Numerical analytic continuation
or
a nice interpolation and extrapolation method

Ralf-Arno Tripolt

Palaver

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JOHANN WOLFGANG GOETHE
UNIVERSITÄT
FRANKFURT AM MAIN

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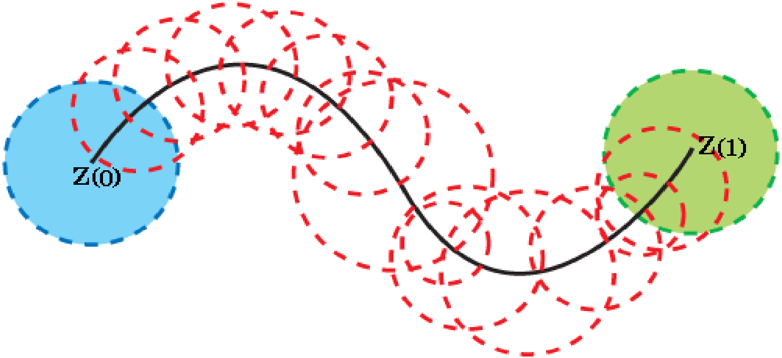
Analytic continuation - the idea

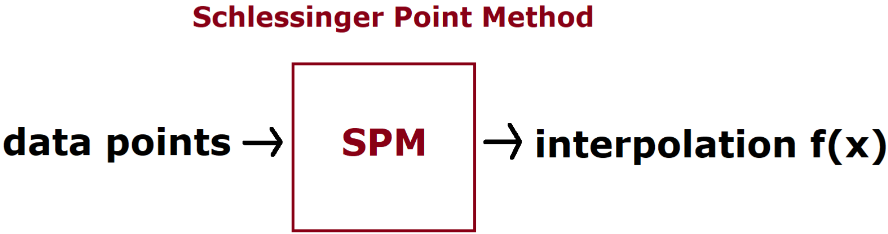


[commons.wikimedia.org]

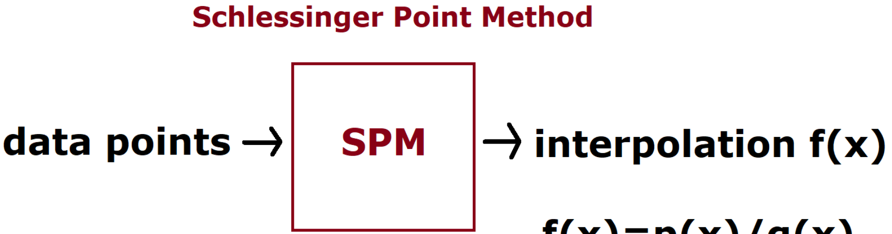
The interpolation method

data points → **SPM** → **interpolation f(x)**

The interpolation method



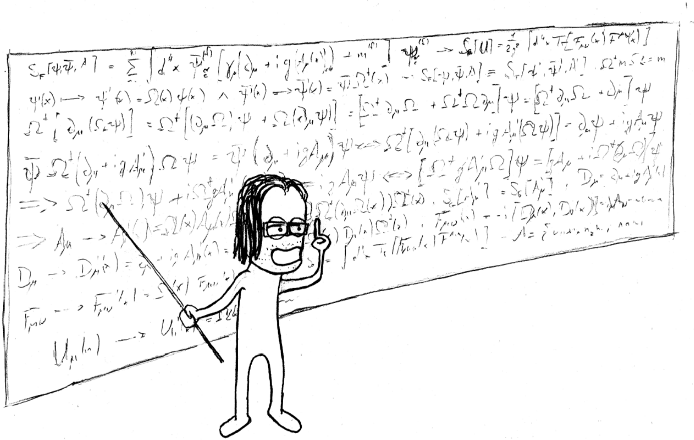
The interpolation method



$$f(x)=p(x)/q(x)$$

p(x) and q(x) are polynomials

For details...



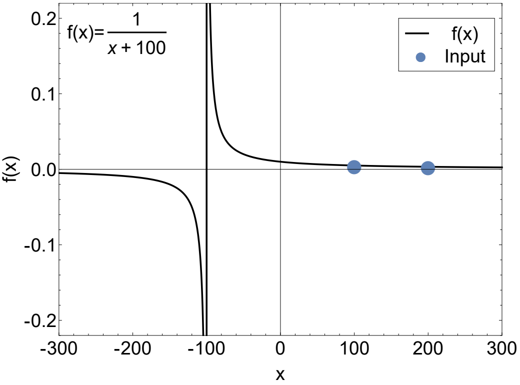
[courtesy L. Holicki]

arXiv: 1610.03252

arXiv: 1801.10348

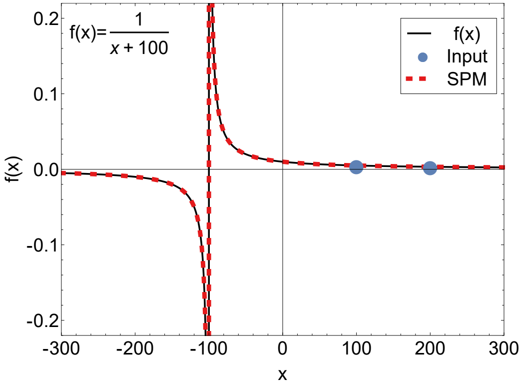
Simple Example: $f(x) = 1/(x + 100)$

- we use 2 input points for the SPM
- can we reconstruct the function?



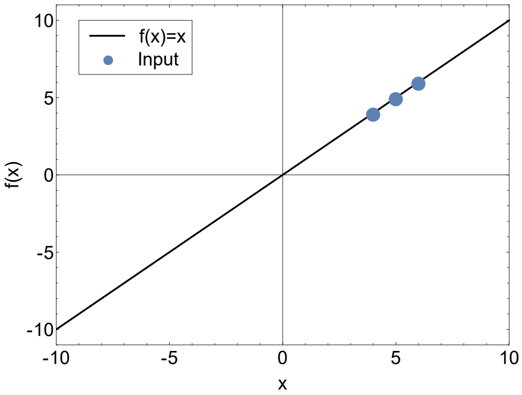
Simple Example: $f(x) = 1/(x + 100)$

- Only 2 input points are needed to reconstruct $f(x) = \frac{1}{x+100}$
- it is the "first guess" of the method



Another simple example: $f(x) = x$

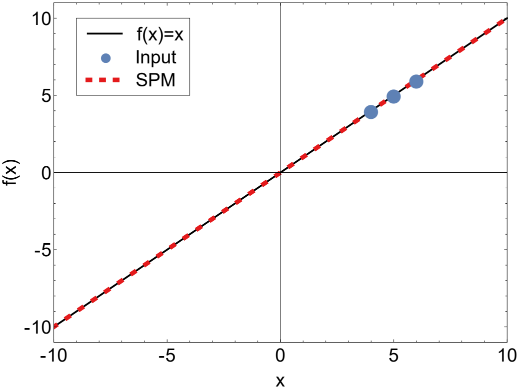
- we use 3 input points for the SPM
- can we reconstruct the function?



Another simple example: $f(x) = x$

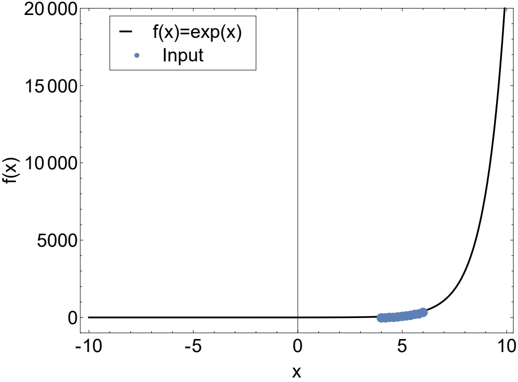
- 3 input points are needed to reconstruct $f(x) = x$
- with 15 digits precision one obtains for example

$$f(x) = \frac{22 + 1.8 \cdot 10^{15} x}{1.8 \cdot 10^{15} - x} \approx x$$



Another example: $f(x) = e^x$

- we use 11 input points
- can we reconstruct the function?



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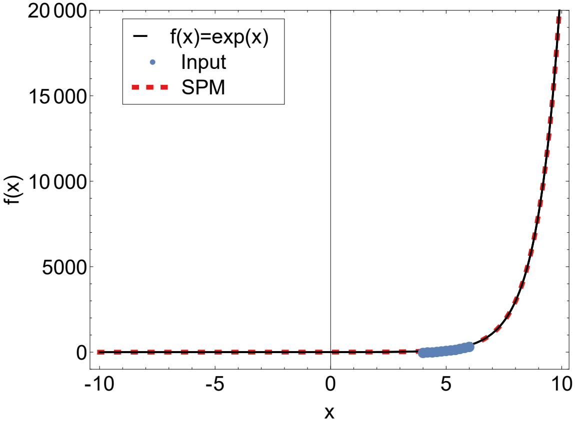
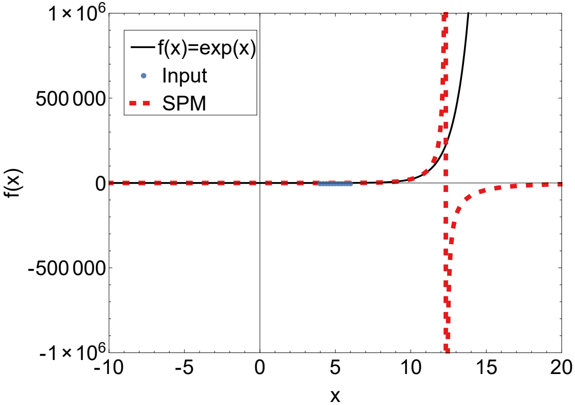


- for 11 input points we obtain

$$f(x) = \frac{263504 + 170536\,x + 46451\,x^2 + 10389\,x^3 + 756\,x^4 + 148\,x^5}{265568 - 98809\,x + 15473\,x^2 - 1274\,x^3 + 55\,x^4 - x^5}$$

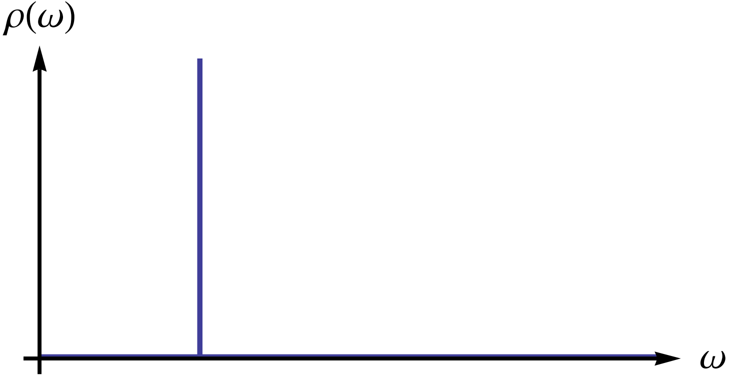
Another example: $f(x) = e^x$



- for 11 input points we obtain

$$C_N(x) = \frac{263504 + 170536\,x + 46451\,x^2 + 10389\,x^3 + 756\,x^4 + 148\,x^5}{265568 - 98809\,x + 15473\,x^2 - 1274\,x^3 + 55\,x^4 - x^5}$$

What is a spectral function?



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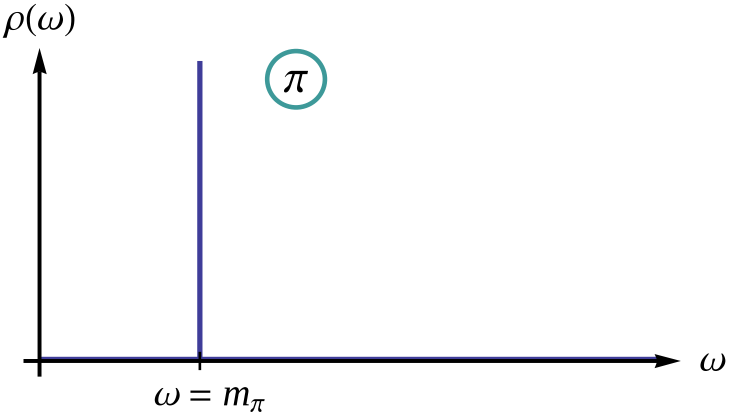

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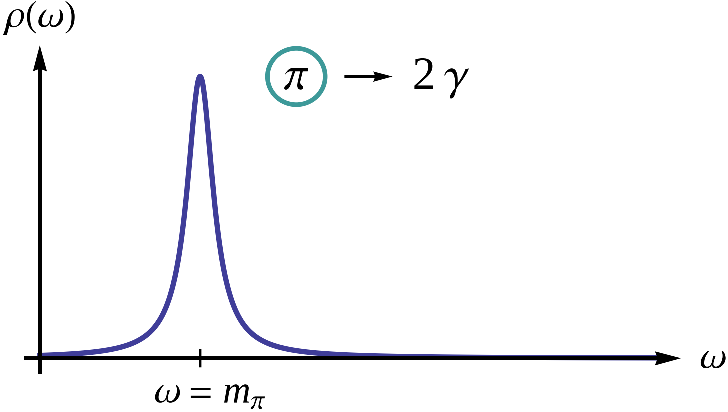

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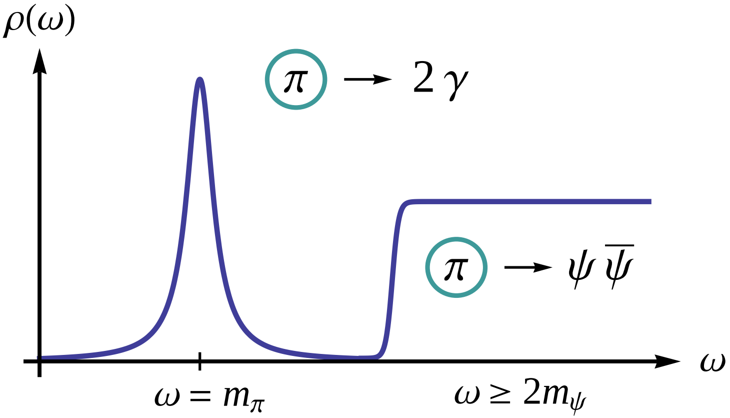


Model for a spectral function

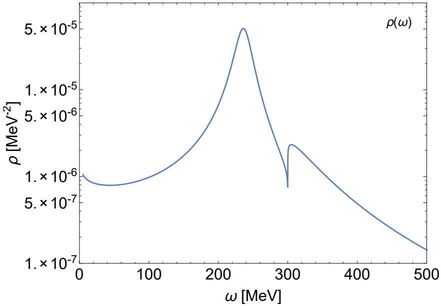
We use

$$\rho(\omega^2) = -\frac{1}{\pi}\mathrm{Im}\left(\frac{1}{\omega^2 - M^2 - \Pi(\omega^2)}\right)$$

with the self energy

$$\Pi(\omega^2) = S_1 \log(T_1^2 - \omega^2) + S_2 \log(T_2^2 - \omega^2)$$

and the parameters
$M = 50$ MeV,
$S_1 = 2000$ MeV^2,
$T_1 = 0$ MeV,
$S_2 = 3000$ MeV^2,
$T_2 = 300$ MeV,
$\omega \to \omega + i\varepsilon$ with $\varepsilon \to 0$



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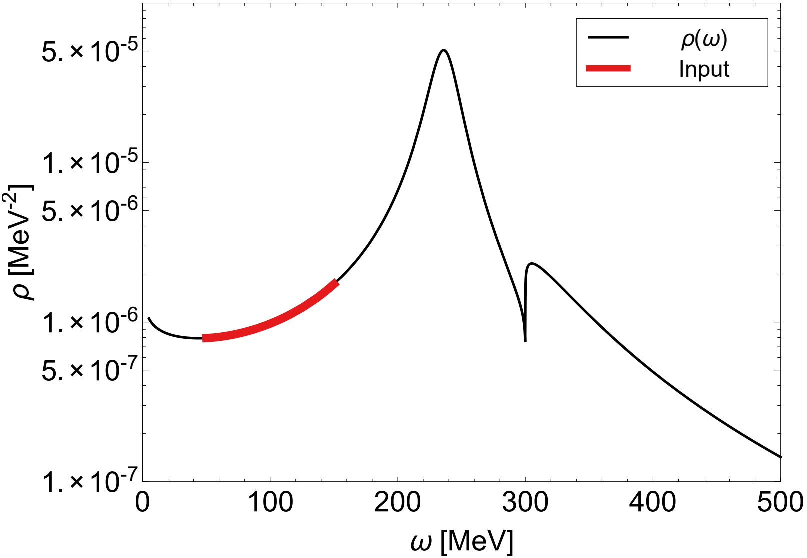

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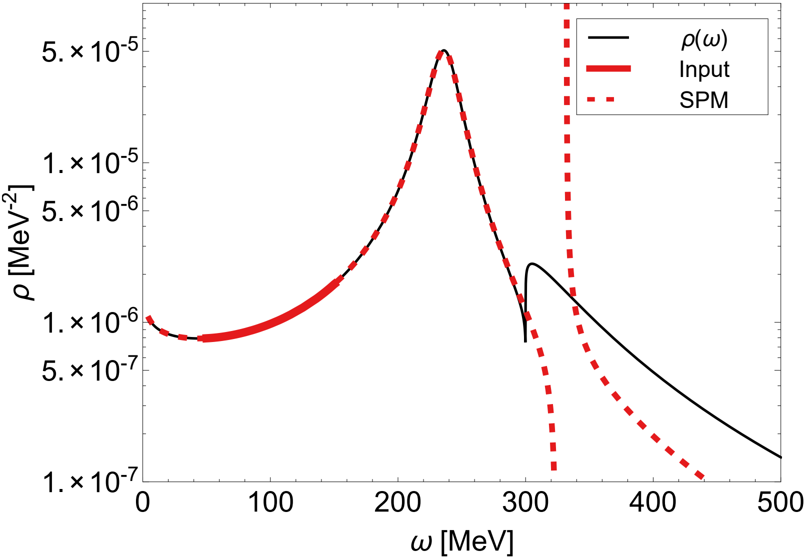

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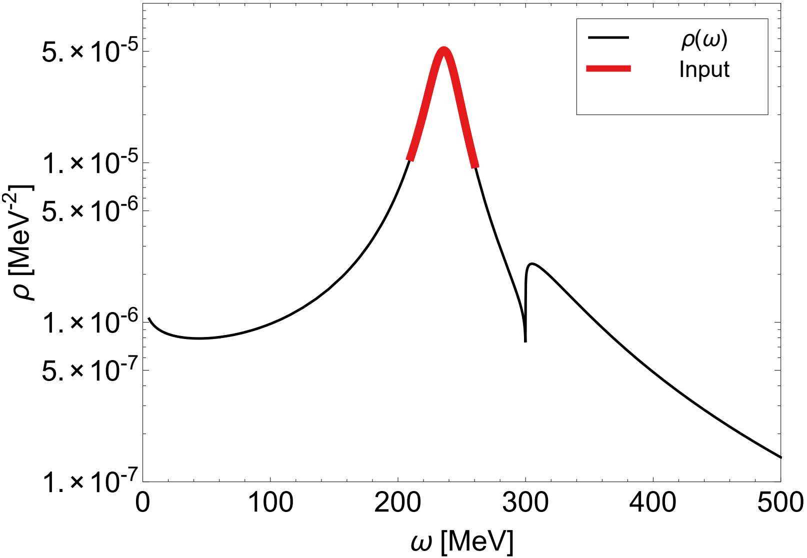

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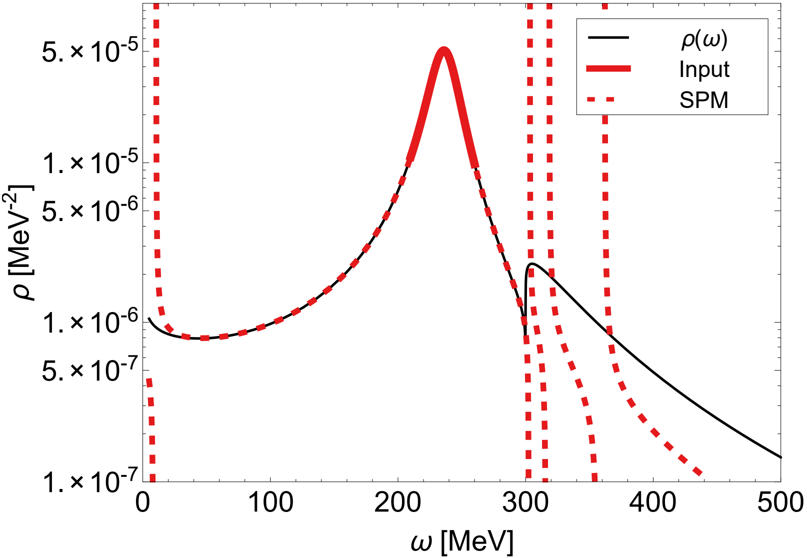

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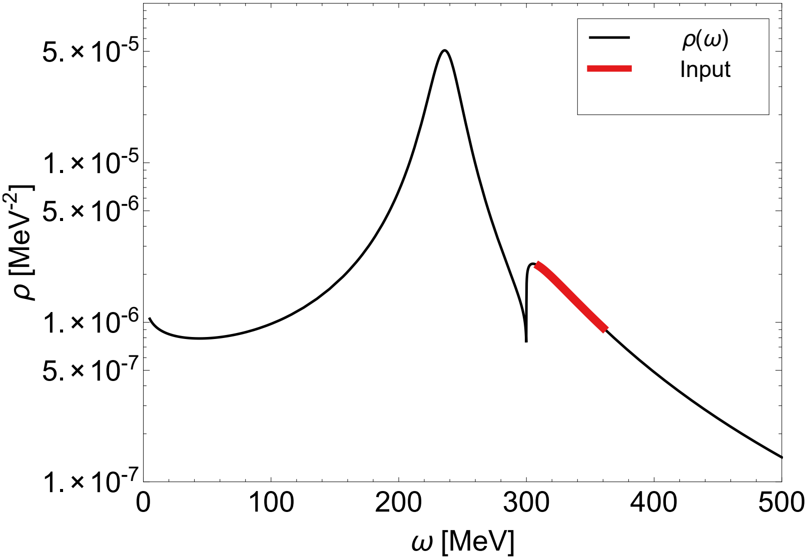
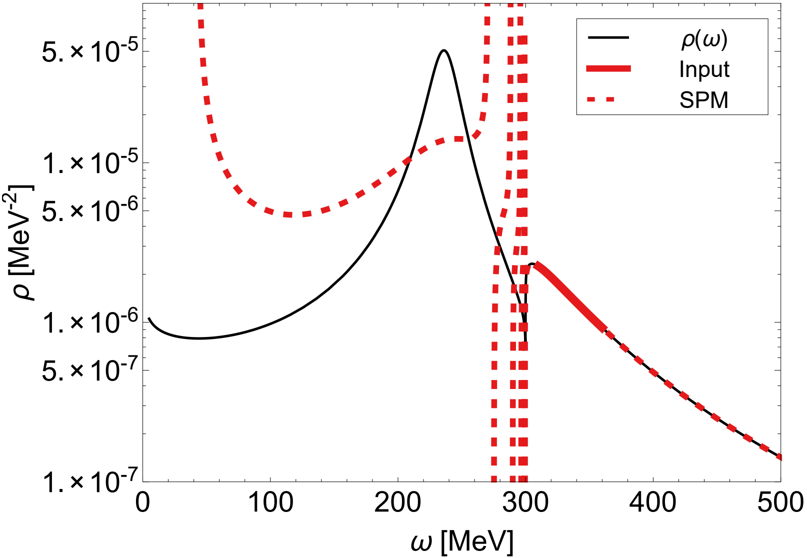


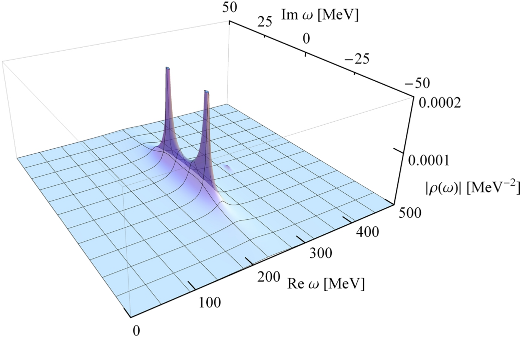
Model for a spectral function



Plot f(x) in the complex plane

We find poles in the complex plane at:
$\omega_P \approx (236.43 \pm i12.64)$ MeV

This pole characterizes the particle which is associated to the peak in the spectral function



Model for overlapping resonances - 3 particles

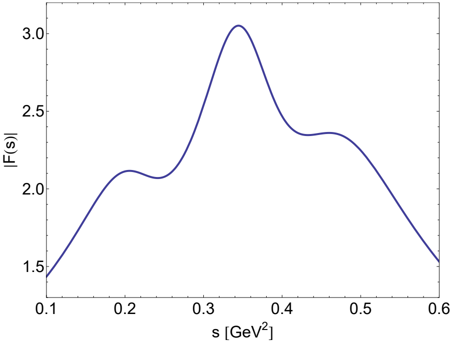
The form factor could be measured by experiment:

$$F(s) = \sum_{i=1}^{3} \frac{M_i^2}{M_i^2 - s - i\Gamma_i \frac{M_i^2}{\sqrt{s}} \left(\frac{k(s)}{k(M_i^2)}\right)^3}$$

with

$$k(s) = \frac{\sqrt{s}}{2}\sqrt{1 - 4m_\pi^2/s}$$

and the parameters
$M_1 = 0.5$ GeV,
$\Gamma_1 = 0.2$ GeV,
$M_2 = 0.6$ GeV,
$\Gamma_2 = 0.1$ GeV,
$M_3 = 0.7$ GeV,
$\Gamma_3 = 0.15$ GeV,
$m_\pi = 0.137$ GeV



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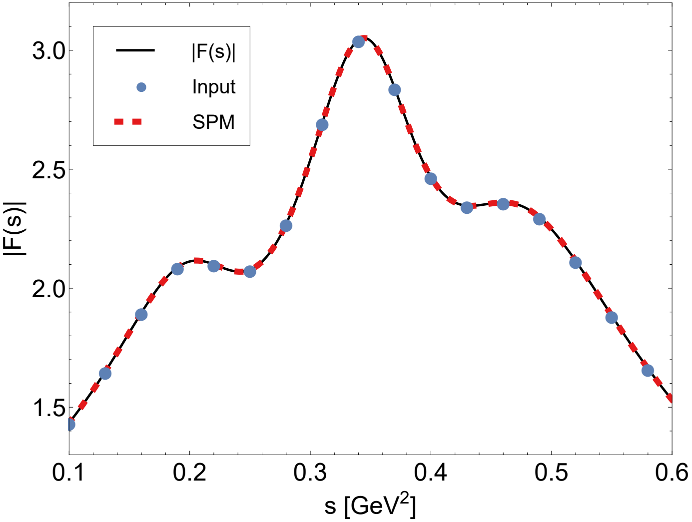
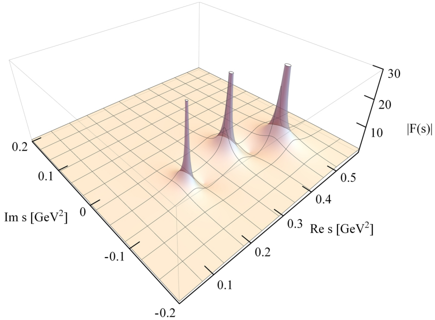
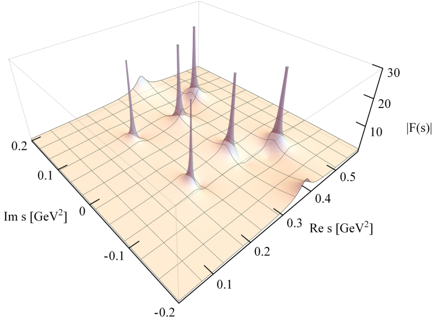


Plot F(s) in the complex plane

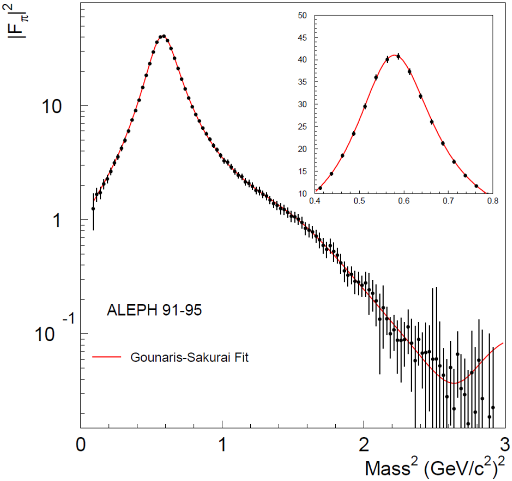
exact result, $F(s)$:



reconstruction, SPM:



Complex pole of the charged $\rho(770)$ meson



[ALEPH collaboration, Phys. Rept. 421, 191-284, 2005, arXiv:hep-ex/0506072]

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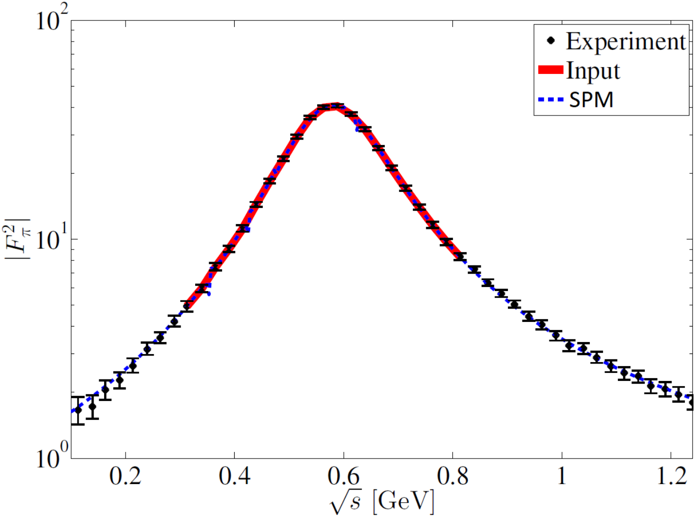


[R.-A. T., I. Haritan, J. Wambach and N. Moiseyev, arXiv:1610.03252]

Complex pole of the charged $\rho(770)$ meson

We find the complex pole of the charged $\rho(770)$ meson to be at $\sqrt{s_\rho} = M_\rho - i\Gamma_\rho/2$ with

$$M_\rho = 761.8 \pm 1.9\,\text{MeV},$$
$$\Gamma_\rho = 139.8 \pm 3.6\,\text{MeV}.$$

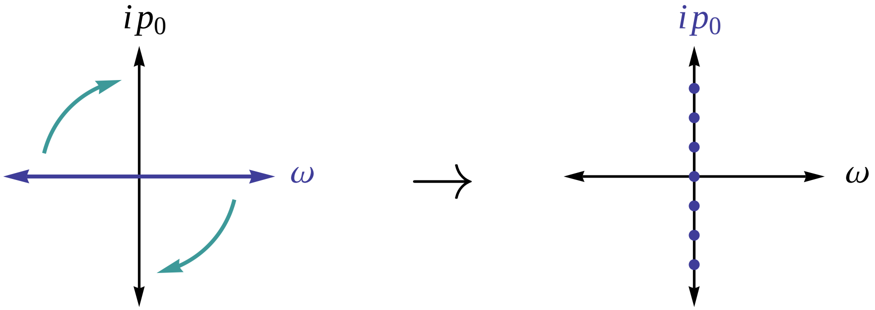
M_ρ (MeV)	Γ_ρ (MeV)	source
762.5 ± 2	142 ± 7	[12]
758.3 ± 5.4	145.1 ± 6.3	[13]
$764.1 \pm 2.7^{+4.0}_{-2.5}$	$148.2 \pm 1.9^{+1.7}_{-5.0}$	[14]
754 ± 18	148 ± 20	[15]
763.0 ± 0.2	139.0 ± 0.5	[16]
760 ± 2	147 ± 6	[17]
761 ± 1	139 ± 2	[18]
763.7 ± 1.2	144 ± 3	[19]
761.8 ± 1.9	139.8 ± 3.6	this work

Table 1: Collection of pole parameter predictions for the $\rho(770)$ meson.

[R.-A. T., I. Haritan, J. Wambach and N. Moiseyev, arXiv:1610.03252]

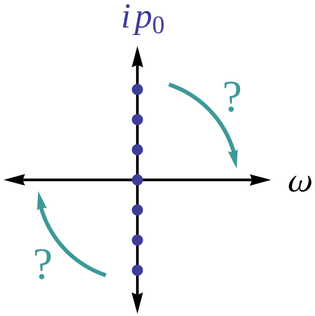
The analytic continuation problem

Calculations at finite temperature are often performed using imaginary energies:

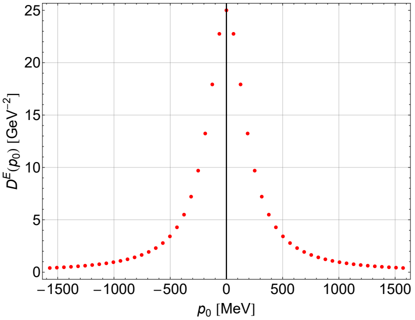


The analytic continuation problem

Analytic continuation problem: How to get back to real energies?



Analytic Continuation - Free Particle



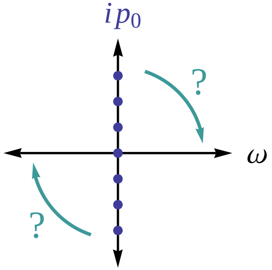
We want to reconstruct the spectral function of a free particle

$$\rho(\omega) = \mathsf{sgn}(\omega)\delta(\omega^2 - m^2)$$

starting from the free propagator

$$D^E(p_0) = \frac{1}{p_0^2 + m^2}$$

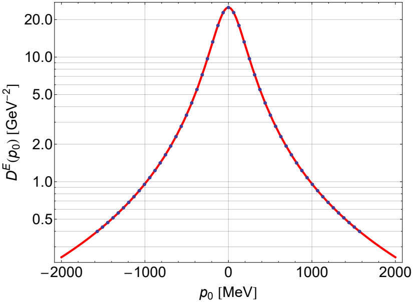
with $p_0 = 2n\pi T$,
$m = 200$ MeV, and
$T = 1/\beta = 10$ MeV.

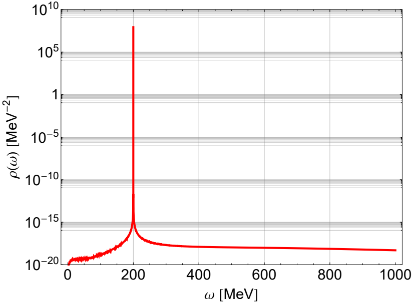


Analytic Continuation - Free Particle

We choose 51 input points from the Euclidean propagator $D^E(p_0)$ and apply the SPM to obtain the real-time propagator $D^R(\omega = ip_0)$ and the spectral function:

$$\rho(\omega) = -\frac{1}{\pi}\text{Im}D^R(\omega)$$





Summary

The Schlessinger point method can be used to obtain the analytic continuation of a function that is given in the form of numerical data.

- one can reconstruct the underlying function not only along the real axis but also in the complex plane
- can be used to identify resonance poles and to “predict” decay thresholds (branch cuts)
- can be used to perform an analytic continuation based on Euclidean data

Schlessinger Point Method (SPM)

Given a finite set of N data points (x_i, f_i) we construct the rational interpolant $p(x)/q(x)$ with polynomials $p(x)$ and $q(x)$ that is given by the continued fraction

$$p(x)/q(x) = C_N(x) = \cfrac{f_1}{1 + \cfrac{a_1(x-x_1)}{1 + \cfrac{a_2(x-x_2)}{\vdots\, a_{N-1}(x-x_{N-1})}}},$$

where the coefficients a_i are given recursively by $a_1 = \frac{f_1/f_2 - 1}{x_2 - x_1}$ and

$$a_i = \frac{1}{x_i - x_{i+1}} \left(1 + \frac{a_{i-1}(x_{i+1} - x_{i-1})}{1+} \frac{a_{i-2}(x_{i+1} - x_{i-2})}{1+} \cdots \frac{a_1(x_{i+1} - x_1)}{1 - f_1/f_{i+1}} \right)$$

The polynomials are of order $(N/2 - 1, N/2)$ for an even number of input points and $((N-1)/2, (N-1)/2)$ for an odd number of input points

[L. Schlessinger, Physical Review, Volume 167, Number 5 (1968)]
[R.W. Haymaker and L. Schlesinger, Mathematics in Science and Engineering, Volume 71, Chapter 11 (1970)]
[H.J. Vidberg and J.W. Serene, Journal of Low Temperature Physics, Vol. 29, Nos. 3/4 (1977)]
[A. Pilaftsis and D. Teresi, Nucl. Phys. B 874 (2013) 594-619]
[G. Markó, U. Reinosa and Z. Szép, arXiv: 1706.08726]

Analytic Continuation and Radius of Convergence

- an analytic continuation into the complex plane can be performed by choosing x in $C_N(x)$ to be complex, i.e. $x = \alpha e^{i\theta}$
- rational interpolants can exactly reproduce polar singularities, thus extending the 'radius of convergence' to the first non-polar singularity, e.g. a branch point
- even non-polar singularities may be well approximated by poles and zeros of the rational fraction
- a rational fraction can have only one sheet in the complex plane - a many-sheeted function can only be reconstructed on a single sheet