Plasma Astrophysics
Chapter 4: Single-Fluid Theory of Plasma - Magnetohydrodynamics

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Exercise 1

- Plasma frequency: $\omega_p = \sqrt{\frac{ne^2}{m_e \epsilon_0}}$
- Debye length: $\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{e^2 n}}$
- Plasma number: $\Lambda = 4\pi n \lambda_D^3 = \frac{1.38 \times 10^6 T_e^{3/2}}{n^{1/2}}$
- Mean free path:

$$\lambda_{mfp} \approx \frac{36\pi}{n} \left(\frac{\epsilon_0 k_B T}{e^2} \right)^2$$

$$\approx 36\pi n \lambda_D^4 \sim \lambda_D N_D$$

Exercise 1 (cont.)

- Gyro frequency: $\omega_c = -\frac{qB}{m}$
- Larmor radius: $r_L = \frac{mv_\perp}{|q|B} = \frac{v_\perp}{\omega_c}$

- Electron volts is energy units = particle's kinetic energy
- For Larmor radius, we need to get perpendicular components of velocity.

Single-Fluid Theory: MHD

- Under certain circumstances, appropriate to consider entire plasma as a single fluid.
- Do not have any difference between ions and electrons.
- Approach is called *magnetohydrodynamics (MHD)*.
- General method for modeling highly conductive fluids, including low-density astrophysical plasmas.
- Single-fluid approach appropriate when dealing with slowly varying conditions.
- MHD is useful when plasma is highly ionized and electrons and ions are forced to act in unison, either because of frequent collisions or by the action of a strong external magnetic field.

Single-fluid equations for fully ionized plasma

- Can combine multiple-fluid equations into a set of equations for a single fluid.
- Assuming two-specials plasma of electrons and ions ($j = e$ or i):

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{v}_j) = 0 \tag{4.1a}$$

$$m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla) \boldsymbol{v}_j \right] = -\nabla \cdot \boldsymbol{P}_j + q_j n_j (\boldsymbol{E} + \boldsymbol{v_j} \times \boldsymbol{B}) + P_{ij} \tag{4.1b}$$

- For a fully ionized two-species plasma, total momentum must be conserved:

$$P_{ei} = -P_{ie}$$

- As $m_i >> m_e$ the time-scales in continuity and momentum equations for ions and electrons are very different. The characteristic frequencies of a plasma, such as plasma frequency or cyclotron frequency are much larger for electrons.

Single-fluid equations for fully ionized plasma (cont.)

- When plasma phenomena are large-scale ($L >> \lambda_D$) and have relatively low frequencies ($\omega << \omega_{plasma}$ and $\omega << \omega_{cyclotron}$), on average plasma is electrically neutral ($n_i \sim n_e$). Independent motion of electrons and ions can then be neglected.
- Can therefore treat plasma as single conducting fluid, whose inertia is provided by mass of ions.
- Governing equations are obtained by combining eqn (4.1)
- First, define macroscopic parameters of plasma fluid:

$$\rho_m = n_e m_e + n_i m_i$$ Mass density

$$\rho_e = n_e q_e + n_i q_i$$ Charge density

$$\boldsymbol{J} = n_e q_e v_e + n_i q_i v_i = n_e q_e (v_e - v_i)$$ Electric current

$$\boldsymbol{v} = (n_e m_e \boldsymbol{v}_e + n_i m_i \boldsymbol{v}_i)/\rho_m$$ Center of Mass Velocity

$$\boldsymbol{P} = \boldsymbol{P}_e + \boldsymbol{P}_i$$ Total pressure tensor

MHD mass and charge conservation

- Using eq (4.1a): $\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \boldsymbol{v}_j) = 0$
- Multiply by q_i and q_e and add continuity equations to get:

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\boldsymbol{J}) = 0$$

Charge conservation

- where J is the electric current density: $\boldsymbol{J} = n_e q_e \boldsymbol{v}_e + n_i q_i \boldsymbol{v}_i$ and the electric charge: $\rho_e = n_e q_e + n_i q_i$
- Multiply eq (4.1a) by m_i and m_e,

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0$$

Mass conservation / continuity equation

- where $\rho_m = n_e m_e + n_i m_i$ is the single-fluid mass density and $\boldsymbol{v}$ is the fluid mass velocity

$$\boldsymbol{v} = (n_e m_e \boldsymbol{v}_e + n_i m_i \boldsymbol{v}_i)/\rho_m$$

MHD equation of motion

- Equation of motion for bulk plasma can be obtained by adding individual momentum transport equations for ions and electrons.
- LHS of eq(4.1b): $m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla)\boldsymbol{v}_j \right]$
- Difficulty is that convective term is non-linear.
- But note that since $m_e << m_i$ contribution of electron momentum is much less than that from ion. So we ignore it in equation
- Approximation: Center of mass velocity is ion velocity: $\boldsymbol{v} \simeq \boldsymbol{v}_i$
- LHS:

$$m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla)\boldsymbol{v}_j \right] \simeq \rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right]$$

MHD equation of motion (cont.)

- RHS of eq(4.1b) : $-\nabla \cdot (\boldsymbol{P}_e + \boldsymbol{P}_i) + (n_e q_e + n_i q_i)\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}$

- In general, second term (Electric body force) is much smaller than ***J*** x ***B*** term. So we ignored.
- Therefore, LHS+RHS:

$$\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla \cdot \boldsymbol{P} + \boldsymbol{J} \times \boldsymbol{B}$$

Equation of motion

- For an isotropic plasma, $\nabla \cdot \boldsymbol{P} = \nabla p$ where total pressure is $p = p_e + p_i$ and

$$\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}$$

Equation of motion

MHD equation of motion (cont.)

- $\rho_e \boldsymbol{E}$ term is generally much smaller than $\boldsymbol{J} \times \boldsymbol{B}$ term. To see this take order of magnitudes.
- from Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = \rho_e/\epsilon_0 \quad \text{so} \quad \rho_e \sim E\epsilon_0/L$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \quad \text{so} \quad \sigma E \sim j \sim B/\mu_0 L$$

- Therefore,

$$\frac{\rho_e E}{jB} \sim \frac{\epsilon_0}{L}\left(\frac{B^2}{\mu_0 \sigma L}\right)^2 \frac{L\mu_0}{B^2} \sim \frac{L^2/c^2}{(\mu_0 \sigma L^2)^2} = \left(\frac{\text{light crossing time}}{\text{resistive skin time}}\right)^2$$

- This is generally very small number.
- Example: small cold plasma, T_e=1eV, L=1cm, this ratio is about 10^{-8}

Generalized Ohm's law

- The final single-fluid MHD equation describes the variation of current density $\boldsymbol{J}$.
- Consider the momentum equations for electron and ions (eq.4.1b):

$$m_j n_j \left[\frac{\partial \boldsymbol{v}_j}{\partial t} + (\boldsymbol{v}_j \cdot \nabla) \boldsymbol{v}_j \right] = -\nabla \cdot \boldsymbol{P}_j + q_j n_j (\boldsymbol{E} + \boldsymbol{v_j} \times \boldsymbol{B}) + P_{ij}$$

- Multiple electron equation by q_e/m_e and ion equation by q_i/m_i and add:

$$\begin{aligned} \frac{\partial \boldsymbol{J}}{\partial t} &= -\frac{q_e}{m_e} \nabla \cdot \boldsymbol{P}_e - \frac{q_i}{m_i} \nabla \cdot \boldsymbol{P}_i \\ &+ \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i} \right) \boldsymbol{E} \\ &+ \left(\frac{n_e q_e^2}{m_e} \boldsymbol{v}_e + \frac{n_i q_i^2}{m_i} \boldsymbol{v}_i \right) \times \boldsymbol{B} \\ &+ \frac{q_e}{m_e} \boldsymbol{P}_{ei} + \frac{q_i}{m_i} \boldsymbol{P}_{ie} \end{aligned}$$

(We ignore second term of LHS as we dealing with small perturbation)

Generalized Ohm's law (cont.)

In forth term of RHS:

$$
\begin{aligned}
&\frac{n_e q_e^2}{m_e}\boldsymbol{v}_e + \frac{n_i q_i^2}{m_i}\boldsymbol{v}_i \\
&= \frac{q_e q_i}{m_e m_i}\left(\frac{n_e q_e m_i}{q_i}\boldsymbol{v}_e + \frac{n_i q_i m_e}{q_e}\boldsymbol{v}_i\right) \\
&= -\frac{q_e q_i}{m_e m_i}\left[n_e m_e \boldsymbol{v}_e + n_i m_i \boldsymbol{v}_i - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e}\right)(q_e n_e \boldsymbol{v}_e + q_i n_i \boldsymbol{v}_i)\right] \\
&= -\frac{q_e q_i}{m_e m_i}\left[\rho_m \boldsymbol{v} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e}\right)\boldsymbol{J}\right] \\
&= \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)\boldsymbol{v} + \left(\frac{q_e}{m_e} + \frac{q_i}{m_i}\right)\boldsymbol{J}
\end{aligned}
$$

Generalized Ohm's law (cont.)

- For an electrically neutral plasma $|q_e n_e| \approx |q_i n_i|$ and using $\boldsymbol{J} = n_e q_e \boldsymbol{v}_e + n_i q_i \boldsymbol{v}_i$ and $\boldsymbol{v} = (n_e m_e \boldsymbol{v}_e + n_i m_i \boldsymbol{v}_i)/\rho_m$, We can write

$$\begin{aligned}\frac{\partial \boldsymbol{J}}{\partial t} \quad = \quad & -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e - \frac{q_i}{m_i}\nabla \cdot \boldsymbol{P}_i \\ & + \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \\ & + \left(\frac{q_e}{m_e} + \frac{q_i}{m_i}\right)(\boldsymbol{J} \times \boldsymbol{B}) \\ & + \left(\frac{q_e}{m_e} - \frac{q_i}{m_i}\right)\boldsymbol{P}_{ei}\end{aligned}$$

- As $m_e \ll m_i \rightarrow q_e/m_e \gg q_i/m_i$ and $n_e q_e^2/m_e \gg n_i q_i^2/m_i$. In thermal equilibrium, kinetic pressures of electrons is similar to ion pressure ($P_{\mathrm{e}} \sim P_{\mathrm{i}}$)

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e + \frac{n_e q_e^2}{m_e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \frac{q_e}{m_e}(\boldsymbol{J} \times \boldsymbol{B}) + \frac{q_e}{m_e}\boldsymbol{P}_{ei} \quad (4.2)$$

Generalized Ohm's law (cont.)

- The collisional term can be written: $\boldsymbol{P}_{ei} = \eta q^2 n_e^2 (\boldsymbol{v}_i - \boldsymbol{v}_e)$ where η is the specific resistivity, q^2 relates to fact that collisions result from Coulomb force between ions (q_i) and electrons (q_e) and total momentum transferred to electrons in an elastic collision with an ion is $\boldsymbol{v}_i - \boldsymbol{v}_e$.
- Now $q_i = - q_e$ and $n_e = n_i$ and $\boldsymbol{J} = n_e q_e(\boldsymbol{v}_e - \boldsymbol{v}_i)$, => $\boldsymbol{P}_{ei} = -n_e q_e \eta \boldsymbol{J}$
- Eq. (4.2) can be written as

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e + \frac{n_e q_e^2}{m_e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \frac{q_e}{m_e}(\boldsymbol{J} \times \boldsymbol{B}) - \frac{n_e q_e^2}{m_e}\hat{\eta} \cdot \boldsymbol{J} \tag{4.3}$$

- Where η is a tensor. This is *generalized Ohm's law*

Generalized Ohm's law (cont.)

- For a steady current in a uniform $\boldsymbol{E}$, $\partial \boldsymbol{J}/\partial t = 0,\ \nabla \cdot \boldsymbol{P} = 0$ and $\boldsymbol{B} = 0$ so that

$$\boldsymbol{E} = \eta \boldsymbol{J} \rightarrow \boldsymbol{J} = 1/\eta \boldsymbol{E}$$

- In general form, the electric field $\boldsymbol{E}$ can be found from Eq (4.3):

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{J} \times \boldsymbol{B}}{n_e q_e} + \frac{\nabla \cdot \boldsymbol{P}}{n_e q_e} + \hat{\eta} \cdot \boldsymbol{J} + \frac{m_e}{n_e q_e} \frac{\partial \boldsymbol{J}}{\partial t}$$

- Consider right hand side of this equation:
  - First term: $\boldsymbol{E}$ associated with plasma motion
  - Second term: Hall effect
  - Third term: Ambipolar diffusion from E-field generated by pressure gradients
  - Fourth term: Ohmic losses/Joule heating by resistivity
  - Fifth term: Electron inertia

One fluid MHD Ohm's law

- Generalized Ohm's law

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\frac{q_e}{m_e}\nabla \cdot \boldsymbol{P}_e + \frac{n_e q_e^2}{m_e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \frac{q_e}{m_e}(\boldsymbol{J} \times \boldsymbol{B}) - \frac{n_e q_e^2}{m_e}\hat{\eta} \cdot \boldsymbol{J}$$

- Now assume plasma is isotropic, so that $\nabla \cdot \boldsymbol{P} = \nabla p$
  Also we neglect Hall effect and Ambipolar diffusion in generalized Ohm's law since not important in one-fluid MHD.
  For slow variations, $\boldsymbol{J}$ = constant, so can write generalized Ohm's law as:

$$0 = \frac{n_e q_e^2}{m_e}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) - \frac{n_e q_e^2}{m_e}\eta \boldsymbol{J}$$

- Rearranging gives,

$$\boxed{\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})}$$ One-fluid MHD Ohm's law

- Where $\sigma = 1/\eta$ is electrical conductivity

Simplified MHD equations

- A set of simplified MHD equations can be written:

$$\begin{aligned} \frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\ \rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= -\nabla p + \boldsymbol{J} \times \boldsymbol{B} \\ \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} &= \eta \boldsymbol{J} \end{aligned}$$

- Fluid equations must be solved with reduced Maxwell equations

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{E} = 0$$

(displacement current term is ignored for low frequency phenomena)

- Here we have assumed that there is no accumulation of charge (i.e., $\rho_e = 0$)
- Complete set of equations only when *equation of state* for relationship between *p* and *n* (ρ) is specified.

$$p\rho_m^{-\gamma} = const$$

Plasma β

- The MHD equation of motion contains *J* x *B* term, which can given rise to effects that are similar to those of the pressure term.
- Current is given by $\boldsymbol{J} = \frac{1}{\mu_0}\nabla \times \boldsymbol{B}$
- Taking cross product with the magnetic field,

$$\boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \frac{1}{\mu_0}\left[(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla\left(\frac{B^2}{2}\right)\right]$$

- Inserting into MHD equation of motion

$$\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right] = \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla\left(p + \frac{B^2}{2\mu_0}\right)$$

- In second term of RHS, the first term acted on by gradient is plasma pressure and the second term is magnetic pressure.
- The dimensionless parameter, plasma β : $\beta \equiv \frac{2\mu_0 p}{B^2}$ *Plasma beta*
- β <<1: dominated by magnetization effects
- β>>1: behaves more like a fluid

The induction equation

- Taking the curl of one-fluid MHD Ohm's law:

$$\nabla \times \boldsymbol{E} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\sigma}\nabla \times \boldsymbol{J}$$

- Assuming σ=*const*. Substituting for $\boldsymbol{J} = \nabla \times \boldsymbol{B}/\mu_0$ from Ampere's law and using the law of induction equations (Faraday's law):

$$-\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma}\nabla \times (\nabla \times \boldsymbol{B})$$

- The double curl can be expanding from vector identity

$$-\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma}\nabla(\nabla \cdot \boldsymbol{B}) - \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B}$$

- The second term in R.H.S. is zero by Gauss's law ($\nabla \cdot \boldsymbol{B} = 0$). So

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma}\nabla^2 \boldsymbol{B}$$

MHD induction equation

The induction equation (cont.)

- The MHD induction equation, together with fluid mass, momentum, and energy equations (EoS), a close set of equations for MHD state variables (ρ_m, $\boldsymbol{v}$, p, $\boldsymbol{B}$)

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\
\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla(p + \frac{B^2}{2\mu_0}) \\
p\rho_m^{\gamma} &= const \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}
\end{aligned}
$$

Here,

$$
\boldsymbol{J} = \nabla \times \boldsymbol{B}/\mu_0
$$

$$
\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \boldsymbol{J}/\sigma
$$

Ideal MHD

- In the case where the conductivity is very high ($\sigma \to \infty$), the electric field is ***E***= -***v*** x ***B*** (motional electric field only). It is known as *ideal Magnetohydrodynamics*.
- A set of equations:

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\
\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla(p + \frac{B^2}{2\mu_0}) \\
p\rho_m^\gamma &= const \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B})
\end{aligned}
$$

- This is the most simplest assumption for MHD. But this is commonly used in Astrophysics.

The pressure equations

- The above formulation of the ideal MHD equations exploits ρ, $\boldsymbol{v}$, p, $\boldsymbol{B}$ as the basic variables
- Equation of states is often replaced by pressure evolution equation.
- It is also work out the evolution equation for the other thermodynamical variables, such as
  - e: internal energy per unit mass (which is equivalent to T)
  - s: entropy per unit mass

$$e \equiv \frac{1}{\gamma - 1} \frac{p}{\rho_m} \approx C_v T$$

C_v: specific heat capacity

$$s \equiv C_v \ln S \ , \qquad S \equiv p/\rho_m^\gamma$$

- Neglect thermal conduction and heat flow, i.e., considering adiabatic processes, the entropy convected by the fluid is constant:

$$\frac{Ds}{Dt} = 0 \ , \quad \text{or} \quad \frac{DS}{Dt} \equiv \frac{D}{Dt}\left(\frac{p}{\rho_m^\gamma}\right) = 0$$

The pressure equations (cont.)

Apply change rule

$$\frac{D}{Dt}\left(\frac{p}{\rho_m^\gamma}\right) = \frac{1}{\rho_m^\gamma}\frac{Dp}{Dt} - \frac{\gamma p}{\rho_m^{\gamma+1}}\frac{D\rho_m}{Dt} = 0$$

Expand D/Dt

$$\frac{1}{\rho_m^\gamma}\frac{\partial p}{\partial t} + \frac{1}{\rho_m^\gamma}(\boldsymbol{v}\cdot\nabla)p - \frac{\gamma p}{\rho_m^{\gamma+1}}\frac{\partial \rho_m}{\partial t} - \frac{\gamma p}{\rho_m^{\gamma+1}}(\boldsymbol{v}\cdot\nabla)\rho_m = 0$$

$$\frac{\partial p}{\partial t} + (\boldsymbol{v}\cdot\nabla)p - \frac{\gamma p}{\rho_m}\left[\frac{\partial \rho_m}{\partial t} + (\boldsymbol{v}\cdot\nabla)\rho_m\right] = 0$$

But $-\left[\frac{\partial \rho_m}{\partial t} + (\boldsymbol{v}\cdot\nabla)\rho_m\right] = \rho_m \nabla\cdot\boldsymbol{v}$

$$\frac{\partial p}{\partial t} + (\boldsymbol{v}\cdot\nabla)p + \frac{\gamma p}{\rho_m}(\rho_m\nabla\cdot\boldsymbol{v}) = 0$$

$$\boxed{\frac{\partial p}{\partial t} + (\boldsymbol{v}\cdot\nabla)p = -\gamma p \nabla\cdot\boldsymbol{v}}$$

Pressure evolution equation

The internal energy equation

- From pressure evolution equations, using equations of state

$$p = (\gamma - 1)\rho_m e$$

we can write the internal energy equations

$$\frac{\partial e}{\partial t} + (\boldsymbol{v} \cdot \nabla)e = -(\gamma - 1)e\nabla \cdot \boldsymbol{v}$$

Internal energy equation

Magnetic field behavior in MHD

- MHD induction equation: $$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \boldsymbol{B}$$
- $\nabla \times (\boldsymbol{v} \times \boldsymbol{B})$ Dominant: convection
  - Infinite conductivity limit: ideal MHD.
  - Flow and field are intimately connected. Field lines convect with the flow. (*flux fleezing*)
  - The flow response to the field motion via ***J*** x ***B*** force
- $(1/\mu_0 \sigma)\nabla^2 \boldsymbol{B}$ Dominant: Diffusion
  - Induction equation takes the form of a diffusion equation.
  - Field lines diffuse through the plasma down any field gradient
  - No coupling between magnetic field and fluid flow
  - Characteristic Diffusion time: $\tau = \mu_0 \sigma L^2 = \mu_0 L^2 / \eta$

Here using $\nabla = 1/L$

- Ratio of the convection term to the diffusion term:

$$R_m = \frac{\boldsymbol{v}\boldsymbol{B}/L}{\boldsymbol{B}/\mu_0 \sigma L^2} = \mu_0 \sigma \boldsymbol{v} L$$

Magnetic Reynold's number

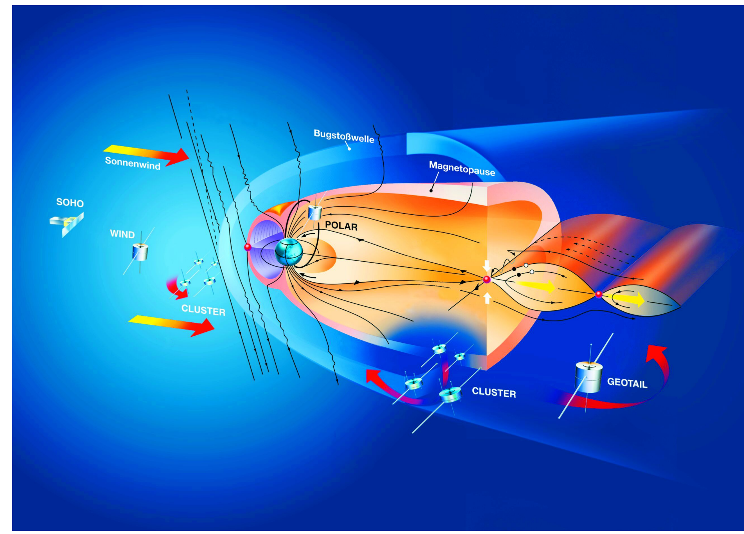
Magnetic field behavior in MHD (cont.)

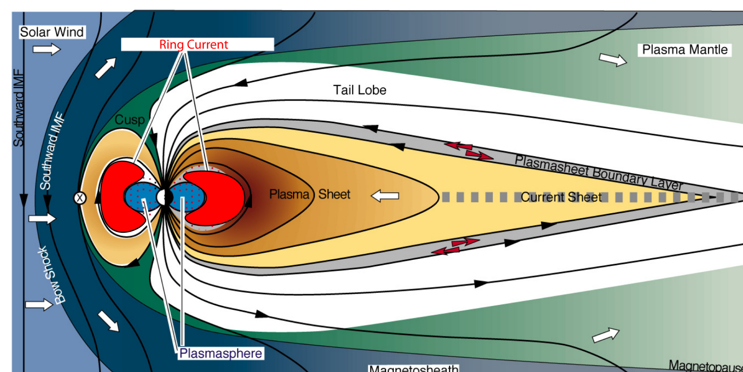
Magnetic Reynold's number

$$R_m = \frac{\boldsymbol{v}\boldsymbol{B}/L}{\boldsymbol{B}/\mu_0\sigma L^2} = \mu_0\sigma\boldsymbol{v}L$$

- If R_m is large, convection dominates, magnetic field frozen into the plasma.
  Else if R_m is small, diffusion dominates.
- In astrophysics generally, R_m is very large.
  - Solar flare: 10^8,
  - planetary magnetosphere: 10^{11}
- But, not large everywhere
  - Thin boundary layers form where $R_m \sim 1$ and ideal MHD breaks down

Earth's magetosphere





Magnetic field behavior in MHD (cont.)

- Rewrite continuity equation:

$$\frac{\partial \rho_m}{\partial t} = -\rho_m(\nabla \cdot \boldsymbol{v}) - (\boldsymbol{v} \cdot \nabla)\rho_m$$

  - first term describes compression (fluid contracts or expansion)
  - Second term describes advection
- The induction equation (ideal MHD) can be written as, using standard vector identities:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{B}(\nabla \cdot \boldsymbol{v}) - (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}$$

- Equation is similar to continuity equation.
  - First term: compression
  - Second term: advection
  - Third term: new term describes stretching. It is related magnetic field amplification

Flux freezing

- Alfven's theorem (1947): "field is frozen into the fluid"
- This is extremely important concept in MHD, since it allows us to study the evolution of the field by finding out about the plasma flow

- MHD induction equation: $\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$

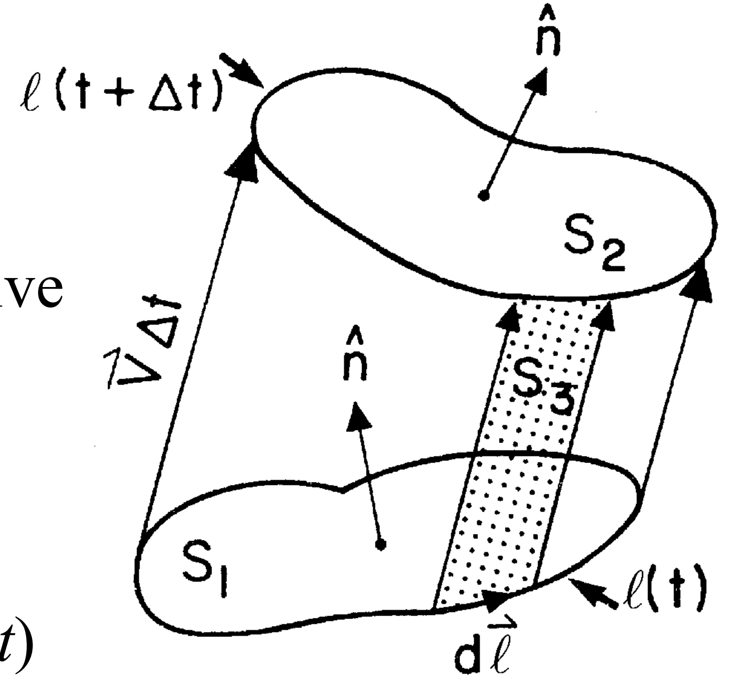
- The magnetic flux though a closed loop *l* : $\Phi_B \equiv \oint_l \boldsymbol{B} \cdot \hat{n} dS$

Where *dS* is the area element of any surfaces which has *l* as a perimeter. The quantity Φ_B is independent of the specific surface chosen, as can be proven from $\nabla \cdot \boldsymbol{B} = 0$.

- So the flux freezing law is expressed as: $\frac{d\Phi_B}{dt} = 0$

where use total derivative *d*/*dt* to indicate that the time derivative is calculated with respect to fluid elements moving with the flow

Flux freezing (cont.)



- The quantity Φ_B is not locally defined.
  So explicit calculation for its time derivative
- Consider a loop of fluid elements *l* at two instants in time, *t* and *t+Δt*
- Two surfaces S_1 and S_2 have $l(t)$ and $l(t+\Delta t)$
- "cylinder" S_3 generated by the fluid motion between the two instants of the elements making up *l*.
- Let Φ_B be the flux enclosed by *l* and Φ_{B1} be the flux through surface S_1 (similarity for S_2 and S_3)
- Then $$\frac{d\Phi_B}{dt} = \lim_{\Delta t \to 0} \left(\frac{\Phi_{B2}(t+\Delta t) - \Phi_{B1}(t)}{\Delta t} \right)$$

Flux freezing (cont.)

- From $\nabla \cdot \boldsymbol{B} = 0$ the net flux through the surfaces at any time is zero

  $-\Phi_{B1}(t+\Delta t) + \Phi_{B2}(t+\Delta t) + \Phi_{B3}(t+\Delta t) = 0$
- (Note that negative sign indicated as inward into the volume)

- We can eliminate $\Phi_{B2}(t+\Delta t)$ and use definition of flux in expressing Φ_{B1} & Φ_{B3}

$$\frac{d\Phi_B}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\iint_{S1} (\boldsymbol{B}(t+\Delta t) - \boldsymbol{B}(t)) \cdot \hat{n} dS - \iint_{S3} \boldsymbol{B} \cdot \hat{n} dS\right] \tag{4.4}$$

- The first term in RHS in eq (4.4):

$$\iint_{S1} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \hat{n} dS$$

Flux freezing (cont.)

- The area element for S_3 can be written $\hat{n}dS = (\boldsymbol{d}l \times \boldsymbol{v})\Delta t$, where dl is a line element of the loop of fluid elements.
- The second term in RHS of eq (4.4):

$$\iint_{S3} \boldsymbol{B} \cdot \hat{n}dS = \oint_{l(t)} \boldsymbol{B} \cdot (\boldsymbol{l} \times \boldsymbol{v})\Delta t = \oint_{l(t)} (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{d}l\Delta t$$

- By using Stokes theorem to convert the line integral to a surface integral

$$\iint_{S3} \boldsymbol{B} \cdot \hat{n}dS = \iint_{S1} \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \cdot \hat{n}dS\Delta t$$

- So finally putting these results into eq(4.4) :

$$\frac{d\Phi_B}{dt} = \iint \left[\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B})\right] \cdot \hat{n}dS = 0$$

Magnetic pressure and curvature force

- Lorentz force:

$$\boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \frac{1}{\mu_0}\left[(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla\left(\frac{B^2}{2}\right)\right]$$

- First term: *magnetic curvature force*, which relates to rate of change of **B** along the direction of **B**.
- Second term: *magnetic pressure*

- To show the role of magnetic curvature force, we consider

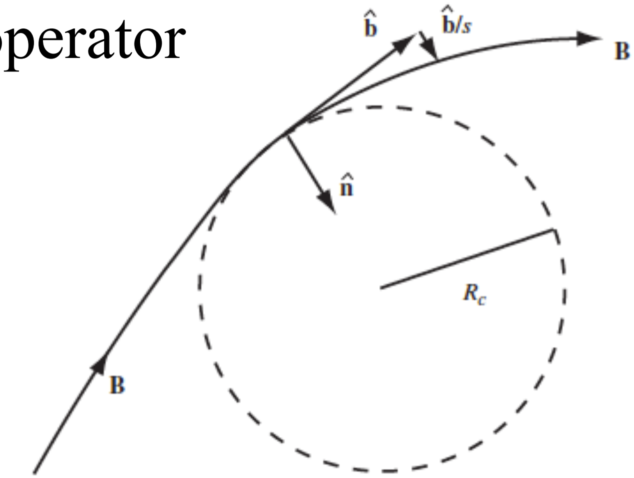
  $\boldsymbol{B} = B\hat{\boldsymbol{b}}$,where B is the local intensity of $\boldsymbol{B}$ and $\hat{\boldsymbol{b}}$ is unit vector
- The Lorentz force then becomes

$$\boldsymbol{F}_L = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\hat{\boldsymbol{b}} \cdot \nabla\hat{\boldsymbol{b}}$$

Magnetic pressure and curvature force (cont.)

- Combine first two term:

$$\boldsymbol{F}_L = -\nabla_\perp\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\hat{\boldsymbol{b}}\cdot\nabla\hat{\boldsymbol{b}}$$



- Where $\nabla_\perp$ is the projection of the gradient operator on a plane perpendicular to $\boldsymbol{B}$
- Second term contains the effects of field line curvature.
- Its magnitude is $\left|\frac{B^2}{\mu_0}\hat{\boldsymbol{b}}\cdot\nabla\hat{\boldsymbol{b}}\right| = \frac{B^2}{\mu_0 R_c}$

  where $R_c = 1/|\hat{\boldsymbol{b}}\cdot\nabla\hat{\boldsymbol{b}}|$ is radius of curvature of path $\hat{\boldsymbol{b}}$
- ($\hat{\boldsymbol{b}}\cdot\nabla \equiv \partial/\partial s$ is the derivative along a field line)
- The curvature force is directed toward a center of curvature ($\hat{n}$). It is often referred as *hoop stress*

Magnetic pressure and curvature force (cont.)

- Example of magnetic curvature force
- Consider an pure toroidal (azimuthal) magnetic field, $\boldsymbol{B} = B\hat{\phi}$ in cylindrical coordinates (R, ϕ, z)
- The strength of B is function of R and z only.
- The unit vector in toroidal (azimuthal) direction $\hat{\phi}$ has the property $\hat{\phi} \cdot \nabla \hat{\phi} = -\hat{R}/R$ so that

$$\frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = -\frac{1}{\mu_0}\frac{B^2}{R}\hat{R}$$

- The curvature force is directed inward, toward the center of curvature.

Magnetic stress tensor

- The most useful alternative form of Lorentz force is in terms of *magnetic stress tensor*
- Writing a vector operators in terms of permutation (Levi-Civita) symbol ε, one has

$$\begin{aligned}[(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]_i &= \epsilon_{ijk}\epsilon_{jlm}\frac{\partial B_m}{\partial x_l}B_k \\ &= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\frac{\partial B_m}{\partial x_l}B_k \\ &= \frac{\partial}{\partial x_k}(B_iB_k - \frac{1}{2}B^2\delta_{ik})\end{aligned}$$

Levi-Civita symbol is related to Kronecker delta

where the summing convention over repeated indices and $\nabla \cdot \boldsymbol{B} = 0$ have been used. Define the *magnetic stress tensor* $\boldsymbol{M}$ by its components:

$$M_{ij} = \frac{1}{2\mu_0}B^2\delta_{ij} - \frac{1}{\mu_0}B_iB_j$$

- The Lorentz force is written as:

$$\frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\nabla \cdot \boldsymbol{M} \qquad (4.5)$$

Magnetic stress tensor (cont.)

- If V is a volume bounded by a closed surface S, eq (4.5) yields by the divergence theorem

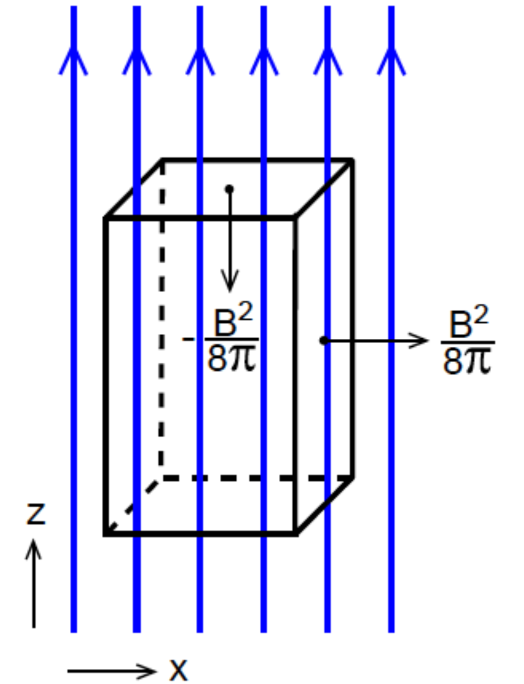
$$\int_V \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} dV = \oint_S -\boldsymbol{n} \cdot \boldsymbol{M} dS$$

- Where $\boldsymbol{n}$ is the outward normal to the surface S.
- This shows how the net Lorentz force acting on a volume V of fluid can be written as an integral of a magnetic stress vector acting on its surface S
- The force $\boldsymbol{F}_S$ exerted by the volume on its surroundings

$$\boldsymbol{F}_S = -\boldsymbol{n} \cdot \boldsymbol{M} = \frac{1}{2\mu_0} B^2 \boldsymbol{n} - \frac{1}{\mu_0} \boldsymbol{B} B_n$$

- Where $B_n = \boldsymbol{B} \cdot \boldsymbol{n}$ is the component of $\boldsymbol{B}$ along the outward normal $\boldsymbol{n}$ to the surface of the volume.

Magnetic stress tensor (cont.)



- To get the behavior of magnetic stresses, consider simple case of a uniform magnetic field, $\boldsymbol{B}=B_z$
- The force $\boldsymbol{F}_S$ in right side of the box is $\boldsymbol{F}_{right} = \hat{\boldsymbol{x}} \cdot \boldsymbol{M}$. The components are

$$\boldsymbol{F}_{right,x} = \frac{1}{2\mu_0}B^2 - \frac{1}{\mu_0}B_x B_z = \frac{1}{2\mu_0}B^2 \qquad \boldsymbol{F}_{right,y} = \boldsymbol{F}_{right,z} = 0$$

- The magnetic field exerts a force in the positive x-direction, away from the volume.
- The force $\boldsymbol{F}_S$ in top of the box is

$$\boldsymbol{F}_{top,z} = \frac{1}{2\mu_0}B^2 - \frac{1}{\mu_0}B_z B_z = -\frac{1}{2\mu_0}B^2 \qquad \boldsymbol{F}_{top,x} = \boldsymbol{F}_{top,y} = 0$$

- The magnetic field exerts a force in the negative z-direction, inward to the volume

Magnetic stress tensor (cont.)

- The magnetic pressure makes the volume of magnetic field expand in the perpendicular directions, x and y. But in the direction along a magnetic field line the volume would contract.
- Along the field lines the magnetic stress thus acts like a negative pressure, as in a stretched elastic wire
- This negative stress is referred to as the tension along the magnetic field lines.
- The stress tensor plays a role analogous like the gas pressure, but unlike gas pressure is extremely anisotropic.

Momentum equation

- From equation of motion and continuity equations

$$\rho_m \frac{\partial \boldsymbol{v}}{\partial t} + \rho_m \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \boldsymbol{v} \nabla \cdot (\rho_m \boldsymbol{v}) + \rho_m \boldsymbol{v} \cdot \nabla \boldsymbol{v}$$

$$= \frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot (\rho_m \boldsymbol{v}\boldsymbol{v})$$

- Using definition of magnetic stress tensor, *the momentum equation* is ($\boldsymbol{B} \to \boldsymbol{B}/\sqrt{\mu_0}$ for SI unit)

$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot \left[\rho_m \boldsymbol{v}\boldsymbol{v} + \left(p + \frac{1}{2} B^2 \right) \boldsymbol{I} - \boldsymbol{B}\boldsymbol{B} \right] = 0$$

$$\frac{\partial \boldsymbol{M}}{\partial t} + \nabla \cdot \boldsymbol{\Pi} = 0$$

$\boldsymbol{I}$ is three-dimensional identity tensor

$$\mathcal{M}_i = \rho_m v_i$$ Momentum density

$$\Pi_{ij} = \rho_m v_i v_j + \left(p + \frac{1}{2} B^2 \right) \delta_{ij} - B_i B_j = 0$$ Stress tensor

Conservation form of ideal MHD equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0$$ *Mass conservation*

$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot \left[\rho_m \boldsymbol{v}\boldsymbol{v} + \left(p + \frac{1}{2}B^2\right)\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}\right] = 0$$ *Momentum conservation*

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m v^2 + \rho_m e + \frac{1}{2}B^2\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho_m v^2 + \rho_m e + p + B^2\right)\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{B})\boldsymbol{B}\right] = 0$$ *Energy conservation*

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) = 0$$ *Magnetic flux conservation*

$$\nabla \cdot \boldsymbol{B} = 0$$

$$p = (\gamma - 1)\rho_m e$$ *Ideal equation of state*

($\boldsymbol{B} \to \boldsymbol{B}/\sqrt{\mu_0}$ for SI unit)

Neglecting gravity force.

This form is often used in numerical simulation.

Poynting flux

- From energy conservation equation, energy flux is

$$\boldsymbol{Y} \equiv \left(\frac{1}{2}\rho_m v^2 + \frac{\gamma}{\gamma - 1}p\right)\boldsymbol{v} + \frac{1}{\mu_0}(B^2\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{B}\boldsymbol{B})$$

- This compose hydrodynamic part and magnetic part.
- The magnetic part can be transformed:

$$\begin{aligned} \boldsymbol{Y}_{em} &\equiv \frac{1}{\mu_0}(B^2\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{B}\boldsymbol{B}) \\ &= -\frac{1}{\mu_0}(\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B} \\ &= \boxed{\boldsymbol{E}\times\boldsymbol{B}} \end{aligned}$$

- This is called *Poynting flux* (*Poynting vector*), which represents the flow of electromagnetic energy

Entropy conservation equation

- The best representation of the conservation form of MHD equation is in terms of the variables, ρ, $\boldsymbol{v}$, e and $\boldsymbol{B}$.
- A peculiar additional variable is the specific entropy s
- For adiabatic process of ideal gas, conservation of entropy is

$$\frac{DS}{Dt} \equiv \frac{\partial S}{\partial t} + (\boldsymbol{v} \cdot \nabla) S = 0$$

- But this is not in conservation form (but expresses the conservation of specific entropy co-moving with the fluid)
- A genuine conservation form is obtained by variable $\rho_m S$, the entropy per unit volume

$$\boxed{\frac{\partial}{\partial t}(\rho_m S) + \nabla \cdot (\rho_m S \boldsymbol{v}) = 0}$$

Entropy conservation equation

Summary

- Single fluid approach is called magnetohydrodynamics (MHD).
- In the case where the conductivity is very high, the electric field is $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$. It is known as ideal MHD.
- In ideal MHD, magnetic field is frozen into the fluid
- Lorentz force divides two different forces: magnetic pressure & curvature force
- The induction equation in ideal MHD shows evolution of magnetic field. It is including compression, advection and stretching
- The induction equation in resistive MHD includes diffusion of magnetic field.
- From energy conservation equation, energy flux composes hydrodynamic part and magnetic part. Magnetic part is called Poynting flux.

Hydro vs MHD

MHD equation is shown the coupling of hydrodynamics with magnetic field

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\
\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla(p + \frac{B^2}{2\mu_0}) \\
p\rho_m^{\gamma} &= const \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B})
\end{aligned}
$$

MHD equation is recovered hydrodynamic equations when B=0.

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + \nabla(\rho_m \boldsymbol{v}) &= 0 \\
\rho_m \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] &= -\nabla p \\
p\rho_m^{\gamma} &= const
\end{aligned}
$$

Hydro vs MHD (cont.)

- Conservation form of hydrodynamic equations

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho_m \boldsymbol{v}) + \nabla \cdot [\rho_m \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I}] = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m v^2 + \rho_m e\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho_m v^2 + \rho_m e + p\right)\boldsymbol{v}\right] = 0$$

$$p = (\gamma - 1)\rho_m e$$

Exercise 2-1

Derivation of conservation form of total energy

From equation of motion: $\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] + \nabla p - \boldsymbol{j} \times \boldsymbol{B} = 0$

=> $\rho \boldsymbol{v} \cdot \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right] + \boldsymbol{v} \cdot \nabla p - \boldsymbol{v} \cdot (\boldsymbol{j} \times \boldsymbol{B}) = 0$

=> $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) - \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} + \frac{1}{2}\rho \boldsymbol{v} \cdot \nabla v^2 + \boldsymbol{v} \cdot \nabla p - \boldsymbol{v} \cdot (\boldsymbol{j} \times \boldsymbol{B}) = 0$

Using continuity equation

=> $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla \cdot \left(\frac{1}{2}\rho v^2 \boldsymbol{v}\right) + \boldsymbol{v} \cdot \nabla p - \boldsymbol{v} \cdot (\boldsymbol{j} \times \boldsymbol{B}) = 0 \qquad (1)$$

Exercise 2-1 (cont.)

From pressure equation: $$\frac{\partial p}{\partial t} + (\boldsymbol{v} \cdot \nabla) p + \gamma p \nabla \cdot \boldsymbol{v} = 0$$

Using ideal EoS and continuity equation,

$$\frac{\partial e}{\partial t} + (\boldsymbol{v} \cdot \nabla) e + (\gamma - 1) e \nabla \cdot \boldsymbol{v} = 0$$

=> $$\rho \frac{\partial e}{\partial t} + \rho (\boldsymbol{v} \cdot \nabla) e + (\gamma - 1) \rho e \nabla \cdot \boldsymbol{v} = 0$$

=> $$\frac{\partial}{\partial t}(\rho e) - e \frac{\partial \rho}{\partial t} + \rho (\boldsymbol{v} \cdot \nabla) e + p \nabla \cdot \boldsymbol{v} = 0$$

Using continuity equation,

=> $$\frac{\partial}{\partial t}(\rho e) + \nabla \cdot (\rho e \boldsymbol{v}) + p \nabla \cdot \boldsymbol{v} = 0 \quad (2)$$

Exercise 2-1 (cont.)

From induction equation: $\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0$

=> $\frac{\boldsymbol{B}}{\mu_0} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \frac{\boldsymbol{B}}{\mu_0} \cdot \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0$

Using D6 =>

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\nabla \cdot [\boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B})] - \frac{1}{\mu_0}(\boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla \times \boldsymbol{B} = 0$$

Using D1 & D2=> $\frac{1}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) = -\boldsymbol{j} \times \boldsymbol{B}$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\nabla \cdot [(\boldsymbol{B} \cdot \boldsymbol{B})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}] + \boldsymbol{v} \cdot \boldsymbol{j} \times \boldsymbol{B} = 0 \quad (3)$$

Exercise 2-1 (cont.)

- (1) + (2) + (3) =0

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho e+\frac{B^2}{2\mu_0}\right)+\nabla\cdot\left[\left(\frac{1}{2}\rho \boldsymbol{v}^2+\rho e+p+\frac{B^2}{\mu_0}\right)\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{B})\frac{\boldsymbol{B}}{\mu_0}\right]=0.$$