Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture IX

Yosuke Mizuno

Winter Semester 2018

Lecture Xi, Exercise 1.

The Carter-Lichnerowicz equation is given by

$$\Omega_{\mu\nu}u^{\mu} = T\nabla_{\mu}s. \tag{1}$$

Here we consider Newtonian limit of the Carter-Lichnerowicz equation. First we rewrite Eq. (1) as

$$\begin{aligned}\Omega_{\mu\nu}u^{\mu} &= u^{\nu}\Omega_{\nu\mu} \\ &= u^{\mu}[\nabla_{\nu}(hu_{\mu}) - \nabla_{\mu}(hu_{\nu})] \\ &= u^0\left[\frac{1}{c}\frac{\partial}{\partial t}(hu_i) - \frac{\partial}{\partial x^i}(hu_0)\right] + u^j\left[\frac{\partial}{\partial x^j}(hu_i) - \frac{\partial}{\partial x^i}(hu_j)\right].\end{aligned} \tag{2}$$

As already discussed in the exercise of Lecture VIII, the covariant components of the four-velocity vector in the Newtonian limit are given by

$$u^{\alpha} \simeq \left(u^0, \frac{v^i}{c}\right) = \left(1 - \frac{\phi}{c^2} + \frac{1}{2}\frac{v_j v^j}{c^2}, \frac{v^i}{c}\right), \tag{3}$$

while the corresponding covariant components are given by

$$u_{\alpha} \simeq \left(u_0, \frac{v_i}{c}\right) = \left(-1 - \frac{\phi}{c^2} - \frac{1}{2}\frac{v_j v^j}{c^2}, \frac{v_i}{c}\right). \tag{4}$$

Similarly the expression for the relativistic specific enthalpy is

$$h = c^2\left(1 + \frac{h_{\rm N}}{c^2}\right), \tag{5}$$

where $h_{\rm N}$ is the specific enthalpy in the Newtonian limit, $h_{\rm N} = \epsilon + p/\rho$. We substitute these relations into Eq (2) to obtain

$$\begin{aligned}\Omega_{\mu\nu}u^{\mu} &= u^0\left\{\partial_t\left[\left(1 + \frac{h_{\rm N}}{c^2}\right)v_i\right] - \partial_i[(c^2 + h_{\rm N})u_0]\right\} \\ &\quad + v^i\left\{\partial_j\left[\left(1 + \frac{h_{\rm N}}{c^2}\right)v_i\right] - \partial_i\left[\left(1 + \frac{h_{\rm N}}{c^2}\right)v_j\right]\right\}.\end{aligned} \tag{6}$$

In tthe Newtonian limit, the terms u^0 and $h_{\rm N}/c^2$ can be set to 1 and 0 respectively, so that the second term in the RHS of Eq (6) can be changed as

$$\begin{aligned}\partial_i[(c^2+h_{\rm N})u_0] &= -\partial_i\left[(c^2+h_{\rm N})\left(1+\frac{\phi}{c^2}+\frac{v_jv^j}{2c^2}\right)\right] \\ &\simeq -\partial_i\left(\phi+\frac{1}{2}v_jv^j+h_{\rm N}\right). \end{aligned} \tag{7}$$

Finally we get

$$\begin{aligned}&\partial_t v_i+\partial_i\left(h_{\rm N}+\frac{1}{2}v_jv^j+\phi\right)+v^i(\partial_j v_i-\partial_i v_j)=T\partial_i s \\ \Rightarrow\quad &\frac{\partial\vec{\boldsymbol{v}}}{\partial t}+\vec{\nabla}\cdot\left(\frac{1}{2}v^2+\epsilon+\frac{p}{\rho}+\phi\right)-\vec{\boldsymbol{v}}\times(\vec{\nabla}\times\vec{\boldsymbol{v}})=T\vec{\nabla}s. \end{aligned} \tag{8}$$

This equation is known as the Crocco equation of motion.

Lecture IX, Exercise 2.

The vorticity four-vector is written as

$$\Omega^\mu = {}^*\Omega^{\mu\nu}u_\nu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Omega_{\alpha\beta}u_\nu. \tag{9}$$

The kinetic vorticity four-vector is given by

$$\omega^\mu = {}^*\omega^{\mu\nu}u_\nu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta}u_\nu \tag{10}$$

Writing out Eq (9) explicitly we obtain

$$\begin{aligned}\Omega_{\alpha\beta}u_\nu &= [\nabla_\beta(hu_\alpha)u_\nu-\nabla_\alpha(hu_\beta)u_\nu] \\ &= [h\nabla_\beta(u_\alpha)u_\nu+u_\alpha u_\nu\nabla_\beta h-h\nabla_\alpha(u_\beta)u_\nu-u_\beta u_\nu\nabla_\alpha h] \\ &= hu_\nu(\nabla_\beta u_\alpha-\nabla_\alpha u_\beta)+u_\alpha u_\nu\nabla_\beta h-u_\beta u_\nu\nabla_\alpha h \\ &= hu_\nu 2\nabla_{[\beta}u_{\alpha]}, \end{aligned} \tag{11}$$

where the terms including $u_\alpha u_\nu$ and $u_\beta u_\nu$ vanish because of the symmetry in the indices and the antisymmetry of the Levi-Civita tensor.

From the definition of the kinetic vorticity tensor, we instead obtain

$$\begin{aligned}&\omega_{\mu\nu}=\nabla_{[\mu}u_{\nu]}+a_{[\mu}u_{\nu]} \\ \Rightarrow\quad &\nabla_{[\mu}u_{\nu]}=\omega_{\mu\nu}-a_{[\mu}u_{\nu]}. \end{aligned} \tag{12}$$

Therefore connecting these two results, the vorticity four-vector can be given by

$$\begin{aligned}\Omega^\mu &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}hu_\nu\omega_{\beta\alpha}-\epsilon^{\mu\nu\alpha\beta}hu_\nu a_{[\beta}u_{\alpha]} \\ &= 2h\omega^\mu, \end{aligned} \tag{13}$$

where the second term of the RHS vanishes because of the symmetries in the four-velocity.