Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture VIII

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Lecture VIII, Exercise 1.

The vorticity tensor is defined as

$$\begin{aligned}\Omega_{\mu\nu} &= 2\nabla_{[\mu}\omega_{\nu]} & (1)\\ &= \nabla_\nu(hu_\mu) - \nabla_\mu(hu_\nu) & (2)\\ &= h\nabla_\nu u_\mu + u_\mu\nabla_\nu h - h\nabla_\mu u_\nu - u_\nu\nabla_\mu h & (3)\\ &= h(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + u_\mu\nabla_\nu h - u_\nu\nabla_\mu h. & (4)\end{aligned}$$

The kinematic vorticity tensor is defined as

$$\begin{aligned}\omega_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu \nabla_{[\beta}u_{\alpha]} & (5)\\ &= \nabla_{[\mu}u_{\nu]} + a_{[\mu}u_{\nu]} & (6)\\ &= \frac{1}{2}(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + a_{[\mu}u_{\nu]}. & (7)\end{aligned}$$

Thus,

$$\nabla_\nu u_\mu - \nabla_\mu u_\nu = 2(\omega_{\mu\nu} - a_{[\mu}u_{\nu]}). \quad (8)$$

Substituting Eq (8) into Eq (4) we obtain

$$\begin{aligned}\Omega_{\mu\nu} &= 2h(\omega_{\mu\nu} - a_{[\mu}u_{\nu]}) + u_\mu\nabla_\nu h - u_\nu\nabla_\mu h & (9)\\ &= 2h\left[\omega_{\mu\nu} - a_{[\mu}u_{\nu]} + \frac{1}{2}\left(u_\mu\frac{1}{h}\nabla_\nu h - u_\nu\frac{1}{h}\nabla_\mu h\right)\right] & (10)\\ &= 2h[\omega_{\mu\nu} - a_{[\mu}u_{\nu]} + u_{[\mu}\nabla_{\nu]}\ln h]. & (11)\end{aligned}$$

From the equation above it is clear that only for a test fluid (i.e., $e = 0 = p$ and $h = 1$) in geodetic motion (i.e., $a_\mu = 0$) two tensors are directly proportional, $\Omega_{\mu\nu} = 2\omega_{\mu\nu}$.

Lecture VIII, Exercise 2.

The Carter-Lichnerowicz equation is given by

$$\Omega_{\mu\nu}u^\mu = T\nabla_\mu s. \quad (12)$$

Here we consider Newtonian limit of the Carter-Lichnerowicz equation. First we rewrite Eq. (12) as

$$\begin{aligned}\Omega_{\mu\nu}u^{\mu} &= u^{\nu}\Omega_{\nu\mu} &(13)\\ &= u^{\mu}[\nabla_{\nu}(hu_{\mu})-\nabla_{\mu}(hu_{\nu})] &(14)\\ &= u^0\left[\frac{1}{c}\frac{\partial}{\partial t}(hu_i)-\frac{\partial}{\partial x^i}(hu_0)\right]+u^j\left[\frac{\partial}{\partial x^j}(hu_i)-\frac{\partial}{\partial x^i}(hu_j)\right]. &(15)\end{aligned}$$

As already discussed in the exercise of Lecture VII, the covariant components of the four-velocity vector in the Newtonian limit are given by

$$u^{\alpha}\simeq\left(u^0,\frac{v^i}{c}\right)=\left(1-\frac{\phi}{c^2}+\frac{1}{2}\frac{v_jv^j}{c^2},\frac{v^i}{c}\right), \tag{16}$$

while the corresponding covariant components are given by

$$u_{\alpha}\simeq\left(u_0,\frac{v_i}{c}\right)=\left(-1-\frac{\phi}{c^2}-\frac{1}{2}\frac{v_jv^j}{c^2},\frac{v_i}{c}\right). \tag{17}$$

Similarly the expression for the relativistic specific enthalpy is

$$h=c^2\left(1+\frac{h_{\mathrm{N}}}{c^2}\right), \tag{18}$$

where h_{N} is the specific enthalpy in the Newtonian limit, $h_{\mathrm{N}}=\epsilon+p/\rho$. We substitute these relations into Eq (15) to obtain

$$\begin{aligned}\Omega_{\mu\nu}u^{\mu} &= u^0\left\{\partial_t\left[\left(1+\frac{h_{\mathrm{N}}}{c^2}\right)v_i\right]-\partial_i[(c^2+h_{\mathrm{N}})u_0]\right\}\\ &\quad +v^i\left\{\partial_j\left[\left(1+\frac{h_{\mathrm{N}}}{c^2}\right)v_i\right]-\partial_i\left[\left(1+\frac{h_{\mathrm{N}}}{c^2}\right)v_j\right]\right\}. &(19)\end{aligned}$$

In tthe Newtonian limit, the terms u^0 and h_{N}/c^2 can be set to 1 and 0 respectively, so that the second term in the RHS of Eq (19) can be changed as

$$\begin{aligned}\partial_i[(c^2+h_{\mathrm{N}})u_0] &= -\partial_i\left[(c^2+h_{\mathrm{N}})\left(1+\frac{\phi}{c^2}+\frac{v_jv^j}{2c^2}\right)\right] &(20)\\ &\simeq -\partial_i\left(\phi+\frac{1}{2}v_jv^j+h_{\mathrm{N}}\right). &(21)\end{aligned}$$

Finally we get

$$\begin{aligned}&\partial_tv_i+\partial_i\left(h_{\mathrm{N}}+\frac{1}{2}v_jv^j+\phi\right)+v^i(\partial_jv_i-\partial_iv_j)=T\partial_is &(22)\\ \Rightarrow\quad &\frac{\partial\vec{\boldsymbol{v}}}{\partial t}+\vec{\nabla}\cdot\left(\frac{1}{2}v^2+\epsilon+\frac{p}{\rho}+\phi\right)-\vec{\boldsymbol{v}}\times(\vec{\nabla}\times\vec{\boldsymbol{v}})=T\vec{\nabla}s. &(23)\end{aligned}$$

This equation is known as the Crocco equation of motion.

Lecture VIII, Exercise 3.

The vorticity four-vector is written as

$$\Omega^\mu = {}^*\Omega^{\mu\nu} u_\nu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\Omega_{\alpha\beta}u_\nu. \tag{24}$$

The kinetic vorticity four-vector is given by

$$\omega^\mu = {}^*\omega^{\mu\nu} u_\nu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta}u_\nu \tag{25}$$

Writing out Eq (24) explicitly we obtain

$$\begin{aligned}
\Omega_{\alpha\beta}u_\nu &= [\nabla_\beta(hu_\alpha)u_\nu - \nabla_\alpha(hu_\beta)u_\nu] && (26)\\
&= [h\nabla_\beta(u_\alpha)u_\nu + u_\alpha u_\nu \nabla_\beta h - h\nabla_\alpha(u_\beta)u_\nu - u_\beta u_\nu \nabla_\alpha h] && (27)\\
&= hu_\nu(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta) + u_\alpha u_\nu \nabla_\beta h - u_\beta u_\nu \nabla_\alpha h && (28)\\
&= hu_\nu 2\nabla_{[\beta}u_{\alpha]}, && (29)
\end{aligned}$$

where the terms including $u_\alpha u_\nu$ and $u_\beta u_\nu$ vanish because of the symmetry in the indices and the antisymmetry of the Levi-Civita tensor.

From the definition of the kinetic vorticity tensor, we instead obtain

$$\begin{aligned}
\omega_{\mu\nu} &= \nabla_{[\mu}u_{\nu]} + a_{[\mu}u_{\nu]} && (30)\\
\Rightarrow \quad \nabla_{[\mu}u_{\nu]} &= \omega_{\mu\nu} - a_{[\mu}u_{\nu]}. && (31)
\end{aligned}$$

Therefore connecting these two results, the vorticity four-vector can be given by

$$\begin{aligned}
\Omega^\mu &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}hu_\nu\omega_{\beta\alpha} - \epsilon^{\mu\nu\alpha\beta}hu_\nu a_{[\beta}u_{\alpha]} && (32)\\
&= 2h\omega^\mu, && (33)
\end{aligned}$$

where the second term of the RHS in Eq. (32) vanishes because of the symmetries in the four-velocity.