Lattice investigation of an inhomogeneous phase of the $2+1$-dimensional Gross-Neveu model in the limit of infinitely many flavors

Marc Winstel, Jonas Stoll and Marc Wagner
Institut für Theoretische Physik, Goethe Universität Frankfurt am Main, 60438 Frankfurt am Main, Germany

E-mail: winstel@itp.uni-frankfurt.de, stoll@itp.uni-frankfurt.de, mwagner@th.physik.uni-frankfurt.de

Abstract. We investigate the phase structure of the $2+1$ dimensional Gross-Neveu model in the large-N_f limit, where N_f denotes the number of fermion flavors. We discuss two different fermion representations and their implication on the interpretation of a discrete symmetry of the action. We present numerical results, which indicate the existence of an inhomogeneous phase similar as in the 1+1-dimensional Gross-Neveu model.

1. Introduction, $2+1$-dimensional Gross-Neveu model in the large-N_f limit

The calculation of the full phase diagram of QCD, either analytically or numerically, is a very challenging and still unsolved problem. In particular lattice QCD is at the moment restricted to small chemical potential, because of the sign problem (see e.g [1] and references therein). Therefore, it is of interest to study the phase structure of simpler quantum field theories, which share some features with QCD and, thus, might serve as crude models or even effective theories for QCD. A common and very simple example is the Gross-Neveu (GN) model in $1+1$ spacetime dimensions [2], which describes N_f fermion flavors and exhibits a discrete chiral symmetry, which can be spontaneously broken. In the large-N_f limit, i.e. $\lim_{N_f\to\infty}$, an inhomogeneous phase was found, i.e. a phase where the chiral order parameter is a periodic function of the spatial coordinate [3, 4]. Recently, numerical evidence was presented, that a similar inhomogeneous phase also exists at finite N_f [5].

As an intermediate step towards $3+1$ spacetime dimensions we consider in this work the GN model in $2+1$ dimensions in the large-N_f limit. The "homogeneous phase diagram", where the chiral condensate is assumed to be constant, was calculated in ref. [6]. In ref. [7] a particular stripe-like 1-dimensional modulation of the condensate has been investigated, but it turned out that it is not favored with respect to a constant. Here we explore the phase structure numerically using lattice field theory with particular focus on the possible existence of an inhomogeneous phase. The Euclidean action and the partition function are

$$S[\bar{\psi},\psi] = \int d^3x \left(\sum_{n=1}^{N_f} \bar{\psi}_n\Big(\gamma_\nu\partial_\nu + \gamma_0\mu\Big)\psi_n - \frac{g^2}{2}\Big(\sum_{n=1}^{N_f}\bar{\psi}_n\psi_n\Big)^2\right) \quad , \quad Z = \int D\bar{\psi}\, D\psi\, e^{-S[\bar{\psi},\psi]}. \tag{1.1}$$

ψ_n denotes a fermionic field with flavor index n, μ is the chemical potential and g is the coupling constant. Representations for the spin matrices γ_0, γ_1 and γ_2 are discussed in section 2. Introducing a real scalar field σ and performing the integration over the Grassmann-valued fermionic fields one obtains the equivalent effective action and partition function

$$S_{\text{eff}}[\sigma] = N_f \left(\frac{1}{2\lambda} \int d^3x \, \sigma^2 - \ln \Big(\det(Q) \Big) \right) \quad , \quad Z = \int D\sigma \, e^{-S_{\text{eff}}[\sigma]}, \tag{1.2}$$

where $\lambda = N_f g^2$ and $Q = \gamma_\nu \partial_\nu + \gamma_0 \mu + \sigma$. In this work we restrict the dependence of σ to the spatial coordinates, i.e. either $\sigma = \sigma(x)$ (see section 3.2) or $\sigma = \sigma(x, y)$ (see section 3.3). Then one can show that $S_{\text{eff}}[\sigma]$ is real, which is essential for numerical calculations. Moreover, because of the factor $N_f \to \infty$ on the right hand side of $S_{\text{eff}}[\sigma]$, only field configurations σ corresponding to a global minimum of $S_{\text{eff}}[\sigma]$ contribute to the partition function Z.

2. Discrete symmetry $\sigma \to -\sigma$ and fermion representations

One can show that the effective action in eq. (1.2) has a discrete symmetry $\sigma \to -\sigma$, i.e. $S_{\text{eff}}[\sigma] = S_{\text{eff}}[-\sigma]$. Moreover, $\sigma \propto \langle \sum_{n=1}^{N_f} \bar{\psi}_n \psi_n \rangle$.

In $1+1$ spacetime dimensions a possible irreducible 2×2 representation for the γ matrices is

$$\gamma_0 = \sigma_1 \quad , \quad \gamma_1 = \sigma_2, \tag{2.1}$$

where σ_j denote the Pauli matrices. A non-vanishing σ would indicate spontaneous breaking of the symmetry

$$\psi_n \to \sigma_3 \psi_n. \tag{2.2}$$

Since σ_3 anticommutes with γ_0 and γ_1, it is appropriate to define $\gamma_5 = \sigma_3$ and to interpret the symmetry (2.2) as discrete chiral symmetry.

A possible irreducible 2×2 representation for the γ matrices in $2+1$ spacetime dimensions is

$$\gamma_0 = \sigma_1 \quad , \quad \gamma_1 = \sigma_2 \quad , \quad \gamma_2 = \sigma_3. \tag{2.3}$$

One can show that it is impossible to find a corresponding appropriate γ_5 matrix, i.e. a matrix, which anticommutes with γ_0, γ_1 and γ_2. Consequently, a non-vanishing σ cannot be interpreted as a signal for chiral symmetry breaking. A possibility to retain the interpretation of σ as chiral order parameter is to use a reducible 4×4 representation for the γ matrices,

$$\gamma_0 = \begin{pmatrix} +\sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} \quad , \quad \gamma_1 = \begin{pmatrix} +\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \quad , \quad \gamma_2 = \begin{pmatrix} +\sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}. \tag{2.4}$$

For a detailed discussion of fermion representations in $2+1$ dimensions and its implicatons we refer to refs. [8, 9].

3. Numerical results

In this section we explore the phase structure of the $2+1$-dimensional GN model in the large-N_f limit numerically. We have performed the majority of computations using the 2×2 representation (2.3) for the γ matrices, but also some computations using the 4×4 representation (2.4). Within numerical precision the results are in agreement. All results shown in the following were obtained using the 2×2 representation (2.3). Based on known analytical results for the phase diagram of the $1+1$-dimensional GN model [3, 4] we expect the following three phases:

- a symmetric phase, characterized by $\sigma = 0$;
- a homogeneously broken phase, characterized by $\sigma = \text{const} \neq 0$;
- an inhomogeneous phase, where $\sigma = \sigma(x, y)$ depends on the spatial coordinates.

The spactime volume in our computations is finite with temporal extent $1/T$, where T is the temperature, and spatial volume L^2. We discretize the effective action (1.2) using a plane wave expansion for the temporal direction and lattice field theory for the two spatial directions, where we decided for naive fermions for the fermionic determinant. Technical aspects are similar as in refs. [10, 11, 12] and will be discussed in detail in an upcoming publication.

3.1. $\sigma = const$

In a first step we determined the phase diagram for homogeneous σ, i.e. not allowing any spatial modulation for σ. As a byproduct we obtained $\sigma_0 = \sigma|_{\mu=0,T=0}$, which we use to set the scale. It is known that there is a symmetric phase and a homogeneously broken phase [6]. We determined the boundary between these two phases by numerically minimizing S_{eff} with respect to the constant σ using a standard algorithm for 1-dimensional minimization. Our result, which is shown in Fig. 1, is consistent with the result from ref. [6].

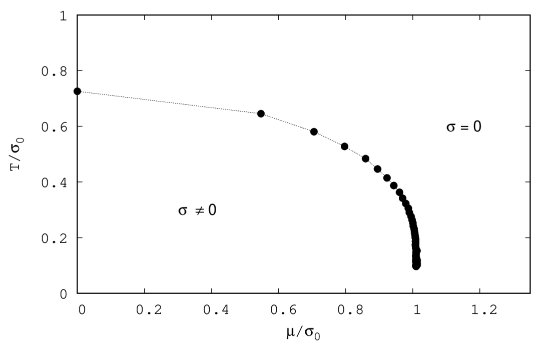

Figure 1: Phase diagram of the $2 + 1$-dimensional GN model in the μ-T plane for $\sigma = \text{const}$ (black dots are our numerical results, while the dashed line is just a guide to the eye).

3.2. $\sigma = \sigma(x)$

In a second step, we allowed σ to depend on one of the two spatial coordinates, i.e. $\sigma = \sigma(x)$. Then S_{eff} depends on N_s variables σ_j, which represent $\sigma(x)$ on the lattice sites $x = ja$, where $a = L/N_s$ denotes the lattice spacing. Since finding the global minimum of a function in many variables is a very challenging task, we started by performing stability analyses with respect to $\sigma = 0$. This amounts to finding the eigenvalues and eigenvectors of the Hessian matrix

$$H_{jk} = \frac{\partial^2}{\partial \sigma_j \partial \sigma_k} S_{\text{eff}} \Big|_{\sigma_0 = \sigma_1 = \ldots = \sigma_{N_s - 1} = 0}, \tag{3.1}$$

where negative eigenvalues indicate directions, in which S_{eff} decreases.

The red dots in Fig. 2 separate a region, where $\sigma = 0$ is stable (i.e. no negative eigenvalues of H), from another region, where $\sigma = 0$ is not stable (i.e. at least one negative eigenvalue of H). Note that the red dots do not necessarily correspond to a phase boundary (even though they could coincide with a phase boundary, as it is the case e.g. for the $1 + 1$ dimensional GN model, where they separate the symmetric phase and the inhomogeneous phase). Nevertheless, they unambiguously signal the existence of an inhomogeneous phase, which covers the triangular region between the red dots and the black dots (the latter represent the phase boundary for $\sigma = \text{const}$ already shown in Fig. 1). This inhomogeneous phase might, however, have a larger extension, which we are currently investigating by performing full multi-dimensional minimizations of S_{eff} using a conjugate gradient algorithm. Note that the red dots do not seem to correspond to a smooth curve. This could be a finite-volume effect, similar to that observed in a lattice field theory study of the $1+1$ dimensional GN model [10]. In Fig. 3

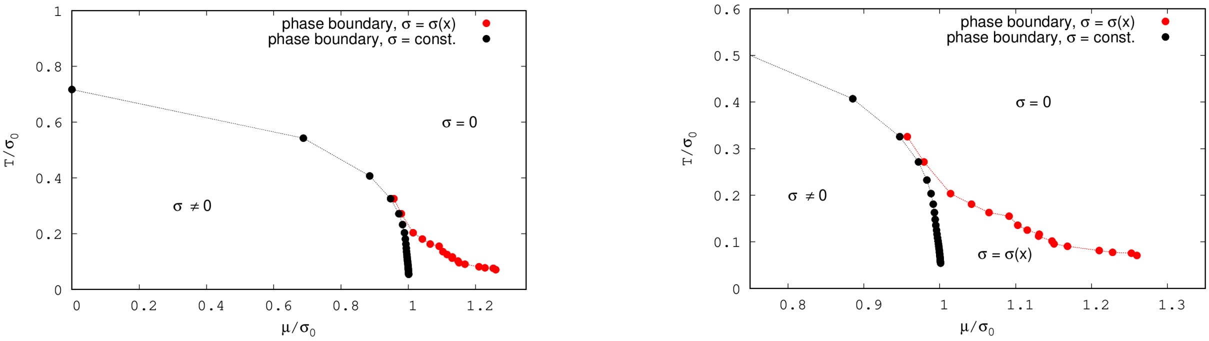

Figure 2: "Phase diagram" of the $2+1$-dimensional GN model in the μ-T plane for $\sigma = \sigma(x)$. The black dots represent the phase boundary for $\sigma = \text{const.}$ also shown in Fig. 1. The red dots are obtained by stability analyses with respect to $\sigma = 0$. The triangular region between the red dots and the black dots is part of an inhomogeneous phase. The right plot is a zoomed version of the left plot.

we show two examples of eigenvectors of the Hessian matrix (3.1) corresponding to negative eigenvalues. These eigenvectors indicate the shape of spatial modulations of $\sigma(x)$, which lower $S_{\text{eff}}[\sigma]$ with respect to $\sigma = 0$. The wave number increases for increasing chemical potential and fixed temperature, a behavior also observed for the $1+1$-dimensional GN model [3, 4].

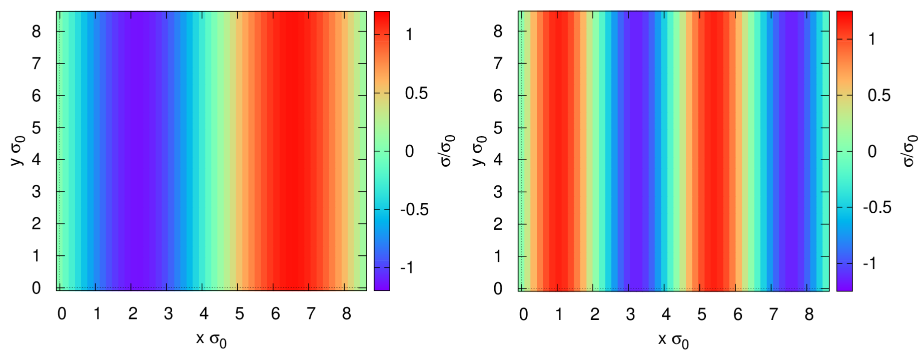

Figure 3: Eigenvectors of the Hessian matrix (3.1) corresponding to negative eigenvalues. **(left)** $\mu/\sigma_0 = 1.011$, $T/\sigma_0 = 0.198$. **(right)** $\mu/\sigma_0 = 1.064$, $T/\sigma_0 = 0.118$.

3.3. $\sigma = \sigma(x,y)$

We also started to explore the phase structure for σ depending on both spatial coordinates, i.e. $\sigma = \sigma(x,y)$. We are not yet in a position to show a phase diagram. It is, however, worthwhile to note that we found negative eigenvalues of the Hessian matrix corresponding to eigenvectors, which oscillate both in x and in y direction (see Fig. 4 for an example). This might indicate that the inhomogeneous phase is larger than in Fig. 2. Note, however, that these first results have been obtained on a rather small lattice with very coarse lattice spacing and might, thus, suffer from sizeable finite volume corrections and discretization errors. We plan to perform similar, but more precise computations on larger and finer lattices in the near future. It will also be interesting to explore, whether there are regions in the μ-T plane, where the global minimum of S_{eff} is related to such 2-dimensional structures of $\sigma(x, y)$.

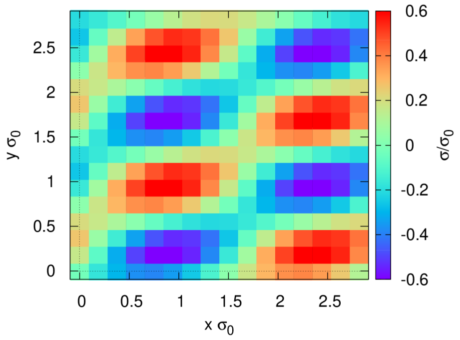

Figure 4: Eigenvector of the Hessian matrix corresponding to a negative eigenvalue. $\mu/\sigma_0 = 5.5$, $T/\sigma_0 = 0.19$.

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