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></text></text></text></equation-block></text></text></text>	$\begin{split} \log \left(23 \right) & \approx 12.18 \right) (Abbits an equivalence on 10^{-1}(5^{-1}/5^{-1}) & \qquad $	<text><text><text><text><equation-block><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></equation-block></text></text></text></text>
<equation-block><equation-block><equation-block><equation-block><text><text><text><text><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></text></text></text></text></equation-block></equation-block></equation-block></equation-block>	<text><text><equation-block><equation-block><text><text><text><text><text><text><equation-block><equation-block><equation-block><text></text></equation-block></equation-block></equation-block></text></text></text></text></text></text></equation-block></equation-block></text></text>	<text><text><text><equation-block><text><text><text><text><text><text><text><text><text><text><text><text></text></text></text></text></text></text></text></text></text></text></text></text></equation-block></text></text></text>	$\begin{split} \log \left(3 \right) & \operatorname{der} \left(3 \right) & \operatorname{der} \left(3 \right) & \operatorname{der} \left(- \frac{3 }{3 \left(3 \right)} + \frac{1}{3 \left(3 \right)} + 1$	<equation-block><equation-block><equation-block><equation-block><equation-block><equation-block><text><text><equation-block><equation-block><equation-block></equation-block></equation-block></equation-block></text></text></equation-block></equation-block></equation-block></equation-block></equation-block></equation-block>

$Z_{Q,V}$ (Z at fixed Q and finite V) (5)

• Result for $Z_{Q,V}$ up to $1/V^3$:

$$\begin{split} Z_{Q,V} &= \\ &= \frac{1}{\sqrt{2\pi\mathcal{E}_2 V}} \Big(\exp\left(-\mathcal{E}_0(0,V_s)T - \frac{1}{\mathcal{E}_2 V}\frac{1}{2}Q^2 - \frac{1}{(\mathcal{E}_2 V)^3}\frac{\mathcal{E}_4}{24\mathcal{E}_2}Q^4 \right) \\ &\quad \left(1 - \frac{1}{(\mathcal{E}_2 V)^2}\frac{\mathcal{E}_4}{2\mathcal{E}_2}Q^2 \right)^{-1/2} \\ &\quad \left(1 - \frac{1}{\mathcal{E}_2 V}\frac{\mathcal{E}_4}{8\mathcal{E}_2} + \frac{1}{(\mathcal{E}_2 V)^2} \left(-\frac{\mathcal{E}_6}{48\mathcal{E}_2} + \frac{35\mathcal{E}_4^2}{384\mathcal{E}_2^2}\right) \right. \\ &\quad + \frac{1}{(\mathcal{E}_2 V)^3} \left(-\frac{\mathcal{E}_8}{384\mathcal{E}_2} + \frac{7\mathcal{E}_4\mathcal{E}_6}{256\mathcal{E}_2^2} - \frac{385\mathcal{E}_4^3}{3072\mathcal{E}_2^3} + \left(\frac{\mathcal{E}_6}{16\mathcal{E}_2} - \frac{\mathcal{E}_4^2}{3\mathcal{E}_2^2}\right)Q^2\right) \\ &\quad + \mathcal{O}\left(\frac{1}{\mathcal{E}_2^4 V^4}, \frac{1}{\mathcal{E}_2^4 V^4}Q^2, \frac{1}{\mathcal{E}_2^4 V^4}Q^4\right) \Big). \end{split}$$

- Parameters also present at unfixed topology: $E_0(0, V_s)$.
- New "fixed topology parameters": $\mathcal{E}_2 = \chi_t$, \mathcal{E}_4 , \mathcal{E}_6 .

$$C_{Q,V}(t)$$
 ($C(t)$ at fixed Q and finite V) (1)

• In the following: expansion of

$$C_{Q,V}(t) \equiv \frac{1}{Z_{Q,V}} \int DA \, D\psi \, D\bar{\psi} \, \delta_{Q,Q[A]} O^{\dagger}(t) O(0) e^{-S_E[A,\bar{\psi},\psi]},$$

the two-point correlation function at fixed topological charge Q and spacetime volume V, in powers of 1/V.

• O is a suitably normalized hadron creation operator,

$$O \equiv \frac{1}{\sqrt{V_s}} \int d^3 r \, O'(\mathbf{r}) \quad \Big(\text{e.g. } O \equiv \frac{1}{\sqrt{V_s}} \int d^3 r \, \bar{d}(\mathbf{r}) \gamma_5 u(\mathbf{r}) \text{ for the pion} \Big),$$

where $O'(\mathbf{r})$ is a local operator (parity P is not a symmetry at $\theta \neq 0$).

• Then

$$\alpha(\theta) \equiv \left| \langle H; \theta | O | 0; \theta \rangle \right|^2 = \sum_{k=0}^{\infty} \frac{\alpha^{(2k)}(0)\theta^{2k}}{(2k)!} = \alpha(0) \exp\left(\sum_{k=1}^{\infty} \frac{\beta^{(2k)}(0)\theta^{2k}}{(2k)!}\right)$$

$C_{Q,V}(t)$ (C(t) at fixed Q and finite V) (2)

• Result for $C_{Q,V}(t)$ up to $1/V^3$:

$$\begin{split} C_{Q,V}(t) &= \frac{\alpha(0)}{\sqrt{1+x_2/\mathcal{E}_2 V}} \exp\left(-M_H(0)t - \frac{1}{\mathcal{E}_2 V} \left(\frac{1}{1+x_2/\mathcal{E}_2 V} - 1\right) \frac{1}{2} Q^2 - \frac{1}{(\mathcal{E}_2 V)^3} \frac{\mathcal{E}_4}{24\mathcal{E}_2} \left(\frac{1+x_4/\mathcal{E}_4 V}{(1+x_2/\mathcal{E}_2 V)^4} - 1\right) Q^4\right) \\ &\left(1 - \frac{1}{(\mathcal{E}_2 V)^2} \frac{\mathcal{E}_4}{2\mathcal{E}_2} Q^2\right)^{+1/2} \left(1 - \frac{1}{(\mathcal{E}_2 V)^2} \frac{\mathcal{E}_4(1+x_4/\mathcal{E}_4 V)}{2\mathcal{E}_2(1+x_2/\mathcal{E}_2 V)^3} Q^2\right)^{-1/2} \frac{G_C}{G} + \mathcal{O}\left(\frac{1}{(\mathcal{E}_2 V)^4} Q^4\right) \\ G_C &= 1 - \frac{1}{\mathcal{E}_2 V} \frac{\mathcal{E}_4(1+x_4/\mathcal{E}_4 V)}{8\mathcal{E}_2(1+x_2/\mathcal{E}_2 V)^2} + \frac{1}{(\mathcal{E}_2 V)^2} \left(-\frac{\mathcal{E}_6(1+x_6/\mathcal{E}_6 V)}{48\mathcal{E}_2(1+x_2/\mathcal{E}_2 V)^3} + \frac{35\mathcal{E}_4^2(1+x_4/\mathcal{E}_4 V)^2}{384\mathcal{E}_2^2(1+x_2/\mathcal{E}_2 V)^4}\right) \\ &+ \frac{1}{(\mathcal{E}_2 V)^3} \left(-\frac{\mathcal{E}_8(1+x_8/\mathcal{E}_8 V)}{384\mathcal{E}_2(1+x_2/\mathcal{E}_2 V)^4} + \frac{7\mathcal{E}_4(1+x_4/\mathcal{E}_4 V)\mathcal{E}_6(1+x_6/\mathcal{E}_6 V)}{25\mathcal{E}_2^2(1+x_2/\mathcal{E}_2 V)^5} - \frac{385\mathcal{E}_4^3(1+x_4/\mathcal{E}_4 V)^3}{3072\mathcal{E}_3^3(1+x_2/\mathcal{E}_2 V)^6} \right) \\ &+ \left(\frac{\mathcal{E}_6(1+x_6/\mathcal{E}_6 V)}{16\mathcal{E}_2(1+x_2/\mathcal{E}_2 V)^4} - \frac{\mathcal{E}_4^2(1+x_4/\mathcal{E}_4 V)^2}{3\mathcal{E}_2^2(1+x_2/\mathcal{E}_2 V)^5}\right) Q^2\right) + \mathcal{O}\left(\frac{1}{(\mathcal{E}_2 V)^4} , \frac{1}{(\mathcal{E}_2 V)^4} Q^2\right) \\ G &= 1 - \frac{1}{\mathcal{E}_2 V} \frac{\mathcal{E}_4}{8\mathcal{E}_2} + \frac{1}{(\mathcal{E}_2 V)^2} \left(-\frac{\mathcal{E}_6}{48\mathcal{E}_2} + \frac{35\mathcal{E}_4^2}{384\mathcal{E}_2^2}\right) \\ &+ \frac{1}{(\mathcal{E}_2 V)^3} \left(-\frac{\mathcal{E}_8}{384\mathcal{E}_2} + \frac{7\mathcal{E}_4 \mathcal{E}_6}{256\mathcal{E}_2^2} - \frac{385\mathcal{E}_4^3}{3072\mathcal{E}_3^3} + \left(\frac{\mathcal{E}_6}{16\mathcal{E}_2} - \frac{\mathcal{E}_4^2}{3\mathcal{E}_2^2}\right) Q^2\right) + \mathcal{O}\left(\frac{1}{(\mathcal{E}_2 V)^4} , \frac{1}{(\mathcal{E}_2 V)^4} Q^2\right). \end{split}$$

- Parameters also present at unfixed topology: $M_H(0)$, $\alpha(0)$.
- New "fixed topology parameters": \mathcal{E}_n ($\mathcal{E}_2 = \chi_t$), $x_n \equiv M_H^{(n)}(0)t + \beta^{(n)}(0)$, n = 2, 4, 6, 8 (i.e. 12 parameters).

$$C_{Q,V}(t)$$
 ($C(t)$ at fixed Q and finite $V\)$ (3)

• For some applications it might be of interest to have $C_{Q,V}(t)$ up to $1/V^3$ in the form

 $C_{Q,V}(t) = \text{const} \times \exp\left(-M_H(0)t + \text{fixed topology corrections as a power series in } 1/\mathcal{E}_2 V\right),$

which can be obtained by straightforward expansions in 1/V:

$$\begin{split} C_{Q,V}(t) &= \alpha(0) \exp\left(-M_H(0)t - \frac{1}{\mathcal{E}_2 V} \frac{x_2}{2} - \frac{1}{(\mathcal{E}_2 V)^2} \left(\frac{x_4 - 2(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{8} - \frac{x_2}{2} Q^2\right) \\ &- \frac{1}{(\mathcal{E}_2 V)^3} \left(\frac{16(\mathcal{E}_4/\mathcal{E}_2)^2 x_2 + x_6 - 3(\mathcal{E}_6/\mathcal{E}_2)x_2 - 8(\mathcal{E}_4/\mathcal{E}_2)x_4 - 12x_2x_4 + 18(\mathcal{E}_4/\mathcal{E}_2)x_2^2 + 8x_2^3}{48} - \frac{x_4 - 3(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{4} Q^2\right) \\ &+ \mathcal{O}\left(\frac{1}{(\mathcal{E}_2 V)^4} , \frac{1}{(\mathcal{E}_2 V)^4} Q^2 , \frac{1}{(\mathcal{E}_2 V)^4} Q^4\right). \end{split}$$

- Parameters also present at unfixed topology: $M_H(0)$, $\alpha(0)$.
- New "fixed topology parameters": \mathcal{E}_n ($\mathcal{E}_2 = \chi_t$), $x_n \equiv M_H^{(n)}(0)t + \beta^{(n)}(0)$, n = 2, 4, 6 (i.e. 9 parameters).

$$C_{Q,V}(t)$$
 ($C(t)$ at fixed Q and finite $V\)$ (4)

• To reduce the number of parameters, one can also consider lower orders in 1/V, e.g. $C_{Q,{\it V}}(t)$ up to 1/V,

$$\begin{split} C_{Q,V}(t) &= \frac{\alpha(0)}{\sqrt{1+x_2/\mathcal{E}_2 V}} \exp\left(-M_H(0)t - \frac{1}{\mathcal{E}_2 V} \left(\frac{1}{1+x_2/\mathcal{E}_2 V} - 1\right) \frac{1}{2} Q^2\right) \frac{G_{\mathcal{C}}}{G} \\ G_{\mathcal{C}} &= 1 - \frac{1}{\mathcal{E}_2 V} \frac{\mathcal{E}_4(1+x_4/\mathcal{E}_4 V)}{8\mathcal{E}_2(1+x_2/\mathcal{E}_2 V)^2} \quad , \quad G &= 1 - \frac{1}{\mathcal{E}_2 V} \frac{\mathcal{E}_4}{8\mathcal{E}_2}. \end{split}$$

- Parameters: $M_H(0)$, $\alpha(0)$, \mathcal{E}_n ($\mathcal{E}_2 = \chi_t$), $x_n \equiv M_H^{(n)}(0)t + \beta^{(n)}(0)$, n = 2, 4.
- Another strategy is to set certain parameters to zero (i.e. to just ignore the corresponding fixed topology corrections), e.g.

$$C_{Q,V}(t) = \frac{\alpha(0)}{\sqrt{1 + x_2/\mathcal{E}_2 V}} \exp\left(-M_H(0)t - \frac{1}{\mathcal{E}_2 V} \left(\frac{1}{1 + x_2/\mathcal{E}_2 V} - 1\right) \frac{1}{2} Q^2\right),$$

which is the $1/V^3$ result with $x_2 \equiv M_H^{(2)}(0)t$ and $\mathcal{E}_n = 0$, $M_H^{(n)}(0) = 0$, $n = 4, 6, 8, \beta^{(n)}(0) = 0$, n = 2, 4, 6, 8.

- Parameters: $M_H(0)$, $\alpha(0)$, $\mathcal{E}_2 = \chi_t$, $M_H^{(2)}(0)$.

Validity of the expansion of $C_{Q,V}(t)$

- The presented expansions are good approximations, if the following four conditions are fulfilled:
- (C1) $1/\mathcal{E}_2 V \ll 1$, $|Q|/\mathcal{E}_2 V \ll 1$. The volume V must be large, the topological charge Q may not be too large.
- (C2) $|x_2| = |M_H^{(2)}(0)t + \beta^{(2)}(0)| \lesssim 1.$ The temporal separation may not be too large.
- (C3) $m_{\pi}(\theta)L \gtrsim 3...5 \gg 1$
 - No ordinary finite size effects.
- (C4) $(M_H^*(\theta) M_H(\theta))t \gg 1$, $M_H(\theta)(T 2t) \gg 1$. No contamination from excited states or particles propagating backwards in time.

Part 2

Quantum mechanics on a circle, the Schwinger model and SU(2) Yang-Mills theory at fixed topology

QM on a circle

• Lagrangian parameterized by the angle φ :

$$L \equiv \frac{mr^2}{2}\dot{\varphi}^2 - U(\varphi) = \frac{I}{2}\dot{\varphi}^2 - U(\varphi),$$

(*m*: mass; *r*: radius; $I \equiv mr^2$: moment of inertia; $U(\varphi)$: potential).

• A periodic time with extension T implies $\varphi(t+T) = \varphi(t) + 2\pi Q$, $Q \in \mathbb{Z}$, and gives rise to topological charge

$$\frac{1}{2\pi} \int_0^T dt \,\dot{\varphi} = \frac{1}{2\pi} (\varphi(T) - \varphi(0)) = Q$$

• QM on a circle at fixed topology;

$$Z_{Q,T} \equiv \int D\varphi \, \delta_{Q,Q(\varphi)} e^{-S_E[\varphi]}.$$



QM on a circle, free particle

- Can be solved analytically, also at fixed topology (hadron creation operator $O \equiv \sin(\varphi)$, hadron mass $M_H(\theta) \equiv E_1(\theta) E_0(\theta)$).
- The expansions of $C_{Q,V}(t)$ from part 1 disagree with the analytical results.
- Reason: either assumption

 $E_n(+\theta) = E_n(-\theta)$

or condition

(C2) $|M_H^{(2)}(0)t + \beta^{(2)}(0)| \lesssim 1$

not fulfilled (a particularity of the free case).

• One can derive the expansions of $C_{Q,V}(t)$ from part 1 for the more general case $E_n(+\theta) \neq E_n(-\theta)$; then there is perfect agreement.

[A. Dromard and M. Wagner, arXiv:1404.0247 [hep-lat]]



QM on a circle, square well potential (1)

• Square well potential:

$$U(\varphi) \equiv \left\{ \begin{array}{ll} 0 & \mbox{if} \ -\rho/2 < \varphi < +\rho/2 \\ U_0 & \mbox{otherwise} \end{array} \right.$$

 Can be solved numerically up to arbitrary precision (no simulations required), also at fixed topology
 → ideal to test the expansions of C_{Q,V}(t) from part 1.

•

• In the following: $\hat{U}_0 = U_0 I = 5.0$, $\rho = 0.9 \times 2\pi$.

n	$\hat{\mathcal{E}}_n$	$\hat{M}_{H}^{(n)}(0)$	$lpha^{(n)}(0)$	$eta^{(n)}(0)$	3.5 energy levels
0	+0.11708	+0.40714	+0.50419		3.0
2	+0.00645	-0.03838	-0.00357	+0.00709	2.0 n=-2 n=+2
4	-0.00497	+0.04983	+0.00328	-0.00636	1.5
6	+0.00042	-0.13191	-0.04721	+0.09308	0.5
8	+0.00834	+0.95631	+0.91037	-1.77931	-3 -2 -1 1 2 3 1

QM on a circle, square well potential (2)

• Effective masses

$$\hat{M}_{Q,\hat{T}}^{\text{eff}}(\hat{t}) \equiv -\frac{d}{d\hat{t}} \ln\left(C_{Q,\hat{T}}(\hat{t})\right)$$

for different topological sectors Q and $\hat{T} = T/I = 6.0/\hat{\mathcal{E}}_2 \approx 930.2$.

- At small temporal separations $\hat{M}_{Q,\hat{T}}^{\mathrm{eff}}(\hat{t})$ quite large and strongly decreasing, due to the presence of excited states.
- At large temporal separations severe deviations from a constant behavior.



QM on a circle, square well potential (3)

- Effective masses $\hat{M}_{Q,\hat{T}}^{\text{eff}}$ derived from the 1/V expansions of two-point correlation functions for different topological sectors Q and $\hat{T} = 6.0/\hat{\mathcal{E}}_2 \approx 930.2.$
 - $\mathcal{O}(1/V)$: 7 parameters.
 - $\mathcal{O}(1/V^2)$: 10 parameters.
 - $\mathcal{O}(1/V^3)$: 13 parameters.





QM on a circle, square well potential (4)

• Similar as before ... this time using expansions of the form

 $C_{Q,V}(t) = \text{const} \times \exp(-M_H(0)t + \text{fixed topology corrections}).$

- $\mathcal{O}(1/V)$: 4 parameters.
- $\mathcal{O}(1/V^2)$: 7 parameters.
- $\mathcal{O}(1/V^3)$: 10 parameters.





QM on a circle, square well potential (5)

- Similar as before ... this time using only three parameters ($M_H(0)$, $\mathcal{E}_2 = \chi_t$, $M_H^{(2)}(0)$), all other parameters set to zero (seems to be a good compromise).
 - (3.18) \rightarrow expansion as sketched in part 1.
 - (3.20) \rightarrow const $\times \exp(-M_H(0)t + \text{fixed topology corrections}).$
 - hep-lat/0302005 \rightarrow
 - [R. Brower, S. Chandrasekharan, J. W. Negele and U. J. Wiese, Phys. Lett. B 560, 64 (2003) [hep-lat/0302005]]



QM on a circle, square well potential (6)

- Determination of physical hadron masses (hadron masses at unfixed topology) from fixed topology simulations based on the 1/V expansions from part 1:
 - 1. Perform simulations at fixed topology for different topological charges Q and spacetime volumes V. Determine "fixed topology hadron masses" defined e.g. via

$$M_{Q,\mathbf{V}} \equiv M_{Q,\mathbf{V}}^{\text{eff}}(t_M) = -\frac{d}{dt} \ln \left(C_{Q,\mathbf{V}}(t) \right) \Big|_{t=t_M}$$

(t_M should be chosen such that the expansions used in step 2 are good approximations).

2. Determine $M_H(0)$ (the hadron mass at unfixed topology), $\mathcal{E}_2 = \chi_t$, $M_H^{(2)}(0)$, ... by fitting an effective mass expression derived e.g. from $C_{Q,V}(t_M) = \frac{\alpha(0)}{\sqrt{1+x_2/\mathcal{E}_2 V}} \exp\left(-M_H(0)t_M - \frac{1}{\mathcal{E}_2 V}\left(\frac{1}{1+x_2/\mathcal{E}_2 V} - 1\right)\frac{1}{2}Q^2\right)$, $(x_2 \equiv M_H^{(2)}(0)t_M)$ to $M_{Q,V}$ obtained in step 1.

QM on a circle, square well potential (7)

- Mimic the method to determine a physical hadron mass (at unfixed topology) from fixed topology computations:
 - 1. Use the exact result for the effective mass to generate $\hat{M}_{Q,\hat{T}}$ values (at $\hat{t}_M = 20.0$) for several topological charges Q = 0, 1, 2, 3, 4 and temporal extensions $\hat{T} = 2.0/\hat{\mathcal{E}}_2, 3.0/\hat{\mathcal{E}}_2, \ldots, 10.0/\hat{\mathcal{E}}_2$.
 - 2. Perform a single fit as explained on the previous slide (only those masses $\hat{M}_{Q,\hat{T}}$ enter the fit, for which the conditions **(C1)** and **(C2)** are fulfilled).



QM on a circle, square well potential (8)

- Alternatively one could perform the fit directly on the "correlator level":
 - 1. Perform simulations at fixed topology for different topological charges Q and spacetime volumes V. Determine $C_{Q,V}(t)$ for each simulation.
 - 2. Determine the physical hadron mass $M_H(0)$ by performing a single χ^2 minimizing fit of one of the 1/V expansions of $C_{Q,V}(t)$ to the numerical results obtained in step 1. This input from step 1 is limited to those Q, V and t values, for which the conditions **(C1)**, **(C2)** and **(C4)** are fulfilled.

QM on a circle, square well potential (9)

• $\hat{M}_H(0)$ from fixed topology computations (exact result: $M_H = 0.40714$):

		fitting to .	$\hat{M}_{Q,\hat{T}}$	fitting to correlators		
	expansion	$\hat{M}_H(0)$ result	rel. error	$\hat{M}_H(0)$ result	rel. error	
$\frac{1}{\chi_t V}, \frac{ Q }{\chi_t V} \le 0.5$	hep-lat/0302005	0.40733	0.047%	0.40702	0.029%	
	$1/V^3$, 3 param.	0.40708	0.014%	0.40706	0.019%	
$\frac{1}{\chi_t V}, \frac{ Q }{\chi_t V} \le 0.3$	hep-lat/0302005	0.40739	0.062%	0.40732	0.044%	
	$1/V^3$, 3 param.	0.40695	0.046%	0.40713	0.002%	

• $\hat{\chi}_t$ from fixed topology computations (exact result: $\hat{\chi}_t = 0.00645$):

		fitting to $\hat{M}_{Q,\hat{T}}$		fitting to correlators	
	expansion	$\hat{\chi}_t$ result	rel. error	$\hat{\chi}_t$ result	rel. error
$\frac{1}{\chi_t V}, \frac{ Q }{\chi_t V} \le 0.5$	hep-lat/0302005	0.00586	9.1%	0.00629	2.5%
	$1/V^3$, 3 param.	0.00631	2.2%	0.00633	1.9%
$\frac{1}{\chi_t V}, \frac{ Q }{\chi_t V} \le 0.3$	hep-lat/0302005	0.00590	8.5%	0.00627	2.8%
	$1/V^3$, 3 param.	0.00592	8.2%	0.00630	2.3%

- Expansions give rather accurate results for $\hat{M}_H(0)$ (relative errors are below 0.1%) and reasonable results for $\hat{\chi}_t$ (relative errors of a few percent).
- Smaller relative errors for both $\hat{M}_{H}(0)$ and $\hat{\chi}_{t}$, when using " $1/V^{3}$, 3 param.".

Schwinger model (1)

• Schwinger model (2D Euclidean quantum electrodynamics):

$$\mathcal{L}(\psi, \bar{\psi}, A) = \sum_{f=1}^{N_f} \bar{\psi}^{(f)} (\gamma_\mu (\partial_\mu + igA_\mu) + m) \psi^{(f)} + \frac{1}{4} F_{\mu\nu} F_{\mu\nu}.$$

- Several similarities to QCD:
 - Topological charge:

$$Q = \frac{1}{\pi} \int \mathsf{d}^2 x \,\epsilon_{\mu\nu} F_{\mu\nu}.$$

- For $N_f = 2$ there is a rather light iso-triplet (pions).

- Fermion confinement.
- First determination of physical hadron masses (pion mass) from fixed topology simulations by Bietenholz et. al.
 - [W. Bietenholz, I. Hip, S. Shcheredin and J. Volkholz, Eur. Phys. J. C 72, 1938 (2012) [arXiv:1109.2649 [hep-lat]]]

Schwinger model (2)

- Schwinger model on the lattice:
 - Periodic spacetime lattice with N_L^2 lattice sites (extension of $L = N_L a$, spacetime volume $V = L^2$).
 - $-N_f = 2$ flavors of Wilson fermions and the Wilson plaquette gauge action.
 - Dimensionful quantities are expressed in units of a, e.g. $\hat{g} = ga$ and $\hat{m} = ma$ (it is common to also use the dimensionless inverse squared coupling constant $\beta = 1/\hat{g}^2$).
 - Approach the continuum limit by increasing N_L , while keeping the dimensionless ratios $gL = \hat{g}N_L$ and $M_{\pi}L = \hat{M}_{\pi}N_L$ fixed (M_{π} : pion mass); this requires to decrease both \hat{g} and \hat{M}_{π} proportional to $1/N_L$ (for the latter \hat{m} has to be adjusted appropriately).

Schwinger model (3)

- Schwinger model on the lattice:
 - Geometric definition of topological charge on the lattice,

$$Q = \frac{1}{2\pi} \sum_{P} \phi(P),$$

where \sum_{P} denotes the sum over all plaquettes $P = e^{i\phi(P)}$ with $-\pi < \phi(P) \le +\pi$ (with this definition $Q \in \mathbb{Z}$).

- Simulations at various values of β , \hat{m} and N_L using a Hybrid Monte Carlo (HMC) algorithm with multiple timescale integration and mass preconditioning.

[https://github.com/urbach/schwinger]

- Figure: probability for a transition to another topological sector per HMC trajectory, plotted versus $\hat{g} = 1/\sqrt{\beta}$ (proportional to a) and $\hat{m}/\hat{g} = \hat{m}\sqrt{\beta}$ (proportional to \hat{m}/a), while $gL = \hat{g}N_L = N_L/\sqrt{\beta} = 24/\sqrt{5} = \text{constant.}$



Schwinger model (4)

- As for "QM on a circle" determine physical hadron masses from fixed topology computations.
 - Pion mass and static potential, hadron creation operators

$$O_{\pi} = \sum_{x} \bar{\psi}^{(u)}(x) \gamma_{1} \psi^{(d)}(x) \quad , \quad O_{\bar{q}q} = \bar{q}(x_{1}) U(x_{1}, x_{2}) q(x_{2}).$$



Schwinger model (5)

- Hadron masses (the pion mass, the static potential) can be determined rather precisely (uncertainty $\ll 1\%$).
- There is a rather large error associated with the topological susceptibility (uncertainty up to $\approx 20\%$).

observable	β	\hat{m}	\hat{M} (fixed top.)	\hat{M} (conv.)	$\hat{\chi}_t$ (fixed top.)	$\hat{\chi}_t \; (\langle Q^2 angle / \hat{V})$
M_{π}			0.2659(3)	0.2663(3)	0.00292(54)	
$V_{Q\bar{Q}}(1)$	3.0	-0.07	0.1708(1)	0.17108(5)	0.0051(13)	0.00454(6)
$V_{Q\bar{Q}}(2)$			0.2914(3)	0.2927(2)	0.00247(20)	
M_{π}			0.2743(6)	0.2743(3)	0.00228(39)	
$V_{O\bar{O}}(1)$	4.0	-0.03	0.12552(7)	0.12551(4)	0.00313(26)	0.00353(14)
$V_{Q\bar{Q}}(2)$			0.2250(2)	0.2247(2)	0.00329(15)	

SU(2) Yang-Mills theory

• SU(2) Yang-Mills theory:

$$\mathcal{L}(A_{\mu}) \equiv \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}.$$

- Left plot: there is a significant discrepancy between the static potential from computations restricted to a single topological sector and corresponding results obtained at unfixed topology.
- Right plot: comparison of the static potential obtained from Wilson loops at fixed topology and from standard lattice simulations.



Conclusions, outlook

• The presented equations and techniques might be a starting point/might help to overcome the problem of topology freezing in QCD (present for overlap quarks and at small values of the lattice spacing).

[A. Dromard and M. Wagner, PoS LATTICE 2013, 339 (2013) [arXiv:1309.2483 [hep-lat]]]

[C. Czaban and M. Wagner, PoS LATTICE 2013, 465 (2013) [arXiv:1310.5258 [hep-lat]]]

[A. Dromard and M. Wagner, arXiv:1404.0247 [hep-lat]]

• Future plans are mainly focused on first tests in QCD.