Lecture VI, Exercise 1.

The sound speed $c_s$ is given by

$$c_s^2 = \left( \frac{\partial p}{\partial e} \right)_s. \quad (1)$$

We consider the first law of thermodynamics with following forms,

$$dp = \rho dh - \rho T ds, \quad (2)$$
$$de = hd\rho + \rho T ds \quad (3)$$

Divide both equations and we get

$$\frac{dp}{de} = \frac{\rho}{h} \frac{dh}{d\rho}. \quad (4)$$

Therefore eq (1) can be written as

$$hc_s^2 = \rho \left( \frac{dh}{d\rho} \right) = \frac{dp}{d\rho}, \quad (5)$$

because $\rho dh = dp$ if $ds = 0$.

We consider the pressure $p$ is a function of density and of the specific internal energy, $p = p(\rho, \epsilon)$. Taking derivative, we obtain

$$dp = \frac{\partial p}{\partial \rho} d\rho + \frac{\partial p}{\partial \epsilon} d\epsilon, \quad (6)$$

which can be divided by $d\rho$ to yield

$$\frac{dp}{d\rho} = \frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial \epsilon} \frac{d\epsilon}{d\rho}. \quad (7)$$

From the first law of thermodynamics if $ds = 0$,

$$de = hdp. \quad (8)$$

1
Using following relations $e = \rho + \rho \epsilon$ and $h = (e + p)/\rho = 1 + \epsilon + p/\rho$, we obtain

\[ d(\rho + \rho \epsilon) = \frac{e + p}{\rho} d\rho \]  
(9)

\[ dp + \rho d\epsilon + \epsilon dp = \frac{e + p}{\rho} d\rho \]  
(10)

\[ dp \left( 1 + \epsilon - \frac{e + p}{\rho} \right) = -\rho d\epsilon \]  
(11)

\[ dp \left( \frac{\rho + \rho \epsilon - \rho - \rho \epsilon - p}{\rho} \right) = -\rho d\epsilon \]  
(12)

\[ \frac{de}{d\rho} = \frac{p}{\rho^2}. \]  
(13)

Adding Eqs. (6) and (13) to Eq. (5), the sound speed is written as

\[ \frac{h c_s^2}{d\rho} = \gamma \frac{dp}{d\rho} = \left[ \frac{\partial p}{\partial \rho} \right]_s + \rho \frac{\partial (\rho \epsilon)}{\partial \rho}. \]  
(14)

First we consider the ideal-fluid equation of state, $p = \rho \epsilon (\gamma - 1)$. We take a differential

\[ dp = (\gamma - 1)(\rho d\epsilon + \epsilon d\rho), \]  
(15)

and divide by $d\rho$,

\[ \frac{dp}{d\rho} = (\gamma - 1) \left[ \rho \frac{d\epsilon}{d\rho} + \epsilon \right]. \]  
(16)

From the definition of energy density $e = \rho + \rho \epsilon$, we take a derivative and using the first law of thermodynamics,

\[ de = d\rho + \rho d\epsilon + \epsilon d\rho \]  
(17)

\[ = (1 + \epsilon) d\rho + \rho d\epsilon = h d\rho, \]  
(18)

we rewrite it as

\[ 1 + \epsilon + \rho \frac{de}{d\rho} = h = 1 + \gamma \epsilon \]  
(19)

\[ \epsilon + \rho \frac{de}{d\rho} = \gamma \epsilon. \]  
(20)

Adding Eq. (20) to Eq. (16) we obtain

\[ \frac{dp}{d\rho} = (\gamma - 1)[\epsilon(\gamma - 1) + \epsilon] = (\gamma - 1)\gamma \epsilon = \frac{\gamma p}{\rho}. \]  
(21)

Therefore the square of the sound speed using ideal-fluid equation of state is written as

\[ c_s^2 = \frac{1}{h} \frac{dp}{d\rho} = \frac{\gamma p}{h} = \frac{(\gamma - 1)\gamma \epsilon}{1 + \gamma \epsilon} = (h - 1)(\gamma - 1) \]  
(22)
Second we consider the polytropic equation of state, \( p = K \rho^\Gamma \). Taking a differential we obtain

\[
dp = \left( \frac{\Gamma p}{\rho} \right) d\rho
\]  

(23)

The energy density for polytropic equation of state is written as

\[
e = \rho + \frac{1}{\Gamma - 1} p = \rho + \rho \epsilon.
\]  

(24)

Using Eqs. (23) and (24), the square of sound speed using the polytropic equation of state is obtained as

\[
c^2_s = \frac{1}{h} \frac{dp}{d\rho} = \frac{\Gamma p}{\rho h} = \frac{\Gamma p}{\rho + \rho \epsilon + p}
\]  

(25)

\[
= \frac{\Gamma p}{\rho + \frac{1}{\Gamma - 1} p} = \frac{\Gamma (\Gamma - 1) p}{\rho (\Gamma - 1) + p \Gamma}.
\]  

(26)

**Lecture VI, Exercise 2.**

The pressure has the following relation,

\[
p = \rho \epsilon (\gamma - 1) = n m \epsilon (\gamma - 1) = n k_B T
\]  

(27)

Therefore the temperature is given by

\[
T = \frac{m}{k_B} (\gamma - 1) \epsilon.
\]  

(28)

From the first law of thermodynamics,

\[
d\epsilon = T ds + \frac{p}{\rho^2} d\rho.
\]  

(29)

Using Eq. (28), it can be rewritten as

\[
ds = \frac{1}{T} d\epsilon - \frac{p}{\rho^2 T} d\rho
\]  

(30)

\[
= \frac{k_B}{m (\gamma - 1) \epsilon} d\epsilon - \frac{p k_B}{\rho^2 m (\gamma - 1) \epsilon} d\rho.
\]  

(31)

Using Eq. (27), Eq (31) is also written as

\[
\frac{m}{k_B} ds = \frac{de}{\epsilon (\gamma - 1)} - \frac{d\rho}{\rho
\]  

(32)

\[
= \frac{d \ln \epsilon}{\gamma - 1} - d \ln \rho
\]  

(33)

\[
= d \ln \epsilon^{1/\gamma - 1} - d \ln \rho
\]  

(34)

\[
= d \left[ \ln \left( \frac{\epsilon^{1/\gamma - 1}}{\rho} \right) \right].
\]  

(35)
We can now integrate Eq. (35) to obtain

\[ s = \frac{k_B}{m} \left[ \ln \left( \frac{\epsilon^{1/\Gamma - 1}}{\rho} \right) + \tilde{K} \right]. \]  

(36)

Here we consider the polytropic equation of state \((p = K\rho^\Gamma)\). The specific internal energy is given by

\[ \epsilon = \frac{K\rho^{\Gamma-1}}{\Gamma - 1}. \]  

(37)

Thus

\[ \frac{\epsilon^{1/\Gamma - 1}}{\rho} = \left( \frac{K}{\Gamma - 1} \right)^{1/\Gamma - 1} \frac{\rho}{\rho} = \left( \frac{K}{\Gamma - 1} \right)^{1/\Gamma - 1}. \]  

(38)

As a result, Eq. (36) can be written as

\[ s = \frac{k_B}{m} \left[ \ln \left( \frac{K}{\Gamma - 1} \right)^{1/\Gamma - 1} + \tilde{K} \right]. \]  

(39)

**Lecture VI, Exercise 3.**

Let’s start from the first law of thermodynamics

\[ dp = \rho dh - \rho T ds. \]  

(40)

Here we consider polytropic equation of state which pressure is a function of density only \((p = p(\rho))\). Therefore

\[ dp = \frac{\partial p}{\partial \rho} d\rho = P'_\rho d\rho. \]  

(41)

The specific enthalpy \(h = e + p/\rho = 1 + e + p/\rho\). Taking the differential we obtain

\[ dh = de + d \left( \frac{p}{\rho} \right). \]  

(42)

From the polytropic equation of state, we know that the pressure is a function of density only, so that the internal energy is a function of density only \((e = e(\rho))\). Thus,

\[ de = \frac{\partial e}{\partial \rho} d\rho = \epsilon'_\rho d\rho. \]  

(43)

Using Eq (41), the second term of RHS in Eq (42) can be expressed as

\[ d \left( \frac{p}{\rho} \right) = \frac{1}{\rho} dp - \frac{p}{\rho^2} d\rho = \frac{p'_\rho}{\rho} d\rho - \frac{p}{\rho^2} d\rho = \frac{d\rho}{\rho} \left( \rho'_{\rho} - \frac{p}{\rho} \right). \]  

(44)
Therefore Eq. (40) is given by

\[ p'_\rho d\rho = \rho \left[ \epsilon'_\rho d\rho + \frac{dp}{\rho} \left( p'_\rho - \frac{p}{\rho} \right) \right] - \rho T ds \]  
(45)

\[ \rightarrow 0 = \rho \epsilon'_\rho d\rho - \frac{p}{\rho} d\rho - \rho T ds \]  
(46)

\[ \rightarrow d\rho \left( \epsilon'_\rho - \frac{p}{\rho^2} \right) = T ds \]  
(47)

\[ \rightarrow d\rho \left( \frac{\partial \epsilon}{\partial \rho} - \frac{p}{\rho^2} \right) = T ds. \]  
(48)

This equation shows that if \( \partial \epsilon / \partial \rho = p / \rho^2 \), the polytropic equation of state is isentropic (\( ds = 0 \)).