Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture XII

Yosuke Mizuno

Winter Semester 2016

Lecture XII, Exercise 1.

The gyro frequencies and Larmor radii are given by

$$\omega_c := \frac{qB}{m}, \quad (1)$$

$$r_L := \frac{mv_\perp}{qB}, \quad (2)$$

where $v_\perp$ is the velocity perpendicular to the magnetic field. We assume the plasma is in thermal equilibrium and the particles have the thermal velocity. Therefore the velocity is given by

$$v_\perp \approx v_{th} = \sqrt{\frac{k_B T}{m}}. \quad (3)$$

Here we compare the gyro frequencies with plasma frequencies which can be obtained

$$\omega_P = \sqrt{\frac{4\pi n q^2}{m}}. \quad (4)$$

We list the results in different physical conditions (1 rad/s $= 1/2\pi$ Hz).

		ω_c [Hz]	r_L (cm)	ω_p [Hz]
fusion machine	electron	8.4×10^{20}	2.3×10^{-13}	2.3×10^{12}
	proton	4.6×10^{17}	1.0×10^{-11}	5.3×10^{10}
Earth's magnetosphere	electron	8.4×10^{14}	2.3×10^{-9}	2.3×10^{6}
	proton	4.6×10^{11}	1.0×10^{-7}	5.3×10^{4}
center of the Sun	electron	8.4×10^{22}	3.0×10^{-15}	2.3×10^{17}
	proton	4.6×10^{19}	1.3×10^{-13}	5.3×10^{15}
solar corona	electron	8.4×10^{16}	7.4×10^{-10}	2.3×10^{8}
	proton	4.6×10^{13}	3.2×10^{-8}	5.3×10^{6}
solar wind	electron	8.4×10^{11}	2.3×10^{-5}	7.1×10^{4}
	proton	4.6×10^{8}	1.0×10^{-3}	1.7×10^{3}
neutron star's atmosphere	electron	8.4×10^{28}	2.3×10^{-21}	2.3×10^{10}
	proton	4.6×10^{25}	1.0×10^{-19}	5.3×10^{-8}

Lecture XII, Exercise 2.

Consider the non relativistic motion of particles moving under the combined influence of uniform magnetic and gravity fields. The equation of motion can be given as

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \vec{\boldsymbol{g}} + \frac{q}{m}(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}), \tag{5}$$

where $m\vec{\boldsymbol{g}}$ is the gravitational force on a particle of mass, m. This equation is identical to that for motion in an ”effective” electric field and the same magnetic field. Thus in this case, the effective electric field is written as

$$\vec{\boldsymbol{E}}_{\text{eff}} = \frac{m}{q}\vec{\boldsymbol{g}}. \tag{6}$$

From the derivation of the $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ drift we know that $\vec{\boldsymbol{v}}_d = \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}/B^2$, the drift component of the motion is given by

$$\vec{\boldsymbol{v}}_d = \frac{m}{q}\frac{\vec{\boldsymbol{g}} \times \vec{\boldsymbol{B}}}{B^2}. \tag{7}$$

The drift velocity depends on both charge and mass. Electrons and ions are drift in opposite direction, producing a current in the system. For the simple case of uniform plasma consisting only of protons and electrons, the current density is given by

$$\vec{\boldsymbol{j}} = -ne\vec{\boldsymbol{v}}_{d,e} + ne\vec{\boldsymbol{v}}_{d,p}, \tag{8}$$

where $\vec{\boldsymbol{v}}_{d,e}$ and $\vec{\boldsymbol{v}}_{d,p}$ are the drift velocities of the electrons and of the protons, respectively. From eq (7), the current density can then be expressed as

$$\vec{\boldsymbol{j}} = \rho\frac{\vec{\boldsymbol{g}} \times \vec{\boldsymbol{B}}}{B^2}, \tag{9}$$

where $\rho := nm_e + nm_p$.

Lecture XII, Exercise 3.

Since the magnetic field does not produce a work on the particle, particle energy in a static magnetic field is conserved. However, if magnetic field is time-dependent, there must be an accompanying electric field

$$\vec{\nabla} \times \vec{\boldsymbol{E}} = -\frac{\partial\vec{\boldsymbol{B}}}{\partial t}. \tag{10}$$

Clearly the electric field can not be uniform in the space and so we must expect that electric field will change the particle’s energy. Here we focus on the motion normal to the magnetic field. The perpendicular component of particle kinetic energy is given by

$$U_\perp = \frac{1}{2}mv_\perp^2. \tag{11}$$

Taking a time derivative and expressing the acceleration of the particle in terms of the electric fiedl, i.e.,

$$m\frac{d\vec{\boldsymbol{v}}_\perp}{dt} = q\vec{\boldsymbol{E}}. \tag{12}$$

we obtain that

$$\frac{dU_\perp}{dt} = q\vec{\boldsymbol{v}}_\perp \cdot \vec{\boldsymbol{E}}. \tag{13}$$

Expressing the perpendicular velocity as

$$\vec{\boldsymbol{v}}_\perp =: \frac{d\vec{\boldsymbol{X}}}{dt}, \tag{14}$$

where $\vec{\boldsymbol{X}}(t)$ denotes the trajectory of the particle, we can rewrite eq (13) as

$$\frac{dU_\perp}{dt} = q\frac{d\vec{\boldsymbol{X}}}{dt} \cdot \vec{\boldsymbol{E}}. \tag{15}$$

The total change in $U_\perp$ over the one cycle of the orbital motion is given by

$$\Delta U_\perp = \int_0^P q\frac{d\vec{\boldsymbol{X}}}{dt} \cdot \vec{\boldsymbol{E}}dt, \tag{16}$$

where P is period of the motion. The time variation of the magnetic field implies a time variation in both the gyro radius and the gyro period. Therefore the orbit will not be closed. However, if we assume that the change in the magnetic field during one period of the circular motion is small compared to the magnitude of magnetic field, i.e., if

$$P\left|\frac{d\vec{\boldsymbol{B}}}{dt}\right| = \frac{2\pi}{\omega_c}\left|\frac{d\vec{\boldsymbol{B}}}{dt}\right| \ll |\vec{\boldsymbol{B}}|. \tag{17}$$

then the time integral of eq (16) can be replaced by a line integral taken over a fictitious circular orbit of the particle

$$\Delta U_\perp = \oint q\vec{\boldsymbol{E}}dl. \tag{18}$$

By using Stokes' theorem, this can be expressed as

$$\Delta U_\perp = -q\int(\vec{\nabla} \times \vec{\boldsymbol{E}})ds, \tag{19}$$

which shows that the surface integral follows the particle motion as for our assumption. Using eq (10), we rewrite eq (18) as

$$\Delta U_\perp = |q|\int \frac{\partial \vec{\boldsymbol{B}}}{\partial t}ds, \tag{20}$$

where the change in energy is in fact independent of the sign of the charge. If we assume that the magnetic field is uniform, the surface integral is expressed as πr_L^2, then eq (20) yields

$$\Delta U_\perp = |q|\pi r_L^2\frac{d\vec{\boldsymbol{B}}}{dt}. \tag{21}$$

As a result, the rate of change of energy per one gyration period is given by

$$\frac{dU_\perp}{dt} = \frac{\Delta U_\perp}{P} = \frac{1}{2}|q|\omega_c r_L^2 \frac{d\vec{\boldsymbol{B}}}{dt}. \tag{22}$$