Plasma Astrophysics
Chapter 6: Shocks and Discontinuities

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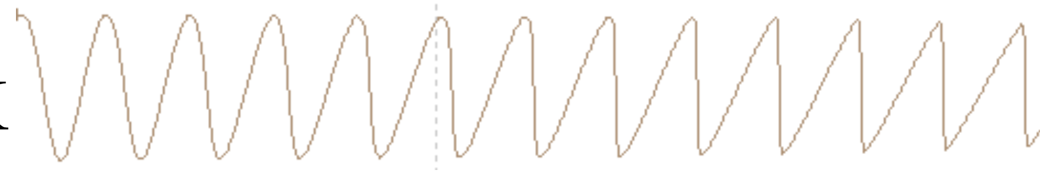
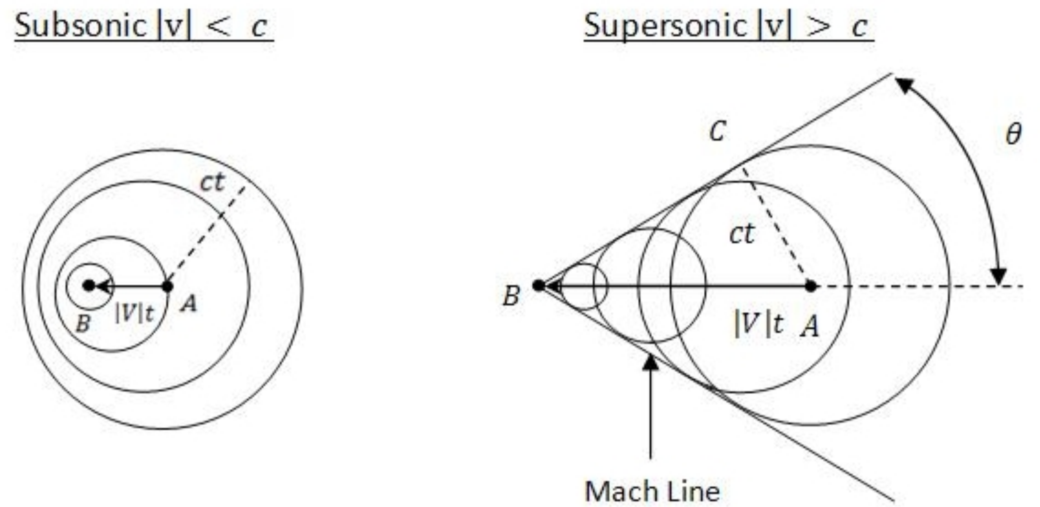
Formation of shock



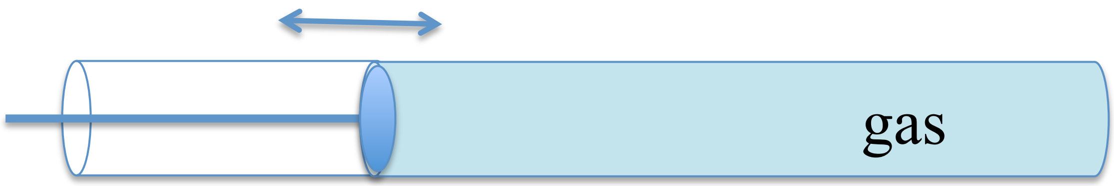
- When the amplitude is so small that linear theory applies, that a disturbance propagates as a sound wave.
- The wave profile maintains a fixed shape, since each part of wave moves with the same speed
- But, when the wave have a finite amplitude, so that nonlinear terms becomes important, the crest of the sound wave moves faster than its leading and trailing edge.
- This causes a progressive steepening of the front portion of wave as the crest catches up
- Ultimately, the gradients of pressure, density, temperature and velocity becomes so large that dissipative processes (viscosity) are no longer negligible
- Then a steady wave-shape is attained, called a *shock wave*, with a balance between the steepening effect of nonlinear convective terms and broadening effect of dissipation.

Shock wave



- The shock wave moves at a speed in excess of the sound speed
- So information cannot be propagated ahead to signal its imminent arrival.
- Since such information would travel at only c_s, relative to the undisturbed medium ahead of the shock.

Shock wave (cont.)



- For example, suppose a long tube contains gas initially at rest and that a piston at one end of tube is accelerated into uniform motion

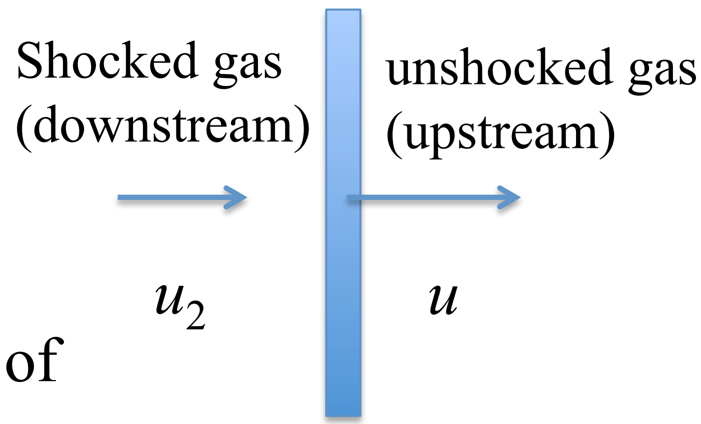
- If the piston is being withdrawn from the tube, an *rarefaction (expansion) wave* travels into a gas (pressure is decreased)

- If the piston is pushed into the tube, a compression wave is generated (pressure is increased) and this eventually steepens into a *shock wave*.

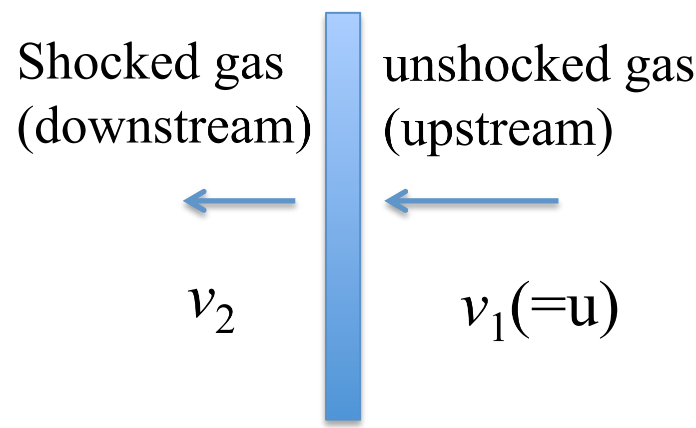
Shock frame

- Models shock front by a plane discontinuity and the two states
- subscripts 1 for unshocked gas (ahead of shock, upstream, pre-shock), 2 for shocked gas (behind of shock, downstream, post-shock)
- In rest frame, shock speed is u, while speed of shocked gas is u_2 ($< u$)
- More convenient to use a frame of reference moving with shock wave.
- So unshocked gas enters the front of shock with speed, $v_1 = u$
- While the shocked gas leaves the back of shock with speed, $v_2 = u - u_2$
- Since u_2 is positive. So $v_2 < v_1$
- When $v_2 = v_1$, there is no shock

Rest frame



Shock frame



Thickness of shock

- A detailed determination of the thickness of the shock and its internal structure is very complicated.
- However, if the dominant dissipation mechanism is known, an order-of-magnitude estimate of the shock width (δx) may be obtained.
- In the case of viscous dissipation, the amount of energy (δE) dissipated during a small time (δt) is give by

$$\frac{\delta E}{\delta t} \approx \rho\nu \left(\frac{\delta v}{\delta x}\right)^2$$

- Where ν is the kinetic viscosity
- $\delta t \sim \delta x / v_1$ as the time for the shock front to move a distance δx and putting $\delta v \sim v_1 - v_2$

$$\delta x \approx \frac{\rho\nu(v_1 - v_2)^2}{v_1 \delta E}$$

Thickness of shock (cont.)

- For strong shock, $\delta E \approx \frac{1}{2}\rho_1 v_1^2$, so that

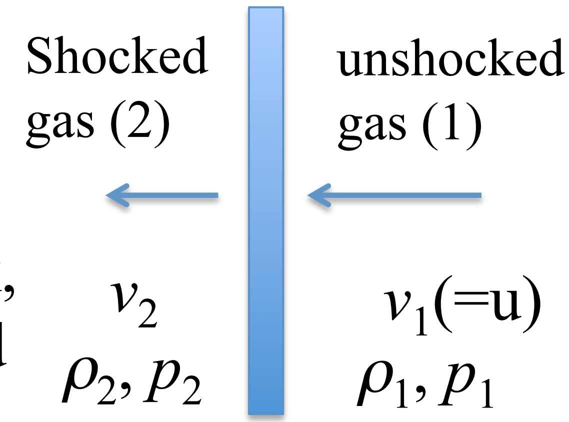
$$\boxed{\delta x \approx \frac{\nu}{v_1}}$$

- In other words, the Reynolds number ($v_1 \delta x / \nu$) is order of unity.

Reynolds number (Re) = inertial force / viscous force = vL/ν

Hydrodynamic shocks

- Consider a plane shock wave propagating steadily with constant speed into a stationary gas with density ρ_1 and pressure p_1
- In a frame of reference moving with the shock, the speed of shocked gas, v_2, its density ρ_2 and pressure p_2



- Jump relation (condition) between the shock surface is given us a set of conservation equations

  $\partial Q/\partial t + \nabla \cdot \boldsymbol{F} = 0$ (Q: conserved quantities, $\boldsymbol{F}$: flux)

- If shock is steady ($\partial/\partial t \equiv 0$) and 1D ($\partial/\partial y \equiv 0, \ \partial/\partial z \equiv 0$),

  $dF_x/dx = 0$

- Which is implies that $(\boldsymbol{F}_1 - \boldsymbol{F}_2) \cdot \hat{n} = 0$

Hydrodynamic shocks (cont.)

- Conservation form of hydrodynamic equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0$$
$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot [\rho_m \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I}] = 0$$
$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m v^2 + \rho_m e\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho_m v^2 + \rho_m e + p\right)\boldsymbol{v}\right] = 0$$

$$p = (\gamma - 1)\rho_m e$$

Hydrodynamic shocks (cont.)

- From conservation of mass, momentum and energy between the shock surface

$$\rho_1 v_1 = \rho_2 v_2 \tag{6.1}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \tag{6.2}$$

$$\left(\frac{1}{2}\rho_1 v_1^2 + \rho_1 e_1 + p_1\right) v_1 = \left(\frac{1}{2}\rho_2 v_2^2 + \rho_1 e_2 + p_2\right) v_2 \tag{6.3}$$

- These equations are referred to as the *Rankine-Hugoniot (jump) relations*

Hydrodynamic shocks (cont.)

- Where (for perfect gas) the internal energy per unit mass is

$$e = p/[(\gamma - 1)\rho]$$

- So eq (6.3) reduces to

$$\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 \qquad (6.4)$$

- Let us define specific volume, $V_1=1/\rho_1$, $V_2=1/\rho_2$ and mass flux $j=\rho_1 v_1=\rho_2 v_2$. From eq (6.1) & (6.2),

$$j^2 = \frac{p_2 - p_1}{V_2 - V_1}$$

Hydrodynamic shocks (cont.)

- In principle this equation allows for two solutions,
  - The post-shock medium has a higher pressure than the pre-shock medium ($p_2 > p_1$),
  - The post-shock medium has a lower pressure than the pre-shock medium ($p_2 < p_1$).
- The latter will turn out to be an unphysical solution. But such a solution will quickly smear out into a *rarefaction (expansion) wave*
- We focus here on the case $p_2 > p_1$ (shock wave case)
- Here we define

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} \equiv x, \quad \frac{p_2}{p_1} \equiv y, \quad c_{s1}^2 = \frac{\gamma p_1}{\rho_1}, \quad M_1^2 \equiv \frac{v_1^2}{c_{s1}^2}$$

- Where M_1 is shock Mach number, c_s is sound speed

Hydrodynamic shocks (cont.)

- Eq (6.2) => $y = 1 + \gamma M_1^2 \left(1 - \frac{1}{x}\right)$ (6.5)

- Eq (6.4) (using eq (6.5)) =>
  $$\{2 + (\gamma - 1)M_1^2\}x^2 - 2(1 + \gamma M_1^2)x + (\gamma + 1)M_1^2 = 0$$
- => $(x - 1)[\{2 + (\gamma - 1)M_1^2\}x - (\gamma + 1)M_1^2] = 0$

- The nontrivial solution can be written

$$x = \frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$$
$$y = \frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$
(6.6)

Properties of hydrodynamic shocks

- From eq (6.6), when M_1=1, density, pressure and velocity do not make any jump. So the shock is developed when M_1>1
- Shock speed (v_1) must exceed the sound speed (c_{s1}) ahead of shock
- In the shock frame, flow is supersonic in front of shock but subsonic behind it ($v_2 \leq c_{s2}$)
- Shock must be compressive with $p_2 \geq p_1, \ \rho_2 \geq \rho_1$
- $T_2 \geq T_1$, shock wave slows the gas down but heat it up (convert flow kinetic energy into thermal energy in the process).
- In $M_1 \to \infty$ (strong shock),

$$x = \frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1}, \quad y \to \infty$$

- When γ=5/3 (mono-atomic gas), $x=\rho_2/\rho_1=4$. So in strong shock, the density jump at the shock (shock compression ratio) is only 4.

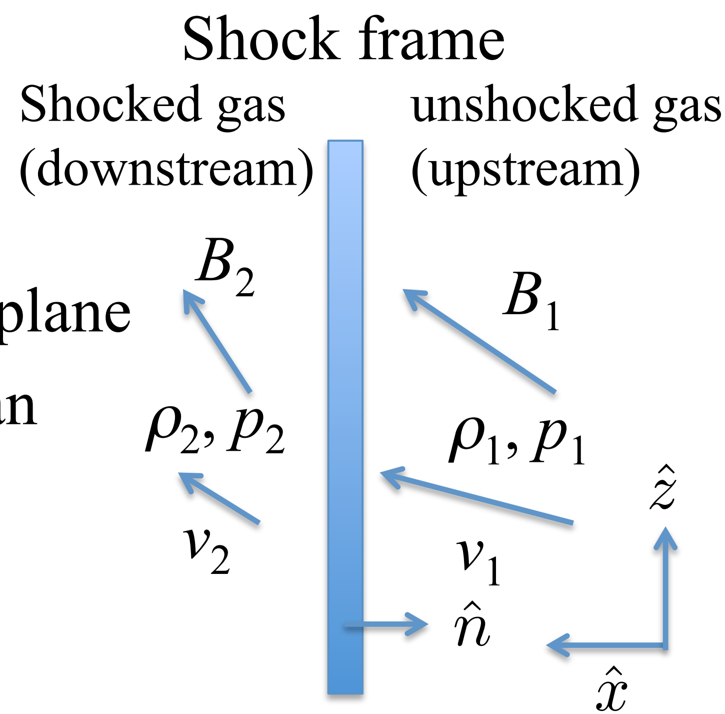
Hydrodynamic discontinuity

- *Contact discontinuity*: $v_1 = v_2 = 0$, Density jump arbitrary, but other quantities (pressure for hydro) are continuous (temperature is also change)
- *Rarefaction (expansion) wave*: a simple wave or progressive disturbance (not shock). density and pressure decrease on crossing the wave

MHD shock

- Consider simple case of a 1D steady shock
- We will work in a frame where shock is stationary (*shock fame*)
- x-axis will be a aligned with the shock normal, so plane of shock is parallel to yz-plane
- The jump across the any quantities of X can be expressed using following notation:

$$[X] = X_u - X_d = X_1 - X_2$$



- MHD jump relation is given us a set of conservation equations

$$\partial Q/\partial t + \nabla \cdot \boldsymbol{F} = 0 \text{ (}Q\text{: conserved quantities, }\boldsymbol{F}\text{: flux)}$$

- If shock is steady ($\partial/\partial t \equiv 0$) and 1D ($\partial/\partial y \equiv 0,\ \partial/\partial z \equiv 0$),

$$dF_x/dx = 0$$

- Which is implies that $(\boldsymbol{F}_1 - \boldsymbol{F}_2) \cdot \hat{n} = 0$=> $[F_n] = 0$

Conservation form of ideal MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad (6.7)$$

Mass conservation

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot \left[\rho \boldsymbol{v}\boldsymbol{v} + \left(p + \frac{B^2}{2\mu_0}\right)\boldsymbol{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{\mu_0}\right] = 0 \quad (6.8)$$

Momentum conservation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}\right)$$

$$+\nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1}p + \frac{B^2}{\mu_0}\right)\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{B})\frac{\boldsymbol{B}}{\mu_0}\right] = 0 \quad (6.9)$$

Energy conservation

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) = 0 \quad (6.10)$$

Magnetic flux conservation

$$\nabla \cdot \boldsymbol{B} = 0 \quad (6.11)$$

$$p = (\gamma - 1)\rho e$$

Ideal equation of state

MHD shock jump relation

- For MHD, from mass conservation (continuity) equation (eq 6.7),
$$d(\rho v_x)/dx = 0$$
- Which leads jump condition for shock:
$$[\rho v_x] = 0$$
- From momentum conservation equation (eq 6.8), we consider two jump condition.
- Firstly, the conservation of momentum normal to shock surface
$$\left[\rho v_x^2 + p + \frac{B^2}{2\mu_0} - \frac{B_x^2}{\mu_0}\right]$$
- Transverse momentum also has to balance,
$$\left[\rho v_x v_t - \frac{B_x}{\mu_0} B_t\right] = 0$$
- Where the *t* subscript indicates transverse component to the shock. This reflects tangential stresses related to bend or kink of B-field

MHD shock jump relation (cont.)

- The shock jump condition from energy conservation (eq.6.9) is

$$\left[\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma-1}p + \frac{B^2}{\mu_0}\right)v_x - (\boldsymbol{v}\cdot\boldsymbol{B})\frac{B_x}{\mu_0}\right] = 0$$

- From $\nabla\cdot\boldsymbol{B} = 0$, the normal component of magnetic field is continuous (B_x=const)

$$[B_x] = 0$$

- From magnetic flux conservation (eq 6.11),

$$[v_x B_t - B_x v_t] = 0$$

MHD Rankine-Hugoniot (jump) relation

$$
\begin{aligned}
&[\rho v_x] = 0 \\
&\left[\rho v_x^2 + p + \frac{B^2}{2\mu_0} - \frac{B_x^2}{\mu_0}\right] = 0 \\
&\left[\rho v_x v_t - \frac{B_x}{\mu_0} B_t\right] = 0 \\
&\left[\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1} p + \frac{B^2}{\mu_0}\right) v_x - (\boldsymbol{v} \cdot \boldsymbol{B}) \frac{B_x}{\mu_0}\right] = 0 \\
&[B_x] = 0 \\
&[v_x B_t - B_x v_t] = 0
\end{aligned}
\tag{6.12}
$$

Possible type of MHD shock

- Shock wave, $v_n \neq 0$: Flow crosses surface of discontinuity accompanied by compression and dissipation

Parallel shock	$B_t = 0$	Magnetic field unchanged by shock (hydrodynamic shock)
Perpendicular shock	$B_n = 0$	Plasma pressure and field strength increases at shock
Oblique shocks	$B_t \neq 0, \;\; B_n \neq 0$	
Fast shock	Plasma pressure and field strength increases at shock, magnetic field bend away from normal.	
Slow shock	Plasma pressure increases and field strength decreases at shock, magnetic field bend towards normal	
Intermediate shock	Only shock-like in anisotropic plasma (magnetic field rotate of 180^{O} in plane of shock, density jump)	

Possible type of MHD discontinuity

Discontinuity

Contact discontinuity	$v_n = 0, \ B_n \neq 0$	Density jump arbitrary, but other quantities are continuous
Tangential discontinuity	$v_n = 0, \ B_n = 0$	Plasma pressure and field change maintaining static pressure balance (total pressure is constant)
Rotational discontinuity	$v_n = B_n/\sqrt{\mu_0 \rho}$	Form of intermediate shock in isotropic plasma, field and flow change direction but not magnitude

Perpendicular shock

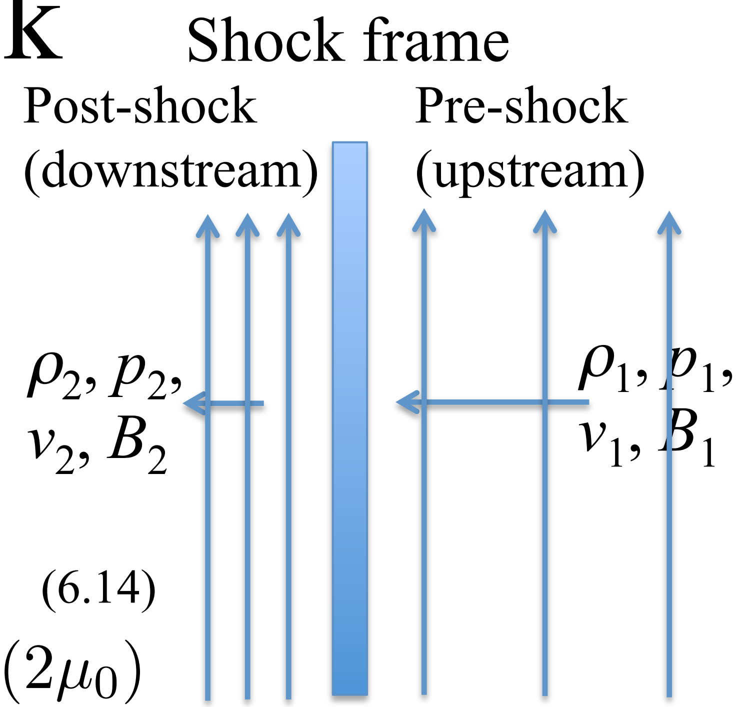
- Consider perpendicular shock (B_n=0). In this case, the velocities of both the shock and plasma are perpendicular to the magnetic field
- The jump relation (eq 6.12) is

$$\rho_1 v_1 = \rho_2 v_2 \qquad (6.13)$$

$$\rho_1 v_1^2 + p_1 + B_1^2/(2\mu_0) = \rho_2 v_2^2 + p_2 + B_2^2/(2\mu_0) \qquad (6.14)$$

$$\left(\frac{1}{2}\rho_1 v_1^2 + \frac{\gamma}{\gamma-1}p_1 + \frac{B_1^2}{\mu_0}\right) v_1 = \left(\frac{1}{2}\rho_2 v_2^2 + \frac{\gamma}{\gamma-1}p_2 + \frac{B_2^2}{\mu_0}\right) v_2 \qquad (6.15)$$

$$B_1 v_1 = B_2 v_2 \qquad (6.16)$$



- Here we define

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{B_2}{B_1} \equiv X, \quad \frac{p_2}{p_1} \equiv Y, \quad M_1 \equiv \frac{v_1}{c_{s1}}, \quad \beta_1 \equiv \frac{2\mu_0 p_1}{B_1^2} \equiv \frac{2c_{s1}^2}{\gamma v_{A1}^2}$$

Perpendicular shock (cont.)

- From eq (6.14),

$$Y = 1 + \beta_1^{-1}(1 - X^2) + \gamma M_1^2(1 - X^{-1}) \quad (6.17)$$

- From eq (6.15),

$$Y = \frac{2(\gamma - 1)}{\gamma\beta_1} X(1 - X) + X + \frac{\gamma - 1}{2} M_1^2 (X - \frac{1}{X}) \quad (6.18)$$

- Combine both eq (6.14) and eq(6.15)

$$(X-1)\left\{-\frac{2(\gamma - 1)}{\gamma\beta_1}X^2 + X + \frac{\gamma - 1}{2}M_1^2(X + 1) + \frac{1}{\beta_1}X(X + 1) - \gamma M_1^2\right\} = 0$$

- X=1 does not make a shock (not our solution). So

$$\begin{aligned} f(x) &= \left\{-\frac{2(\gamma - 1)}{\gamma\beta_1}X^2 + X + \frac{\gamma - 1}{2}M_1^2(X + 1) + \frac{1}{\beta_1}X(X + 1) - \gamma M_1^2\right\} \\ &= \boxed{2(2 - \gamma)X^2 + \gamma\{2\beta_1 + (\gamma - 1)\beta_1 M_1^2 + 2\}X - \gamma(\gamma + 1)\beta_1 M_1^2 = 0} \end{aligned}$$

Perpendicular shock (cont.)

- We need to find the positive solution of $f(x)$
- The fact that $1 < \gamma < 2$ implies that this equation have just one positive root (a quadratic function with a single minimum)
- The solution reduces to the hydrodynamic value (eq 6.6) in the limit of large β_1
- The effect of magnetic field is to reduce X below its hydrodynamic value, since the flow kinetic energy can be converted into magnetic energy as well as heat

- If X=1, $f(1) < 0$ => When X >1, we get a solution of f(X) = 0

$$f(1) = 4 - 2\gamma\beta_1 M_1^2 + 2\gamma\beta_1 < 0 \rightarrow \boxed{M_1^2 > 1 + \frac{2}{\gamma\beta_1}}$$

Perpendicular shock (cont.)

- In terms of sound and Alfven speeds,

$$v_1^2 > c_{s1}^2 + v_{A1}^2$$

- The shock speed (v_1) must exceed the fast magnetosonic speed $(c_{s1}^2 + v_{A1}^2)^{1/2}$ ahead of the shock

- Strong shock limit ($M_1 >> 1$),

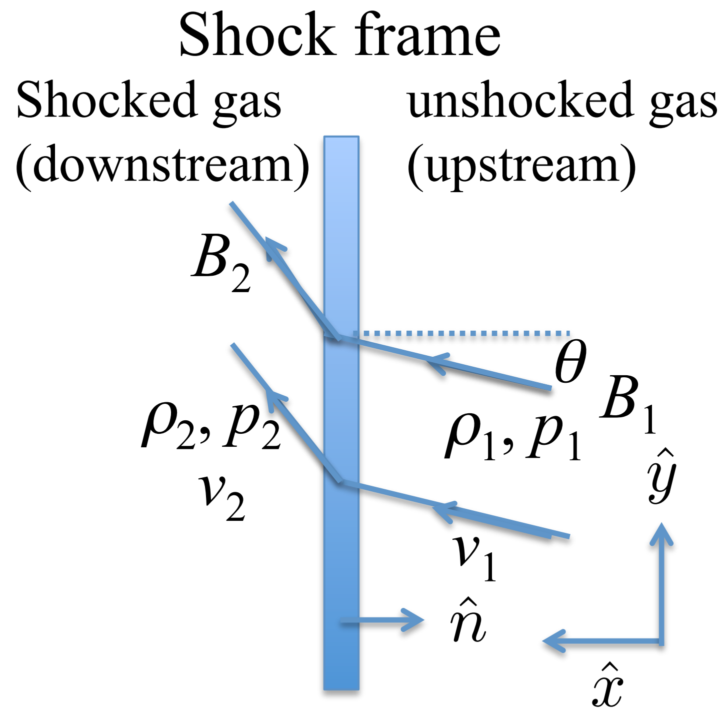
$$X = \frac{\gamma + 1}{\gamma - 1} = \frac{\rho_2}{\rho_1} = \frac{B_2}{B_1}$$

- For γ=5/3 case,

$$\frac{\rho_2}{\rho_1} = \frac{B_2}{B_1} = 4$$

Oblique shock

- In this case, the magnetic field contains components both parallel and normal to the shock front
- Assume the velocity and magnetic field vectors lie in the *xy* plane.



Oblique shock (cont.)

- The jump relation (from eq 6.12) is

$$\rho_1 v_{1x} = \rho_2 v_{2x} \qquad (6.17)$$

$$\rho_1 v_{1x}^2 + p_1 + B_1^2/(2\mu_0) - B_{1x}^2/\mu_0 = \rho_2 v_{2x}^2 + p_2 + B_2^2/(2\mu_0) - B_{2x}^2/\mu_0 \qquad (6.18)$$

$$\rho_1 v_{1x} v_{1y} - B_{1x} B_{1y}/\mu_0 = \rho_2 v_{2x} v_{2y} - B_{2x} B_{2y}/\mu_0 \qquad (6.19)$$

$$\left[\frac{1}{2}\rho_1 v_{1x}^2 + \frac{\gamma}{\gamma - 1} p_1 + \frac{B_1^2}{\mu_0}\right] v_{1x} - (v_{1x} B_{1x} + v_{1y} B_{1y}) \frac{B_{1x}}{\mu_0} = \left[\frac{1}{2}\rho_2 v_{2x}^2 + \frac{\gamma}{\gamma - 1} p_2 + \frac{B_2^2}{\mu_0}\right] v_{2x} - (v_{2x} B_{2x} + v_{2y} B_{2y}) \frac{B_{2x}}{\mu_0} \qquad (6.20)$$

$$B_{1x} = B_{2x} \qquad (6.21)$$

$$v_{1x} B_{1y} - v_{1y} B_{1x} = v_{2x} B_{2y} - v_{2y} B_{2x} \qquad (6.22)$$

Oblique shock (cont.)

- An analysis of the jump relations can be considerably simplified by choosing axis moving along the *y*-axis parallel to the shock front at such a speed that

$$v_{1y} = v_{1x}\frac{B_{1y}}{B_{1x}} \qquad (6.23) \qquad \left(v_{2y} = v_{2x}\frac{B_{2y}}{B_{2x}}\right)$$

- In this frame of reference both side of eq (6.22) vanish, and plasma velocity becomes parallel to the magnetic field on both side of the shock front (***v*** || ***B***) (***v*** x ***B*** =0).

- Using eq (6.17), (6.21), (6.23), we define as

$$\frac{\rho_2}{\rho_1} = \frac{v_{1x}}{v_{2x}} \equiv X, \; \frac{p_2}{p_1} \equiv Y, \; \frac{B_{2y}}{B_{1y}} \equiv Z, \; \frac{v_{2y}}{v_{1y}} \equiv \frac{Z}{X} \qquad (6.24)$$

Oblique shock (cont.)

- From eq (6.19) and (6.23),

$$Z = \frac{(v_{1x}^2 - v_{A1x}^2)X}{v_{1x}^2 - Xv_{A1x}^2}$$

- Here we consider the frame which plasma velocity becomes parallel to the magnetic field. So

$$v_{A1} \equiv B_1/(\mu_0\rho_1)^{1/2}$$

$$Z = \frac{B_{2y}}{B_{1y}} = \frac{(v_1^2 - v_{A1}^2)X}{v_1^2 - Xv_{A1}^2} \quad (6.25)$$

- From eq (6.24) and (6.25),

$$\frac{v_{2y}}{v_{1y}} = \frac{v_1^2 - v_{A1}^2}{v_1^2 - Xv_{A1}^2} \quad (6.26)$$

- From eq (6.20), (6.23) and (6.24),

$$Y = X + \frac{(\gamma - 1)}{2c_{s1}^2} v_1^2 \left(1 - \frac{v_2^2}{v_1^2}\right) X \quad (6.27)$$

Oblique shock (cont.)

- Using eq (6.24) and (6.26),

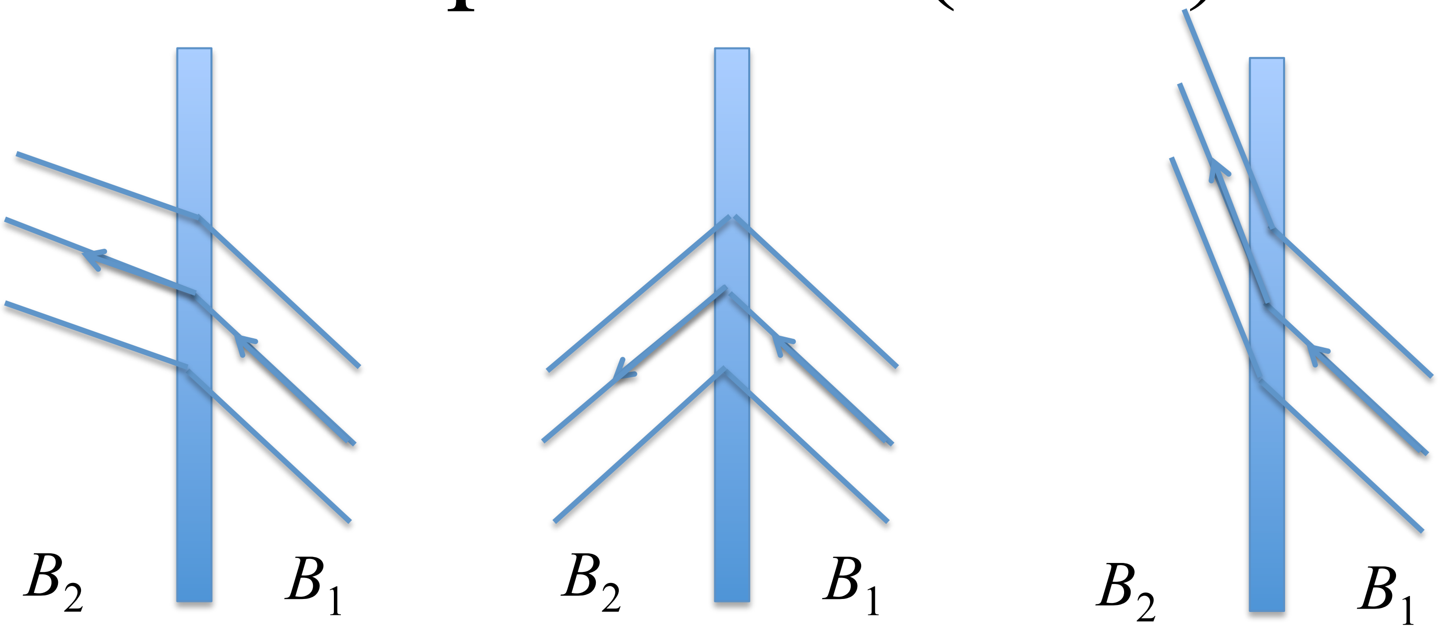
$$v_{2x}^2 = v_1^2 \cos^2\theta \frac{1}{X^2}, \quad v_{2y} = v_1^2 \cos^2\theta \frac{Z^2}{X^2}$$

- Where θ is inclination of upstream magnetic field to the shock normal such that $v_{1x} = v_1 \cos\theta$
- Based on eq (6.27), we can drive the equation related X (need long calculation)

$$(v_1^2 - X v_{A1}^2)^2 \left[X c_{s1}^2 + \frac{1}{2} v_1^2 \cos^2\theta \{X(\gamma - 1) - (\gamma + 1)\} \right]$$
$$+ \frac{1}{2} X v_{A1}^2 v_1^2 \sin^2\theta [\{\gamma + X(\gamma - 2)\} v_1^2 - X v_{A1}^2 \{(\gamma + 1) - X(\gamma - 1)\}] = 0$$

(6.28)

Oblique shock (cont.)



Slow shock | Intermediate shock | fast shock

- Three solutions of eq (6.28), *slow shock*, *intermediate (Alfven) shock* and *fast shock*
- In the limit as X => 1 (no compression), reduce to three waves
  - $v_1^2 = v_{A1}^2$ for Alfven wave
  - $v_{1x}^4 - (c_{s1}^2 + v_{A1}^2)v_{1x}^2 + c_{s1}^2 v_{A1}^2 \cos^2\theta = 0$ for the propagation speeds of slow and fast magnetosonic waves

Slow and Fast shocks

- Consider first, the slow and fast shocks.
- They are compressive, with $X > 1$, which implies that $p_2 > p_1$
- From $X > 1$,

$$\frac{B_{2y}}{B_{1y}} = \frac{v_1^2 - v_{A1}^2}{v_1^2 - X v_{A1}^2} X = \frac{v_1^2 - v_{A1}^2}{v_1^2 / X - v_{A1}^2}$$

- When $(X v_{A1}^2 >) v_{A1}^2 > v_1^2$, this equation implies that $B_{2y}/B_{1y} < 1$, called a *slow shock.*
- Magnetic field is refracted towards the shock normal and its strength decreases as the shock front passes by
- When $v_1^2 > X v_{A1}^2 (> v_{A1}^2)$, this equation implies that $B_{2y}/B_{1y} > 1$, called a *fast shock.*
- Magnetic field is refracted away from the shock normal and its strength increases as the shock front passes by

Slow and Fast shocks (cont.)

- Evolutionary condition: shock speed (v_{1x}) relative to unshocked plasma must exceed the characteristic wave speed (slow magnetosonic wave speed in the case of slow shock, fast magnetosonic wave speed in the case of fast shock)
- The effect of shock is to slow down the flow in x-direction ($v_{2x} < v_{1x}$)
- The flow in y-direction is slowed down for a slow shock ($v_{2y} < v_{1y}$) but speeded up for a fast shock ($v_{2y} > v_{1y}$)

- In the limit B_x => 0, field becomes purely tangential, fast shock becomes a perpendicular shock.
- Slow shock reduces to a tangential discontinuity, for which both flow velocity and B-field are tangential to plane of discontinuity since $v_{1x}=v_{2x}=B_{1x}=B_{2x}=0$.

Slow and Fast shocks (cont.)

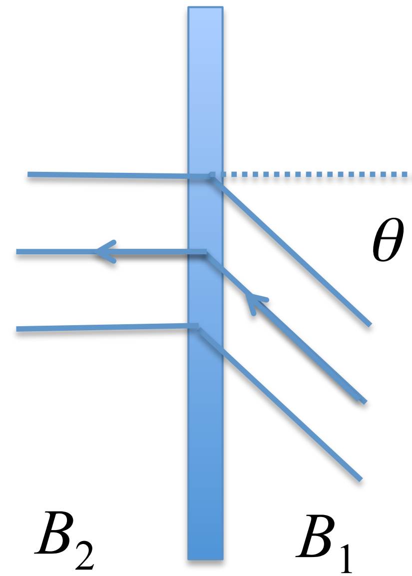
- The tangential discontinuity ($v_{1x}=v_{2x}=B_{1x}=B_{2x}=0$) is a boundary between two distinct plasmas, at which the jumps in the tangential components of velocity (v_y) and B-field (B_y) are arbitrary.
- Subject only to the condition,

$$p_1 + \frac{B_1^2}{2\mu_0} = p_2 + \frac{B_2^2}{2\mu_0}$$

- That total pressure is continuous

- Consider the case: $B_{1x} = B_{2x} \neq 0,\ v_{1x} = v_{2x} = 0$
- Magnetic field lines cross the boundary but there is no flow across it
- One trivial solution: velocity, magnetic field and pressure is continuous but density (temperature) may be discontinuous ($p_2=p_1$).
- This is known as a contact (entropy) discontinuity

Switch-off shock



Switch-off shock

- Two special cases of slow and fast shocks are of particular interest, so-called *switch-off shock* and *switch-on shock*.
- They occur in the limit when $v_1 = v_{A1}$ and $X \neq 1$
- From eq (6.25),

$$\frac{B_{2y}}{B_{1y}} = \frac{(v_1^2 - v_{A1}^2)X}{v_1^2 - Xv_{A1}^2}$$

- Even B_{1y} is non-zero, The tangential magnetic field component behind the shock (B_{2y}) can be vanished (slow shock).
- Such the case, we called *switch-off shock*
- Since $\boldsymbol{v}_1$ and $\boldsymbol{B}_1$ are parallel, a switch-off shock propagates at a Alfven speed $v_{1x} = v_{A1x} (= B_{1x}/(\mu_0 \rho_1)^{1/2})$ based on the normal B-field component

Switch-off shock (cont.)

- Using $v_1 = v_{A1}$ and $X \neq 1$, eq (6.28) reduces to

$$f(X) \equiv (2A + \gamma - 1)X^2 - \{2A + \gamma(1 + \cos^2\theta)\}X + (\gamma + 1)\cos^2\theta = 0 \tag{6.29}$$

Where $A = c_{s1}^2 / v_{A1}^2$

- The solution $f(X) = 0$ is

$$X = \frac{2A + \gamma(1 + \cos^2\theta) \pm \sqrt{(2A + \gamma(1 + \cos^2\theta))^2 - 4A(2A + \gamma - 1)(\gamma + 1)\cos^2\theta}}{2(2A + \gamma - 1)} \tag{6.30}$$

- When $\theta \Rightarrow 0$, ~~$X = 1$~~ or $\dfrac{\gamma + 1}{2A + \gamma - 1}$

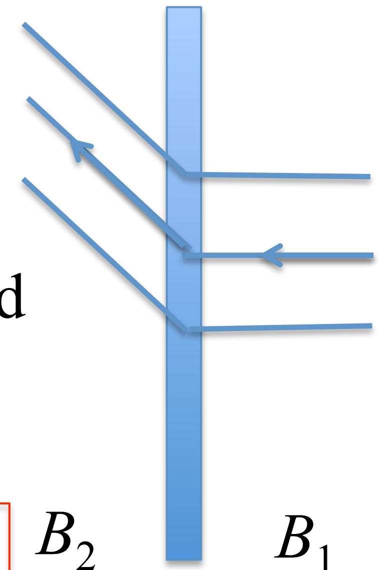
- When $\theta \Rightarrow \pi/2$, ~~$X = 0$~~ or $1 + (2A + \gamma - 1)^{-1}$

Switch-off shock (cont.)

- The behavior of solution is different with the value of $A > 1$, or $0 < A < 1$ $\quad (A = c_{s1}^2/v_{A1}^2)$

- When $A > 1$, one of the solution for θ=0 case is reduced as $(\gamma+1)/(2A+\gamma-1) < 1$
- It is not appropriate solution for shock (because X < 1).
- So $A > 1$ case, the solution for X increases $1 \rightarrow 1+(2A+\gamma-1)^{-1}$ as the angle of incidence θ increase $\theta = 0 \rightarrow \pi/2$
- When $1/2 < A < 1$, the solution for X increases $(\gamma+1)/(2A+\gamma-1) \rightarrow 1+(2A+\gamma-1)^{-1}$ as $\theta = 0 \rightarrow \pi/2$
- When $0 < A < 1/2$, the solution for X decreases $(\gamma+1)/(2A+\gamma-1) \rightarrow 1+(2A+\gamma-1)^{-1}$ as $\theta = 0 \rightarrow \pi/2$

Switch-on shock



- Consider a shock propagating along the magnetic field (so that B_{1y} =0, θ=0)
- The eq (6.28) reduces to

$$\left[c_s^2 X + \frac{1}{2}v_1^2\{X(\gamma - 1) - (\gamma + 1)\}\right](v_1^2 - X v_{A1}^2)^2 = 0 \qquad (6.31)$$

Hydrodynamic shock   fast shock

- $X = \dfrac{v_1^2}{v_{A1}^2}$ corresponding to a *switch-on shock*
- Since $X > 1$, this occur only when the shock speed exceeds the Alfven speed ($v_1 > v_{A1}$)

Switch-on shock (cont.)

- A shock propagating along magnetic field. So from eq (6.21), $B_{2x}=B_1=B_{1x}$ ($B_{1y}=0$)
- Elimination of p_2 from eq(6.18) and (6.20) yields,

$$B_{2y}^2/B_{2x}^2 = (X-1)\{(\gamma+1)-(\gamma-1)X - 2\mu_0\gamma p_1/B_{1x}^2\}$$

- Since the right-hand side must be positive, the density ratio X is

$$1 < X < \frac{\gamma + 1 - 2c_{s1}^2/v_{A1}^2}{\gamma - 1}$$

- The upper limit is $(\gamma+1)/(\gamma-1)$ when $v_{A1} >> c_{s1}$
- The switch-on shock can exist only when the Alfven speed exceeds the sound speed in the unshocked plasma

Switch-on shock (cont.)

- As X increases from 1, the deflection of field line (B_{2y}^2/B_{2x}^2) increases from 0 to a maximum value of

$$4(1 - c_{s1}^2/v_{A1}^2)^2/(\gamma - 1)^2 \ \text{at} \ X = (\gamma - c_{s1}^2/v_{A1}^2)/(\gamma - 1)$$

- Then it decreases to 0 at $X = (\gamma + 1 - 2c_{s1}^2/v_{A1}^2)/(\gamma - 1)$

Intermediate shock

- When the wave-front propagates at the Alfven speed in the unshocked plasma, $v_1=v_{A1}$, one solution of eq (6.28) is $X=1$ (another solution is fast & slow shocks).
- From eq (6.22) and (6.23),

$$v_{2y}/v_{1y} = B_{2y}/B_{1y}$$

- From eq (6.18) and (6.20),

$$p_2 = p_1, \quad B_{2y}^2 = B_{1y}^2$$

- Thus, in addition to the trivial solution $\boldsymbol{B}_2=\boldsymbol{B}_1$, we have

$$B_{2y} = -B_{1y}, \quad B_{2x} = B_{1x},$$
$$v_{2y} = -v_{1y}, \quad v_{2x} = v_{1x},$$

for an *intermediate* (or *transverse*) *wave* (or *rotational discontinuity for no density change*)

Intermediate shock (cont.)

- The tangential magnetic field component is reversed by the wave, and within the wave front magnetic field simply rotates out of the plane maintaining a constant magnitude
- This is just a finite-amplitude Alfven wave
- No change in pressure => not shock

Summery

- Shock wave, $v_n \neq 0$: Flow crosses surface of discontinuity accompanied by compression and dissipation

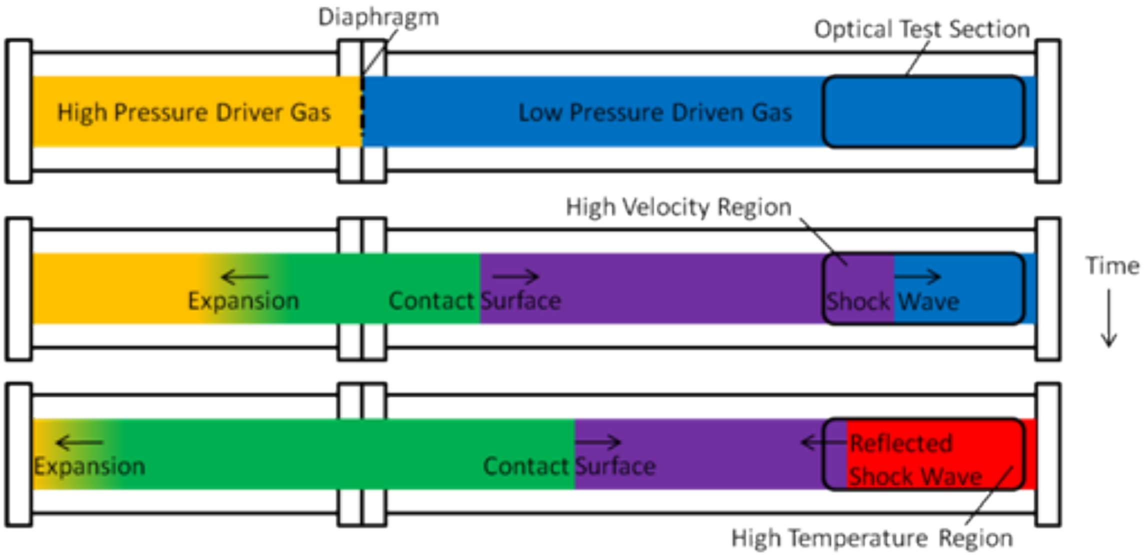
Parallel shock	$B_t = 0$	Magnetic field unchanged by shock (hydrodynamic shock)
Perpendicular shock	$B_n = 0$	Plasma pressure and field strength increases at shock
Oblique shocks	$B_t \neq 0, \; B_n \neq 0$	
Fast shock	Plasma pressure and field strength increases at shock, magnetic field bend away from normal.	
Slow shock	Plasma pressure increases and field strength decreases at shock, magnetic field bend towards normal	
Intermediate shock	Only shock-like in anisotropic plasma (magnetic field rotate of 180^{O} in plane of shock, density jump)	

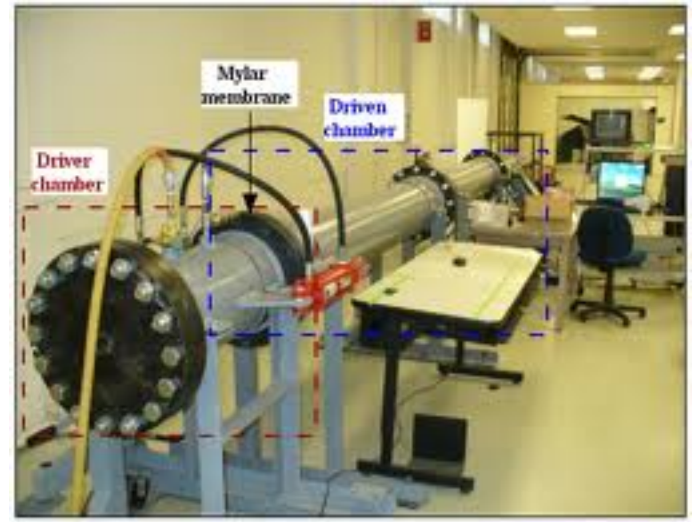
Summary (cont.)

Discontinuity

Contact discontinuity	$v_n = 0, \ B_n \neq 0$	Density jump arbitrary, but other quantities are continuous
Tangential discontinuity	$v_n = 0, \ B_n = 0$	Plasma pressure and field change maintaining static pressure balance (total pressure is constant)
Rotational discontinuity	$v_n = B_n / \sqrt{\mu_0 \rho}$	Form of intermediate shock in isotropic plasma, field and flow change direction but not magnitude

Shock tube problem

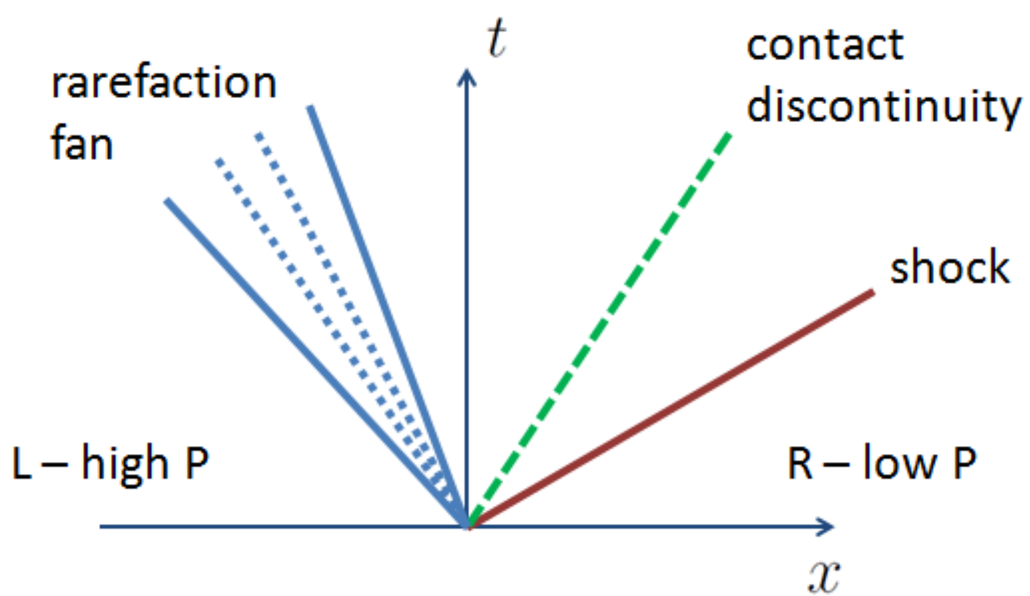




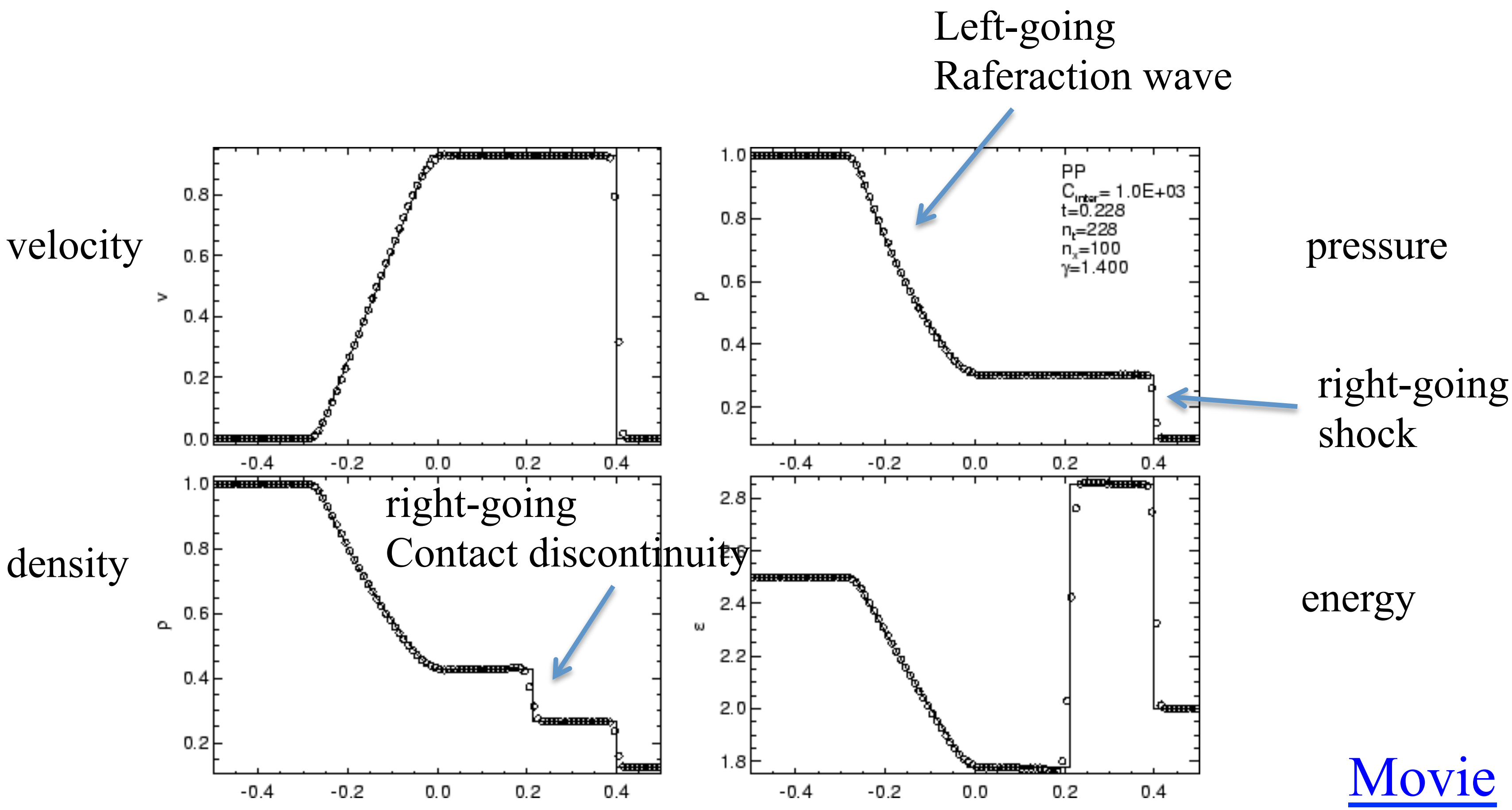
- A common test for the accuracy of computational (magneto-)fluid code and laboratory experiments
- The test consist separated two different states of gas initially
- In time evolution, two different states make propagations of shocks or discontinuities
- We can calculate analytical solutions therefore can test the validity of numerical codes.

Shock tube problem (cont.)

- Sod's shock tube test (the most famous test)
- Initial condition
  - ρ_L=1, p_L=1, v_L=0
  - ρ_R=0.125, p_R=0.1, v_R=0, with γ=1.4
- Results
  - Three Characteristic velocities
  - v \pm c_s (shock/rafefaction to right/left)
  - v (entropy waves)



Shock tube problem (cont.)

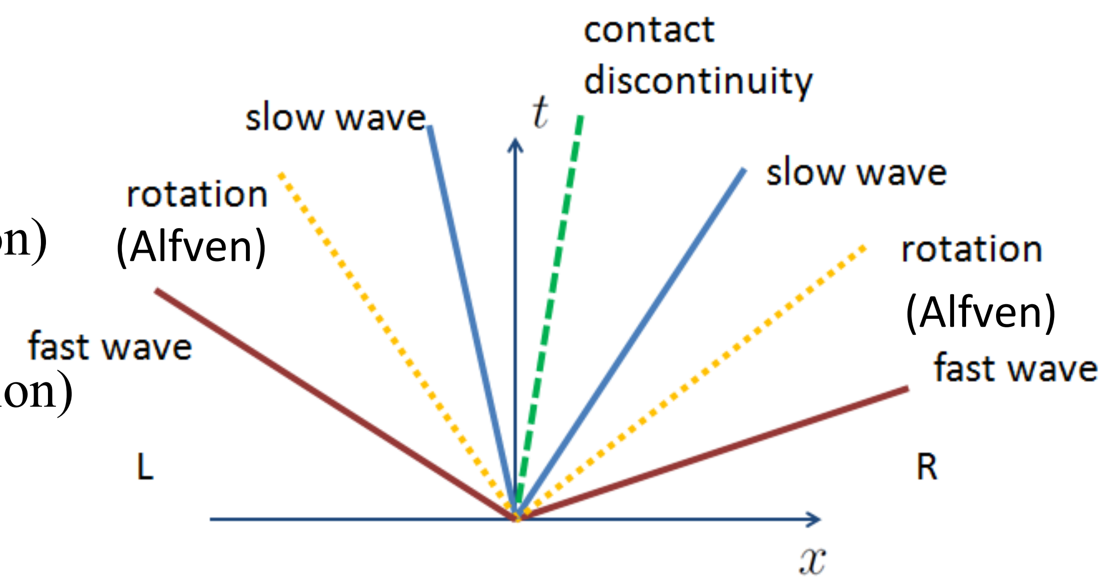


[Movie]

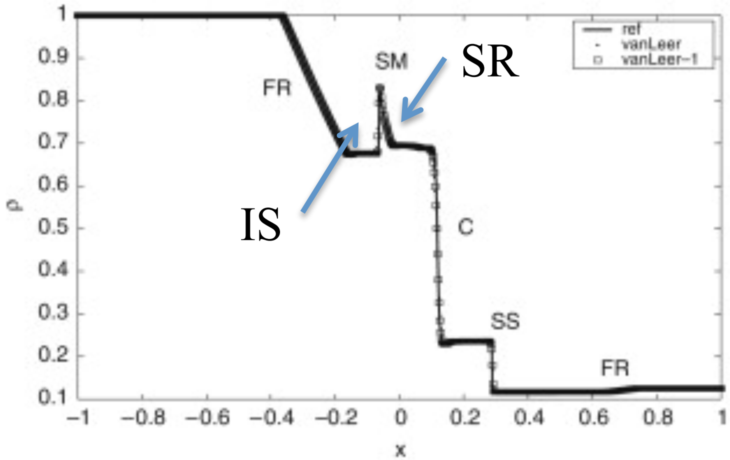
MHD shock tube problem

- Brio & Wu MHD shock tube test (1D)
- Presence of magnetic field, shock structure becomes much complicated
- Initial condition
  - $\rho_L=1$, $p_L=1$, $v_L=0$, $B_{yL}=-1$, $B_{zL}=0$
  - $\rho_R=0.125$, $p_R=0.1$, $v_R=0$, $B_{yR}=1$, $B_{zR}=0$ with $B_x=0.75$ and $\gamma=2$

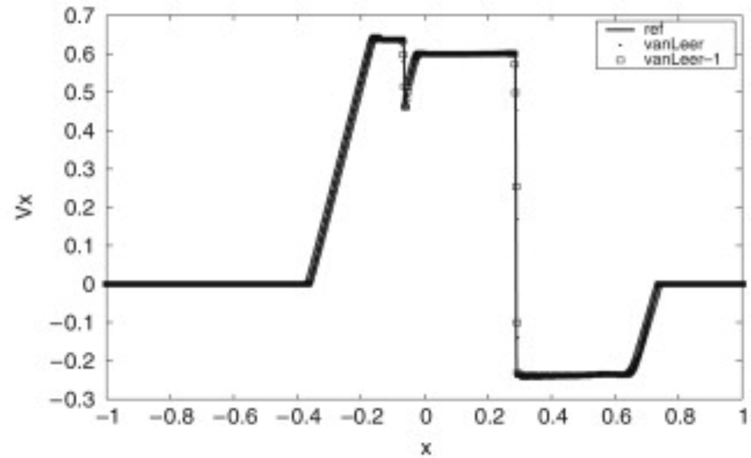
- Results
  - 7 characteristic velocities
  - v \pm v_f (fast shock/rarefaction)
  - v \pm v_A (rot discont.)
  - v \pm c_s (slow shock/rarefaction)
  - v (entropy wave)



density

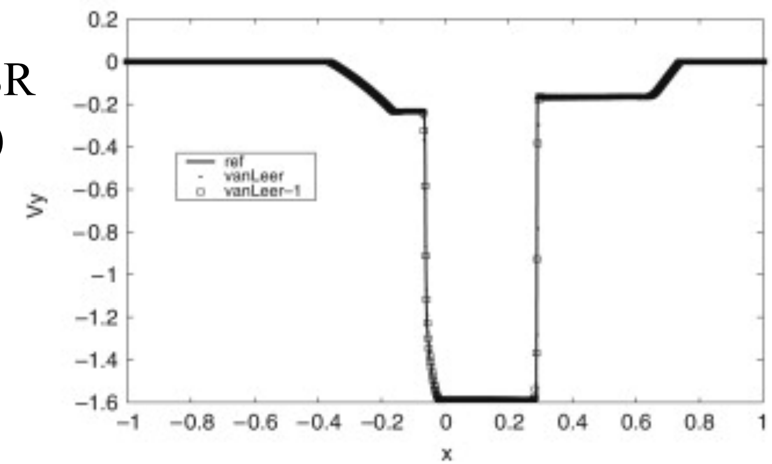


v_x



Regular solution: SS
Non-regular solution: IS + SR
(Takahashi & Yamada 2013)

v_y



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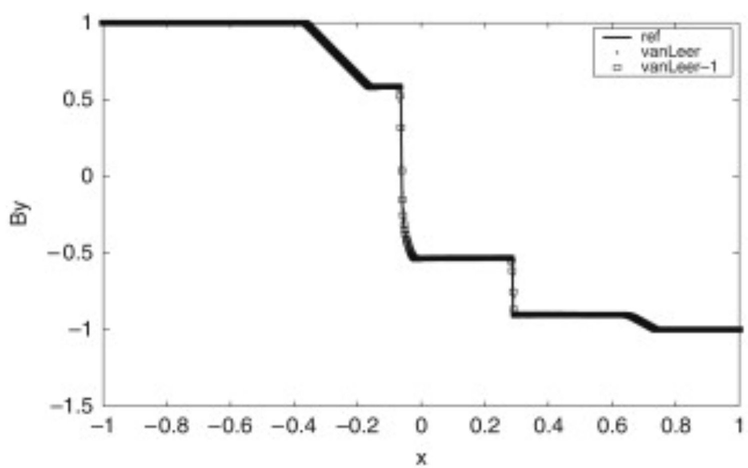
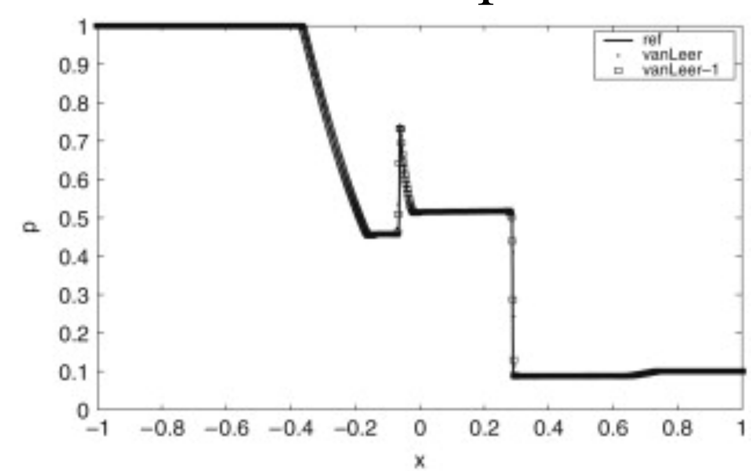
pressure

B_y





Astrophysical shocks

- Shock waves are common in astrophysical environments
  - Bow (termination shock) shock of solar system (interaction between solar wind and interstellar medium)
  - Supernova remnants (blast wave)
  - Shock traveling through a massive star as it explodes in core collapse supernova
  - Shock in insterstellar medium, caused by the collision between molecular clouds or by a gravitational collapse of clouds
  - Accretion shock in cluster of galaxies
  - Gamma-ray bursts (relativistic blast wave)
  - Shocks in astronomical jets
  - Termination shock in pulsar wind nebulae
- Shock is related particle accelerations (Fermi acceleration)