

Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture XIII

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Lecture XIII, Exercise 1.

See Lecture XII, Exercise 2. for the solution of this problem.

Lecture XIII, Exercise 2.

Here we focus on charged particle motion in a non-uniform magnetic field. When particles move into a weaker field, the Larmor radius increases. It decreases again as particle moves back into strong field. Different Larmor radii generates a drift motion of particle. Now we consider the most simple case

$$\vec{E} = 0, \quad \vec{B} = (0, 0, B^z(y)). \quad (1)$$

We assume Larmor radius is much smaller than the lengthscale of the variation of magnetic field, i.e., $r_L/L \ll 1$. The orbit theory also assume the velocity can be decomposed into components with a small drift velocity

$$\vec{v} = \vec{v}_D + \vec{v}_\perp, \quad (2)$$

where we assume the drift velocity to be much smaller than the other components of the velocity

$$v_D \ll v_\perp. \quad (3)$$

The equation of motion is written as

$$m \frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B}). \quad (4)$$

The components of the Lorentz force are given by

$$F_x = qv_y B^z, \quad (5)$$

$$F_y = -qv_x B^z, \quad (6)$$

$$F_z = 0. \quad (7)$$

The magnetic field has the z -component only which is a function of y . the gradient of magnetic field is

$$\frac{dB^z}{dy} \sim \frac{B^z}{L} \ll \frac{B^z}{r_L}. \quad (8)$$

Therefore the gradient of magnetic field is small,

$$r_L \frac{dB^z}{dy} \ll B^z. \quad (9)$$

From this condition, we take the Taylor expansion of the magnetic field

$$B^z(y) = B_0 + yB'_z + O(y^2), \quad (10)$$

where $B'_z = dB^z/dy$. Using this expression of magnetic field, the Lorentz force can be written as

$$F_x = qv_y(B_0 + yB'_z), \quad (11)$$

$$F_y = -qv_x(B_0 + yB'_z). \quad (12)$$

In the uniform magnetic field, the particle performs a circular motion. Therefore the position is given by

$$x = r_L \sin(\omega_c t), \quad (13)$$

$$y = r_L \cos(\omega_c t). \quad (14)$$

The velocity for the circular motion is expressed as

$$v_x = -v_\perp \cos(\omega_c t), \quad (15)$$

$$v_y = \pm v_\perp \sin(\omega_c t). \quad (16)$$

Using eqs (13) -(16), the various components of the Lorentz force can be obtained as

$$F_x = -qv_\perp \sin(\omega_c t)[B_0 \pm r_L \cos(\omega_c t)B'_z], \quad (17)$$

$$F_y = -qv_\perp \cos(\omega_c t)[B_0 \pm r_L \cos(\omega_c t)B'_z]. \quad (18)$$

We take a time average of them over the one period of the circular motion

$$\langle \psi \rangle := \frac{1}{\Delta t} \int_0^P \psi dt, \quad (19)$$

Using the above definition, the each components of Lorentz force are

$$\langle \dot{F}_x \rangle = -qv_\perp [B_0 \langle \sin(\omega_c t) \rangle \pm r_L \langle \sin(\omega_c t) \cos(\omega_c t) \rangle B'_z] \quad (20)$$

$$\langle \dot{F}_y \rangle = -qv_\perp [B_0 \langle \cos(\omega_c t) \rangle \pm r_L \langle \cos^2(\omega_c t) \rangle B'_z], \quad (21)$$

where $\langle \sin(\omega_c t) \rangle$, $\langle \cos(\omega_c t) \rangle$, and $\langle \sin(\omega_c t) \cos(\omega_c t) \rangle$ are zero, and $\langle \cos^2(\omega_c t) \rangle = 1/2$. Therefore

$$\langle \dot{F}_x \rangle = 0, \quad (22)$$

$$\langle \dot{F}_y \rangle = \pm \frac{qv_\perp r_L}{2} B'_z. \quad (23)$$

The drift velocity of the general forces is defined by

$$\vec{v}_F = \frac{1}{q} \frac{\vec{F} \times \vec{B}}{B^2}. \quad (24)$$

Therefore in this case the drift velocity is given by

$$\vec{v}_F = \frac{1}{q} \frac{\langle \dot{F}_y \rangle \hat{y} \times B^z \hat{z}}{B_z^2} \quad (25)$$

$$= \mp \frac{v_\perp r_L}{2B^z} \frac{dB^z}{dy} \hat{x}. \quad (26)$$

In three-dimensions, this results can be generalized to

$$\vec{v}_{\vec{\nabla}B} = \pm \frac{1}{2} v_\perp r_L \frac{\vec{B} \times \vec{\nabla}|\vec{B}|}{B^2}. \quad (27)$$

This drift is the so-called “grad-B” drift.

Lecture XIII, Exercise 3.

- (a) The earth’s magnetic field has a dipole field which contains gradient and curvature. Therefore we need to consider two drift motion, magnetic field gradient (grad-B) drift and curvature drift. The grad-B drift is given by

$$\vec{v}_{\vec{\nabla}B} = \frac{mv_\perp^2}{2qB^3} (\vec{B} \times \vec{\nabla}|\vec{B}|). \quad (28)$$

And the curvature drift is expressed as

$$\vec{v}_c = \frac{mv_\parallel^2}{qB^4} \vec{B} \times [(\vec{B} \cdot \vec{\nabla})\vec{B}]. \quad (29)$$

From the vector identity,

$$(\vec{B} \cdot \vec{\nabla})\vec{B} - \vec{\nabla}B^2/2 = (\vec{\nabla} \times \vec{B}) \times \vec{B} = \mu_0 \vec{J} \times \vec{B} \quad (30)$$

$$= 0 \quad (31)$$

because of the absence of currents. Using this equation, the curvature drift can be written as

$$\vec{v}_c = \frac{mv_\parallel^2}{qB^3} (\vec{B} \times \vec{\nabla}|\vec{B}|). \quad (32)$$

Therefore the drift velocity in this case is given by

$$\vec{v}_d = \vec{v}_{\vec{\nabla}B} + \vec{v}_c = \frac{m}{qB} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{(\vec{B} \times \vec{\nabla}|\vec{B}|)}{B^2} \quad (33)$$

$$= \frac{m}{qB} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \frac{\vec{\nabla}|\vec{B}|}{B}. \quad (34)$$

Here the earth's magnetic field in the equatorial plane is expressed as $B = k/r^3$. The gradient of earth's magnetic field in the equatorial plane is obtained as

$$\vec{\nabla}|\vec{B}| := \left(\vec{\nabla}|\vec{B}|\right) \cdot \vec{e}_r = \frac{\partial B}{\partial r} = -\frac{3k}{r^4}. \quad (35)$$

Therefore

$$\frac{\vec{\nabla}|\vec{B}|}{B} = -\frac{3}{r}. \quad (36)$$

The particle velocity is obtained by the thermal velocity, which in three-dimension is given by

$$v_{th}^2 = \frac{3k_B T}{m} = v_x^2 + v_y^2 + v_z^2. \quad (37)$$

If we consider the magnetic field is on the z -direction, the velocity component parallel and perpendicular of the magnetic field is expressed as

$$v_{\parallel}^2 = v_z^2 = \frac{k_B T}{m}, \quad (38)$$

$$v_{\perp}^2 = v_x^2 + v_y^2 = \frac{2k_B T}{m}. \quad (39)$$

Therefore

$$v_{\parallel}^2 + \frac{v_{\perp}^2}{2} = \frac{2k_B T}{m}. \quad (40)$$

Using eqs (36) and (40), the drift velocity is obtained

$$\vec{v}_d = \pm \frac{m(2k_B T/m)(-3/r)}{eB} = \mp \frac{6k_B T}{eBr}. \quad (41)$$

We now consider particles that are about five Earth's radii, $r/R_E = 5$. The electrons have an energy of 30 keV and the protons an energy of 1 eV. Using the eq (41), the drift velocity of electrons and protons are

$$v_{d,e} = 2.4 \times 10^6 \text{ cm/s}. \quad (42)$$

$$v_{d,p} = 79 \text{ cm/s}, \quad (43)$$

where we have used that $1\text{eV} = 1.16 \times 10^4 K$. The drift motion makes circular motion around the Earth which is on the equatorial plane, with the electrons and protons moving in opposite directions.

- (b) Because the drift motion of electrons and protons is in opposite directions, it leads to a ring current given by

$$\vec{j} = ne(\vec{v}_{d,p} - \vec{v}_{d,e}) = 3.8 \times 10^{-15} \text{ A/cm}^2. \quad (44)$$

- (c) The drift time around the Earth is calculated by

$$t_d = 2\pi r/v_d. \quad (45)$$

Using $r = 5R_E$, the drift time of electrons and protons are

$$\text{electrons: } 8.5 \times 10^4 \text{ s, } \sim 24 \text{ h} \quad (46)$$

$$\text{protons: } 2.8 \times 10^8 \text{ s} \sim 8.8 \text{ yr.} \quad (47)$$