Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture VI

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Winter Semester 2014

Lecture VI, Exercise 1.

u^μ is 4-velocity and a^μ is 4-acceleration, whose contravariant components are defined as

$$a^\mu = u^\nu \nabla_\nu u^\mu. \tag{1}$$

There is a normalization condition

$$u^\mu u_\mu = -1 \tag{2}$$

and the orthogonality condition

$$a^\mu u_\mu = 0. \tag{3}$$

After taking a contravariant derivative of Eq (2), we can obtain the identity

$$u^\mu \nabla_\nu u_\mu = 0. \tag{4}$$

First we consider the expansion scalar Θ. Starting from its definition, the expansion scalar can be rewritten as

$$\begin{aligned}
\Theta &= h^{\mu\nu}\nabla_\nu u_\mu & (5)\\
&= (g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu u_\mu & (6)\\
&= g^{\mu\nu}\nabla_\nu u_\mu + u^\mu u^\nu \nabla_\nu u_\mu & (7)\\
&= \nabla_\nu g^{\mu\nu} u_\mu \quad \text{(here using Eq (4))} & (8)\\
&= \nabla_\nu u^\nu. & (9)
\end{aligned}$$

Next we consider the kinematic vorticity tensor $\omega_{\mu\nu}$. Starting from its definition, the kinematic vorticity tensor can be rewritten as

$$
\begin{aligned}
\omega_{\mu\nu} &= h_\mu^\alpha h_\nu^\beta \nabla_{[\beta,u_\alpha]} &(10)\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)(g_\nu^\beta + u^\beta u_\nu)(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta) &(11)\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[g_\nu^\beta(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta) + u^\beta u_\nu \nabla_\beta u_\alpha - u^\beta u_\nu \nabla_\alpha u_\beta] &(12)\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[(\nabla_\nu u_\alpha - \nabla_\alpha u_\nu) + u^\beta u_\nu \nabla_\beta u_\alpha] \quad \text{(using Eq (4))} &(13)\\
&= \frac{1}{2}g_\mu^\alpha(\nabla_\nu u_\alpha - \nabla_\alpha u_\nu) + \frac{1}{2}[u^\alpha u_\mu(\nabla_\nu u_\alpha - \nabla_\alpha u_\nu)]\\
&\quad + \frac{1}{2}g_\mu^\alpha u_\nu u^\beta \nabla_\beta u_\alpha + \frac{1}{2}u^\alpha u_\mu u^\beta u_\nu \nabla_\beta u_\alpha &(14)\\
&= \frac{1}{2}(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + \frac{1}{2}(u^\beta \nabla_\beta u_\mu u_\nu - u^\alpha \nabla_\alpha u_\nu u_\mu) \quad \text{(using Eq (4))} &(15)\\
&= \frac{1}{2}(\nabla_\nu u_\mu - \nabla_\mu u_\nu) + \frac{1}{2}(a_\mu u_\nu - a_\nu u_\mu) \quad \text{(using Eq (1))} &(16)\\
&= \nabla_{[\nu u_\mu]} + a_{\mu u_\nu}. &(17)
\end{aligned}
$$

Next we consider the shear tensor $\sigma_{\mu\nu}$. Starting from its definition, the shear tensor can be rewritten as

$$
\begin{aligned}
\sigma_{\mu\nu} &= \nabla_{\langle \mu u_\nu \rangle} &(18)\\
&= h_\mu^\alpha h_\nu^\beta \nabla_{(\beta u_\alpha)} - \frac{1}{3}\nabla_\beta u_\alpha h^{\alpha\beta} h_{\mu\nu} &(19)\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)(g_\nu^\beta + u^\beta u_\nu)(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta) - \frac{1}{3}\Theta h_{\mu\nu} &(20)\\
&\quad \text{(using Eq (5))}\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[g_\nu^\beta(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta) + u^\beta u_\nu(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta)]\\
&\quad - \frac{1}{3}\Theta h_{\mu\nu} &(21)\\
&= \frac{1}{2}(g_\mu^\alpha + u^\alpha u_\mu)[(\nabla_\nu u_\alpha + \nabla_\alpha u_\nu) + u^\beta u_\nu \nabla_\beta u_\alpha] - \frac{1}{3}\Theta h_{\mu\nu} &(22)\\
&\quad \text{(using Eq (4))}\\
&= \frac{1}{2}g_\mu^\alpha(\nabla_\nu u_\alpha + \nabla_\alpha u_\nu) + \frac{1}{2}g_\mu^\alpha u^\beta u_\nu \nabla_\beta u_\alpha\\
&\quad + \frac{1}{2}u^\alpha u_\mu(\nabla_\nu u_\alpha + \nabla_\alpha u_\nu) + \frac{1}{2}u^\alpha u_\mu u^\beta u_\nu \nabla_\beta u_\alpha + \frac{1}{3}\Theta h_{\mu\nu} &(23)\\
&= \frac{1}{2}(\nabla_\nu u_\mu + \nabla_\mu u_\nu) + \frac{1}{2}(a_\mu u_\nu + a_\nu u_\mu) - \frac{1}{3}\Theta h_{\mu\nu} &(24)\\
&\quad \text{(using Eq (4) \& (5))}\\
&= \nabla_{(\mu u_\nu)} + a_{(\mu u_\nu)} - \frac{1}{3}\Theta h_{\mu\nu}. &(25)
\end{aligned}
$$

Lecture VI, Exercise 2.

The energy-momentum tensor of a perfect fluid can be written as

$$T^{\mu\nu} = (e+p)u^\mu u^\nu + pg^{\mu\nu}. \tag{26}$$

Now we consider the following projection

$$\begin{aligned}
L_\mu &= -h^\alpha_\mu u^\beta T_{\alpha\beta} && (27)\\
&= -(g^\alpha_\mu + u^\alpha u_\mu)[(e+p)u^\beta u_\alpha u_\beta + pg_{\alpha\beta}u^\beta] && (28)\\
&= -(e+p)g^\alpha_\mu u^\beta u_\alpha u_\beta - pg_{\alpha\beta}g^\alpha_\mu u^\beta \\
&\quad -(e+p)u^\alpha u_\mu u^\beta u_\alpha u_\beta - pg_{\alpha\beta}u^\beta u^\alpha u_\mu && (29)\\
&= (e+p)g^\alpha_\mu u_\alpha - (e+p)u_\alpha u_\alpha u^\beta u_\beta u_\mu \\
&\quad -pg^\alpha_\mu g_{\alpha\beta}u^\beta - u^\alpha u_\mu p u_\alpha && (30)\\
&= (e+p)u_\mu - (e+p)u_\mu - pu_\mu + pu_\mu && (31)\\
&= 0. && (32)
\end{aligned}$$

This projection is identically zero for the case of a perfect fluid. Because in the case of a perfect fluid, the energy-momentum tensor in the local rest frame of a comoving observer is symmetric. Since there are no off-diagonal components, the spatial momentum density is identically zero.

Lecture VI, Exercise 3.

The spatial stress tensor $L_{\mu\nu}$, the spatial momentum density L_μ, and the energy density e can be defined using the energy-momentum tensor $T_{\alpha\beta}$ and the projection

$$\begin{aligned}
L_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu T_{\alpha\beta}, && (33)\\
L_\mu &= -h^\alpha_\mu u^\beta T_{\alpha\beta}, && (34)\\
e &= u^\alpha u^\beta T_{\alpha\beta}. && (35)
\end{aligned}$$

We try to show the following identity is true

$$T_{\mu\nu} = eu_\mu u_\nu + 2u_{(\mu}L_{\nu)} + L_{\mu\nu}. \tag{36}$$

In Eq (36), the each terms are given by

$$\begin{aligned}
eu_\mu u_\nu &= u^\alpha u^\beta T_{\alpha\beta}u_\mu u_\nu, && (37)\\
2u_{(\mu}L_{\mu)} &= 2\times\frac{1}{2}(u_\mu L_\nu + u_\nu L_\mu) = -h^\alpha_\nu u^\beta T_{\alpha\beta}u_\mu - h^\alpha_\mu u^\beta T_{\alpha\beta}u_\nu, && (38)\\
L_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu T_{\alpha\beta}. && (39)
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{Eq (36)} &= T_{\alpha\beta}(u^\alpha u^\beta u_\mu u_\nu - h^\alpha_\nu u^\beta u_\mu - h^\alpha_\mu u^\beta u_\mu + h^\alpha_\mu h^\beta_\nu) &(40)\\
&= T_{\alpha\beta}[u^\alpha u^\beta u_\mu u_\nu - (g^\alpha_\nu + u^\alpha u_\nu)u^\beta u_\mu - (g^\alpha_\mu + u^\alpha u_\mu)u^\beta u_\nu \\
&\quad +(g^\alpha_\mu + u^\alpha u_\mu)(g^\beta_\nu + u^\beta u_\nu)] &(41)\\
&= T_{\alpha\beta}(u^\alpha u^\beta u_\mu u_\nu - g^\alpha_\nu u^\beta u_\mu - u^\alpha u_\nu u^\beta u_\mu - g^\alpha_\mu u^\beta u_\nu - u^\alpha u_\mu u^\beta u_\nu \\
&\quad +g^\alpha_\mu g^\beta_\nu + g^\alpha_\mu u^\beta u_\nu + g^\beta_\nu u^\alpha u_\mu + u^\alpha u_\mu u^\beta u_\nu) &(42)\\
&= T_{\alpha\beta}(-g^\alpha_\nu u^\beta u_\mu - g^\alpha_\mu u^\beta u_\nu + g^\alpha_\mu g^\beta_\nu + g^\alpha_\mu u^\beta u_\nu + g^\beta_\nu u^\alpha u_\mu) &(43)\\
&= T_{\alpha\beta} g^\alpha_\mu g^\beta_\nu &(44)\\
&= T_{\mu\nu}. &(45)
\end{aligned}$$