Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture I

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Lecture I, Exercise 1.

Prove the Newtonian H-theorem, that is,

$$\frac{\partial f_0}{\partial t} = \Gamma(f_0) = 0, \tag{1}$$

where f_0 is the equilibrium distribution function. Condition (1) is fully equivalent to the condition

$$f_0(\vec{\boldsymbol{u}}'_2) f_0(\vec{\boldsymbol{u}}'_1) - f_0(\vec{\boldsymbol{u}}_2) f_0(\vec{\boldsymbol{u}}_1) = 0, \tag{2}$$

where $f_{1,2} := f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{u}}_{1,2})$, $f'_{1,2} := f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{u}}'_{1,2})$ are the distribution functions before and after the collision at time t and position $\vec{\boldsymbol{x}}$.

Here we introduce Boltzmann's H function as

$$H(t) = \int f(t, \vec{\boldsymbol{u}}) \ln(f(t, \vec{\boldsymbol{u}})) d^3u. \tag{3}$$

Taking a time derivative gives

$$\frac{dH(t)}{dt} = \int \frac{\partial f(t, \vec{\boldsymbol{u}})}{\partial t} [1 + \ln f(t, \vec{\boldsymbol{u}})] d^3u. \tag{4}$$

If $\partial f/\partial t = 0$, $dH/dt = 0$. So $dH/dt = 0$ is necessary condition for $\partial f/\partial t = 0$.

Next, we consider binary collisions, which gives

$$\frac{\partial f}{\partial t} = \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_1 - \vec{\boldsymbol{u}}_2| [f(\vec{\boldsymbol{u}}'_2) f(\vec{\boldsymbol{u}}'_1) - f(\vec{\boldsymbol{u}}_2) f(\vec{\boldsymbol{u}}_1)] = 0. \tag{5}$$

By adding Eq. (5) in Eq. (4) we obtain

$$\frac{dH(t)}{dt} = \int d^3u_1 \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_1 - \vec{\boldsymbol{u}}_2| (f'_2 f'_1 - f_2 f_1)[1 + \ln f_1] = 0, \tag{6}$$

which is equivalent to

$$\frac{dH(t)}{dt} = \int d^3u_1 \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1| (f'_2 f'_1 - f_2 f_1)[1 + \ln f_2] = 0, \tag{7}$$

because the cross section $\sigma(\Omega)$ is invariant under the swapping of u_1 with u_2. Thus we can add the two equations to obtain

$$\frac{dH(t)}{dt} = \frac{1}{2} \int d^3u_1 \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1| (f'_2 f'_1 - f_2 f_1)[2 + \ln(f_1 f_2)] = 0. \tag{8}$$

Since for each collision there is an inverse collision with the same cross section, the integral (8) is invariant under change of $\vec{\boldsymbol{u}}_1$, $\vec{\boldsymbol{u}}_2$ with $\vec{\boldsymbol{u}}_1'$, $\vec{\boldsymbol{u}}_2'$. Similarly f_2, f_1 and f_2', f_1', i.e.

$$\frac{dH(t)}{dt} = \frac{1}{2}\int d^3u_1' \int d^3u_2' \int d\Omega\sigma'(\Omega)|\vec{\boldsymbol{u}}_2' - \vec{\boldsymbol{u}}_1'|(f_2 f_1 - f_2' f_1')[2 + \ln(f_1' f_2')] = 0. \tag{9}$$

By adding together Eq. (8) and Eq. (9) using $d^3u_1' d^3u_2' = d^3u_1 d^3u_2$, $|\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1| = |\vec{\boldsymbol{u}}_2' - \vec{\boldsymbol{u}}_1'|$, and $\sigma(\Omega) = \sigma'(\Omega)$ we obtain

$$\frac{dH(t)}{dt} = \frac{1}{4}\int d^3u_1 \int d^3u_2 \int d\Omega\sigma(\Omega)|\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1|(f_2' f_1' - f_2 f_1)[\ln(f_1 f_2) - \ln(f_1' f_2')] = 0. \tag{10}$$

Using $x = (f_1 f_2)/(f_1' f_2')$, this is changed to

$$\frac{dH(t)}{dt} = \frac{1}{4}\int d^3u_1 \int d^3u_2 \int d\Omega\sigma(\Omega)|\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1|(f_2' f_1')[(1-x)\ln x] = 0. \tag{11}$$

The integrand of Eq. (11) is never positive for $x \geq 0$, which implies that

$$\frac{dH}{dt} \leq 0. \tag{12}$$

As a result, $dH/dt = 0$ only when

$$(f_2' f_1' - f_2 f_1) = 0. \tag{13}$$

Lecture I, Exercise 2.

From the properties of H, we can understand the Boltzmann's H function corresponds to the entropy of thermodynamics. Time derivative of H shows the H-theorem is fundamentally irreversible processes from microscopic mechanism. H value is never changed sign (H is never positive).