Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture VIII

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Lecture VIII, Exercise 1.

The vorticity tensor is defined as

$$\begin{aligned}
\Omega_{\mu\nu} &= 2\nabla_{[\mu}\omega_{\nu]} && (1)\\
&= \nabla_\nu(hu_\mu) - \nabla_\mu(hu_\nu) && (2)\\
&= h\nabla_\nu u_\mu + u_\mu\nabla_\nu h - h\nabla_\mu u_\nu - u_\nu\nabla_\mu h && (3)\\
&= h(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + u_\mu\nabla_\nu h - u_\nu\nabla_\mu h. && (4)
\end{aligned}$$

The kinematic vorticity tensor is defined as

$$\begin{aligned}
\omega_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu \nabla_{[\beta}u_{\alpha]} && (5)\\
&= \nabla_{[\mu}u_{\nu]} + a_{[\mu}u_{\nu]} && (6)\\
&= \frac{1}{2}(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + a_{[\mu}u_{\nu]}. && (7)
\end{aligned}$$

Thus,

$$\nabla_\nu u_\mu - \nabla_\mu u_\nu = 2(\omega_{\mu\nu} - a_{[\mu}u_{\nu]}). \quad (8)$$

Substituting Eq (8) into Eq (4) we obtain

$$\begin{aligned}
\Omega_{\mu\nu} &= 2h(\omega_{\mu\nu} - a_{[\mu}u_{\nu]}) + u_\mu\nabla_\nu h - u_\nu\nabla_\mu h && (9)\\
&= 2h\left[\omega_{\mu\nu} - a_{[\mu}u_{\nu]} + \frac{1}{2}\left(u_\mu\frac{1}{h}\nabla_\nu h - u_\nu\frac{1}{h}\nabla_\mu h\right)\right] && (10)\\
&= 2h[\omega_{\mu\nu} - a_{[\mu}u_{\nu]} + u_{[\mu}\nabla_{\nu]}\ln h]. && (11)
\end{aligned}$$

From the equation above it is clear that only for a test fluid (i.e., $e = 0 = p$ and $h = 1$) in geodetic motion (i.e., $a_\mu = 0$) two tensors are directly proportional, $\Omega_{\mu\nu} = 2\omega_{\mu\nu}$.

Lecture VIII, Exercise 2.

The Carter-Lichnerowicz equation is given by

$$\Omega_{\mu\nu}u^\mu = T\nabla_\mu s. \quad (12)$$

Here we consider Newtonian limit of the Carter-Lichnerowicz equation. First we rewrite Eq. (12) as

$$\begin{aligned}
\Omega_{\mu\nu}u^\mu &= u^\nu \Omega_{\nu\mu} && (13)\\
&= u^\mu[\nabla_\nu(hu_\mu) - \nabla_\mu(hu_\nu)] && (14)\\
&= u^0\left[\frac{1}{c}\frac{\partial}{\partial t}(hu_i) - \frac{\partial}{\partial x^i}(hu_0)\right] + u^j\left[\frac{\partial}{\partial x^j}(hu_i) - \frac{\partial}{\partial x^i}(hu_j)\right]. && (15)
\end{aligned}$$

As already discussed in the exercise of Lecture VII, the covariant components of the four-velocity vector in the Newtonian limit are given by

$$u^\alpha \simeq \left(u^0, \frac{v^i}{c}\right) = \left(1 - \frac{\phi}{c^2} + \frac{1}{2}\frac{v_j v^j}{c^2}, \frac{v^i}{c}\right), \tag{16}$$

while the corresponding covariant components are given by

$$u_\alpha \simeq \left(u_0, \frac{v_i}{c}\right) = \left(-1 - \frac{\phi}{c^2} - \frac{1}{2}\frac{v_j v^j}{c^2}, \frac{v_i}{c}\right). \tag{17}$$

Similarly the expression for the relativistic specific enthalpy is

$$h = c^2\left(1 + \frac{h_{\rm N}}{c^2}\right), \tag{18}$$

where $h_{\rm N}$ is the specific enthalpy in the Newtonian limit, $h_{\rm N} = \epsilon + p/\rho$. We substitute these relations into Eq (15) to obtain

$$\begin{aligned}
\Omega_{\mu\nu}u^\mu &= u^0\left\{\partial_t\left[\left(1 + \frac{h_{\rm N}}{c^2}\right)v_i\right] - \partial_i[(c^2 + h_{\rm N})u_0]\right\}\\
&\quad + v^i\left\{\partial_j\left[\left(1 + \frac{h_{\rm N}}{c^2}\right)v_i\right] - \partial_i\left[\left(1 + \frac{h_{\rm N}}{c^2}\right)v_j\right]\right\}. && (19)
\end{aligned}$$

In tthe Newtonian limit, the terms u^0 and $h_{\rm N}/c^2$ can be set to 1 and 0 respectively, so that the second term in the RHS of Eq (19) can be changed as

$$\begin{aligned}
\partial_i[(c^2 + h_{\rm N})u_0] &= -\partial_i\left[(c^2 + h_{\rm N})\left(1 + \frac{\phi}{c^2} + \frac{v_j v^j}{2c^2}\right)\right] && (20)\\
&\simeq -\partial_i\left(\phi + \frac{1}{2}v_j v^j + h_{\rm N}\right). && (21)
\end{aligned}$$

Finally we get

$$\begin{aligned}
& \partial_t v_i + \partial_i\left(h_{\rm N} + \frac{1}{2}v_j v^j + \phi\right) + v^i(\partial_j v_i - \partial_i v_j) = T\partial_i s && (22)\\
\Rightarrow\quad & \frac{\partial \vec{\boldsymbol{v}}}{\partial t} + \vec{\nabla}\cdot\left(\frac{1}{2}v^2 + \epsilon + \frac{p}{\rho} + \phi\right) - \vec{\boldsymbol{v}}\times(\vec{\nabla}\times\vec{\boldsymbol{v}}) = T\vec{\nabla}s. && (23)
\end{aligned}$$

This equation is known as the Crocco equation of motion.

Lecture VIII, Exercise 3.

The vorticity four-vector is written as

$$\Omega^\mu = {}^*\Omega^{\mu\nu} u_\nu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Omega_{\alpha\beta}u_\nu. \tag{24}$$

The kinetic vorticity four-vector is given by

$$\omega^\mu = {}^*\omega^{\mu\nu} u_\nu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta}u_\nu \tag{25}$$

Writing out Eq (24) explicitly we obtain

$$\begin{aligned}
\Omega_{\alpha\beta}u_\nu &= [\nabla_\beta(hu_\alpha)u_\nu - \nabla_\alpha(hu_\beta)u_\nu] && (26)\\
&= [h\nabla_\beta(u_\alpha)u_\nu + u_\alpha u_\nu \nabla_\beta h - h\nabla_\alpha(u_\beta)u_\nu - u_\beta u_\nu \nabla_\alpha h] && (27)\\
&= hu_\nu(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta) + u_\alpha u_\nu \nabla_\beta h - u_\beta u_\nu \nabla_\alpha h && (28)\\
&= hu_\nu 2\nabla_{[\beta}u_{\alpha]}, && (29)
\end{aligned}$$

where the terms including $u_\alpha u_\nu$ and $u_\beta u_\nu$ vanish because of the symmetry in the indices and the antisymmetry of the Levi-Civita tensor.

From the definition of the kinetic vorticity tensor, we instead obtain

$$\begin{aligned}
\omega_{\mu\nu} &= \nabla_{[\mu}u_{\nu]} + a_{[\mu}u_{\nu]} && (30)\\
\Rightarrow \quad \nabla_{[\mu}u_{\nu]} &= \omega_{\mu\nu} - a_{[\mu}u_{\nu]}. && (31)
\end{aligned}$$

Therefore connecting these two results, the vorticity four-vector can be given by

$$\begin{aligned}
\Omega^\mu &= \epsilon^{\mu\nu\alpha\beta}hu_\nu\omega_{\beta\alpha} - \epsilon^{\mu\nu\alpha\beta}hu_\nu a_{[\beta}u_{\alpha]} && (32)\\
&= 2h\omega^\mu, && (33)
\end{aligned}$$

where the second term of the RHS in Eq. (32) vanishes because of the symmetries in the four-velocity.