Hydrodynamics and Magnetohydrodynamics: Exercise Solutions - Lecture III

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Lecture III, Exercise 1.

The four momentum $\vec{p}$ is

$$\vec{\boldsymbol{p}} = mc\vec{\boldsymbol{u}} = (p^0, p^i). \tag{1}$$

The contravariant and covariant forms of four momentum is written as

$$\begin{aligned} p^\mu &= mW(1, v^i), & (2)\\ p_\mu &= mW(-1, v^i), & (3) \end{aligned}$$

where W is Lorentz factor and v^i is the three velocity. The square of the four momentum is

$$p^2 = p^\mu p_\mu = -m^2c^2. \tag{4}$$

Now we consider the frame boosted x-direction. The Lorentz matrix is given by

$$\Gamma^{\nu'}_{\ \mu} = \begin{pmatrix} W & -Wv & 0 & 0 \\ -Wv & W & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

The Lorentz transformation for the four-momentum is obtained

$$p^{\nu'} = \Gamma^{\nu'}_{\ \mu} p^\mu. \tag{6}$$

And it becomes

$$\begin{aligned} p^{0'} &= W(p^0 - vp^1) & (7)\\ p^{1'} &= W(p^1 - vp^0) & (8)\\ p^{2'} &= p^2 & (9)\\ p^{3'} &= p^3 & (10) \end{aligned}$$

$d^3p = dp^1 dp^2 dp^3$ and $d^3p' = dp^{1'} dp^{2'} dp^{3'}$. Taking a derivative in Eq (8) yields

$$\frac{dp^{1'}}{dp^1} = W\left(1 - v\frac{dp^0}{dp^1}\right). \tag{11}$$

From eq (4),

$$\frac{dp^0}{dp^1} = \frac{d}{dp^1}\left(\sum_{i=1,2,3}(p^i)^2 + m^2c^2\right)^{1/2} = p^1\left(\sum_{i=1,2,3}(p^i)^2 + m^2c^2\right)^{-1/2} = \frac{p^1}{p^0}, \tag{12}$$

(where we use $(p^0)^2 = p^i p_i + m^2c^2$.) Using Eq (12), Eq (11) becomes

$$\frac{dp^{1'}}{dp^1} = W\left(1 - v\frac{p^1}{p^0}\right) = \frac{W(p^0 - vp^1)}{p^0} = \frac{p^{0'}}{p^0} \tag{13}$$

It can be written as

$$\frac{dp^{1'}}{p^{0'}} = \frac{dp^1}{p^0}. \tag{14}$$

Because $dp^{2'} = dp^2$ and $dp^{3'} = dp^3$, we obtain

$$\frac{d^3p'}{p^{0'}} = \frac{d^3p}{p^0}. \tag{15}$$

Lecture III, Exercise 2.

Start from the Euler equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho v^2 + \rho\epsilon + p\right)\vec{\boldsymbol{v}}\right] = \frac{\rho}{m}\vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} \tag{16}$$

$$\begin{aligned}
\Rightarrow\quad & \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon\right) + \left(\frac{1}{2}\rho v^2 + \rho\epsilon + p\right)\nabla\cdot\vec{\boldsymbol{v}} \\
& +(\vec{\boldsymbol{v}}\cdot\nabla)\left(\frac{1}{2}\rho v^2 + \rho\epsilon + p\right) = \frac{\rho}{m}\vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} \qquad (17) \\
\Rightarrow\quad & \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon\right) + (\vec{\boldsymbol{v}}\cdot\nabla)\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon\right) \\
& + \left(\frac{1}{2}\rho v^2 + \rho\epsilon + p\right)\nabla\cdot\vec{\boldsymbol{v}} = \frac{\rho}{m}\vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}\cdot\nabla p. \qquad (18)
\end{aligned}$$

Here using $D/Dt = \partial/\partial t + \vec{\boldsymbol{v}}\cdot\nabla$, we obtain

$$\frac{D}{Dt}\left(\frac{1}{2}\rho\vec{\boldsymbol{v}}^2 + \rho\epsilon\right) + \left(\frac{1}{2}\rho v^2 + \rho\epsilon + p\right)\nabla\cdot\vec{\boldsymbol{v}} = \rho\vec{\boldsymbol{v}}\left(\frac{\vec{\boldsymbol{F}}}{m} - \frac{1}{\rho}\nabla p\right). \tag{19}$$

Lecture III, Exercise 3.

First we assume the flow is incompressible

$$\nabla\cdot\vec{\boldsymbol{v}} = 0. \tag{20}$$

And we use mass and momentum conservations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{\boldsymbol{v}}) = 0 \tag{21}$$

$$\frac{\partial\rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{\boldsymbol{v}}\vec{v} + p\mathcal{I}) = -\rho g e_y. \tag{22}$$

In the static state, we assume following condition

$$\rho = \rho_0, \quad \vec{\boldsymbol{v}} = (v_0, 0, 0), \quad p = p_0, \tag{23}$$

where ρ_0, v_0, and p_0 are a function of y-direction only. The set of equations are written as

$$\nabla \cdot \vec{\boldsymbol{v}}_0 = 0 \tag{24}$$

$$\nabla \cdot (\rho_0 \vec{\boldsymbol{v}}_0) = 0 \tag{25}$$

$$\nabla \cdot (\rho_0 \vec{\boldsymbol{v}}_0 \vec{\boldsymbol{v}}_0 + p_0 \mathcal{I}) = -\rho_0 g e_y, \tag{26}$$

where we do not use the partial time derivative term because it is in static state ($\partial/\partial t = 0$).

Here we introduce perturbations in all quantities,

$$\rho = \rho_0 + \delta\rho, \ \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_0 + \delta\vec{\boldsymbol{v}} = (v_0 + \delta v_x, \delta v_y, 0), \ p = p_0 + \delta p. \tag{27}$$

Eq (22) can be written as

$$\text{eq (22)} \Rightarrow \nabla \cdot \vec{\boldsymbol{v}}_0 + \nabla \cdot \delta\vec{\boldsymbol{v}} = \nabla \cdot \delta\vec{\boldsymbol{v}} = 0, \tag{28}$$

where we use eq (26). Eq (23) can be changed as

$$\begin{aligned}
\text{eq (23)} \quad \Rightarrow \quad & \frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \nabla \cdot [(\rho_0 + \delta\rho)(\vec{\boldsymbol{v}}_0 + \delta\vec{\boldsymbol{v}})] && (29)\\
= \quad & \frac{\partial}{\partial t}\delta\rho + \nabla \cdot (\rho_0 \vec{\boldsymbol{v}}_0) + \nabla \cdot (\delta\rho \vec{\boldsymbol{v}}_0 + \rho_0 \delta\vec{\boldsymbol{v}}_0) && (30)\\
= \quad & \frac{\partial}{\partial t}\delta\rho + \nabla \cdot (\delta\rho \vec{\boldsymbol{v}}_0 + \rho_0 \delta\vec{\boldsymbol{v}}) && (31)\\
= \quad & \frac{\partial}{\partial t}\delta\rho + (\vec{\boldsymbol{v}}_0 \cdot \nabla)\delta\rho + \delta\rho(\nabla \cdot \vec{\boldsymbol{v}}_0) + (\delta\vec{\boldsymbol{v}} \cdot \nabla)\rho_0 + \rho_0(\nabla \cdot \delta\vec{\boldsymbol{v}}) && (32)\\
= \quad & \frac{\partial}{\partial t}\delta\rho + (\vec{\boldsymbol{v}}_0 \cdot \delta)\delta\rho + (\delta\vec{\boldsymbol{v}} \cdot \nabla)\rho_0 = 0, && (33)
\end{aligned}$$

where we ignore time derivative of the initial sate and use Eqs (26), (27), and (28). Eq (24) is also changed as

$$\begin{aligned}
\text{eq (24)} \quad \Rightarrow \quad & \frac{\partial}{\partial t}[(\rho_0 + \delta\rho)(\vec{\boldsymbol{v}}_0 + \delta\vec{\boldsymbol{v}})] \\
& + \nabla \cdot [(\rho_0 + \delta\rho)(\vec{\boldsymbol{v}}_0 + \delta\vec{\boldsymbol{v}})(\vec{\boldsymbol{v}}_0 + \delta\vec{\boldsymbol{v}}) + (p_0 + \delta p)\mathcal{I}] && (34)\\
= \quad & \frac{\partial}{\partial t}[(\rho_0 \delta\vec{\boldsymbol{v}} + \delta\rho \vec{\boldsymbol{v}}_0) \\
& + \nabla \cdot [\rho_0 \vec{\boldsymbol{v}}_0 \vec{\boldsymbol{v}}_0 + \delta\rho \vec{\boldsymbol{v}}_0 \vec{\boldsymbol{v}}_0 + \rho_0 \vec{\boldsymbol{v}}_0 \delta\vec{\boldsymbol{v}} + \rho_0 \delta\vec{\boldsymbol{v}} \vec{\boldsymbol{v}}_0 + (p_0 + \delta p)\mathcal{I}] \\
& = -(\rho_0 + \delta\rho) g e_y, && (35)
\end{aligned}$$

where we neglect time derivative of initial state and 2nd-order terms. Using Eq (26), eq (35) is given as

$$\begin{aligned}
\text{eq (35)} \quad \Rightarrow \quad & \frac{\partial}{\partial t}(\rho_0 \delta\vec{\boldsymbol{v}} + \delta\rho \vec{\boldsymbol{v}}_0) \\
& + \nabla \cdot [\delta\rho \vec{\boldsymbol{v}}_0 \vec{\boldsymbol{v}}_0 + \rho_0 \vec{\boldsymbol{v}}_0 \delta\vec{\boldsymbol{v}} + \rho_0 \delta\vec{\boldsymbol{v}} \vec{\boldsymbol{v}}_0 + \delta p \mathcal{I}] = \delta\rho g e_y && (36)
\end{aligned}$$

Eqs (28), (33), and (36) are linearized equations for this problem. Next we divide these linearized equations in each component

$$
\begin{aligned}
\text{eq (28)} \quad &\Rightarrow \quad \frac{\partial}{\partial x}\delta v_x + \frac{\partial}{\partial_y}\delta v_y = 0 && (37)\\
\text{eq (33)} \quad &\Rightarrow \quad \frac{\partial}{\partial t}\delta\rho + v_0\frac{\partial}{\partial x}\delta\rho + \delta v_x\frac{\partial}{\partial x}\rho_0 + \delta v_y\frac{\partial}{\partial y}\rho_0 \\
&= \quad \frac{\partial}{\partial t}\delta\rho + v_0\frac{\partial}{\partial x}\delta\rho + \delta v_y\frac{\partial}{\partial y}\rho_0 = 0, && (38)\\
&\quad\ \text{(where } \rho_0 \text{ is a function of } y \text{ only)}\\
\text{eq (36)}_x \quad &\Rightarrow \quad \rho_0\frac{\partial}{\partial t}\delta v_x + v_0\frac{\partial}{\partial t}\delta\rho + \frac{\partial}{\partial x}[(\delta\rho v_0)_x v_0 + (\rho_0 v_0)_x\delta v_x + (\rho_0\delta v)_x v_0 + \delta p] \\
&\quad\ +\frac{\partial}{\partial y}[(\rho_0 v_0)_x\delta v_y] && (39)\\
&= \quad \rho_0\frac{\partial}{\partial t}\delta v_x - v_0\left(v_0\frac{\partial}{\partial x}\delta\rho + \delta v_y\frac{\partial}{\partial y}\rho_0\right) + \frac{\partial}{\partial x}(v_0^2\delta\rho + 2\rho_0 v_0\delta v_x + \delta p) \\
&\quad\ +\frac{\partial}{\partial y}(\rho_0 v_0\delta v_y) \text{ (where we use eq(38))} && (40)\\
&= \quad \rho_0\frac{\partial}{\partial t}\delta v_x - v_0^2\frac{\partial}{\partial x}\delta\rho - v_0\delta y\frac{\partial}{\partial y}\rho_0 + v_0^2\frac{\partial}{\partial x}\delta\rho + 2\rho_0 v_0\frac{\partial}{\partial x}\delta v_x + \frac{\partial}{\partial x}\delta p \\
&\quad\ +v_0\delta v_y\frac{\partial}{\partial y}\rho_0 + \rho_0\delta v_y\frac{\partial}{\partial y}v_0 + \rho_0 v_0\frac{\partial}{\partial y}\delta v_y && (41)\\
&= \quad \rho_0\frac{\partial}{\partial t}\delta v_x + 2\rho_0 v_0\frac{\partial}{\partial x}\delta v_x + \frac{\partial}{\partial x}\delta p + \rho_0\delta v_y\frac{\partial}{\partial y}v_0 + \rho_0 v_0\frac{\partial}{\partial y}\delta v_y && (42)\\
&= \quad \rho_0\frac{\partial}{\partial t}\delta v_x + \rho_0 v_0\frac{\partial}{\partial x}\delta v_x + \rho_0 v_0\left(\frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y\right) \\
&\quad\ +\frac{\partial}{\partial x}\delta p + p_0\delta v_y\frac{\partial}{\partial y}v_0 && (43)\\
&= \quad \rho_0\frac{\partial}{\partial t}\delta v_x + \rho_0 v_0\frac{\partial}{\partial x}\delta v_x + \frac{\partial}{\partial x}\delta p + p_0\delta v_y\frac{\partial}{\partial y}v_0 = 0 && (44)\\
&\quad\ \text{(where we use Eq (37)),}\\
\text{eq (36)}_y \quad &\Rightarrow \quad \rho_0\frac{\partial}{\partial t}\delta v_y + \frac{\partial}{\partial x}[(\delta\rho v_0)_y v_0 + (\rho_0 v_0)_y\delta v_x + (\rho_0\delta v)_y v_0] \\
&\quad\ +\frac{\partial}{\partial y}[(\delta\rho v_0)_y\cdot 0 + (\rho_0 v_0)_y\delta v_y + (\rho_0\delta v)_y\cdot 0 + \delta p] && (45)\\
&= \quad \rho_0\frac{\partial}{\partial t}\delta v_y + \frac{\partial}{\partial x}(\rho_0\delta v_y v_0) + \frac{\partial}{\partial y}\delta p && (46)\\
&= \quad \rho_0\frac{\partial}{\partial t}\delta v_y + \rho_0 v_0\frac{\partial}{\partial x}\delta v_y + \frac{\partial}{\partial y}\delta p = -\delta\rho g && (47)\\
&\quad\ \text{(where } \rho_0 \text{ and } v_0 \text{ are a function of } y \text{ only)}\\
& && (48)
\end{aligned}
$$

Then we introduce Fourier mode for perturbed state,

$$\delta\rho, \delta\vec{v}, \delta p \propto e^{i(kx-\omega t)}. \tag{49}$$

Eqs (37), (38), (44), and (47) are then written as

$$\begin{aligned}
\text{eq (37)} \quad &\Rightarrow \quad ik\delta v_x + \frac{\partial}{\partial y}\delta v_y = 0 &(50)\\
&\rightarrow \quad \delta v_x = \frac{i}{k}\frac{\partial}{\partial y}\delta v_y &(51)\\
\text{eq (38)} \quad &\Rightarrow \quad -i\omega\delta\rho + \delta v_y \frac{\partial \rho_0}{\partial y} + ikv_0\delta\rho = 0 &(52)\\
&\rightarrow \quad \delta\rho = \frac{i}{kv_0 - \omega}\delta v_y \frac{\partial \rho_0}{\partial y} &(53)\\
\text{eq (44)} \quad &\Rightarrow \quad -i\omega\rho_0\delta v_x + ik\rho_0 v_0 \delta v_x + ik\delta p + \rho_0 \delta v_y \frac{\partial v_0}{\partial y} &(54)\\
&= \quad -i\rho_0(\omega - kv_0)\delta v_x + ik\delta p + \rho_0\delta v_y \frac{\partial v_0}{\partial y} = 0 &(55)\\
&\rightarrow \quad \delta p = \frac{\omega - kv_0}{k}\rho_0\delta v_x + i\frac{\rho_0}{k}\delta v_y \frac{\partial v_0}{\partial y} &(56)\\
&\quad\quad = i\frac{(\omega - kv_0)}{k^2}\rho_0 \frac{\partial}{\partial y}\delta v_y + i\frac{\rho_0}{k}\delta v_y\frac{\partial v_0}{\partial y} \quad \text{(using eq. (51))} &(57)\\
\text{eq (47)} \quad &\Rightarrow \quad -i\omega\rho_0\delta v_y + ik\rho_0 v_0\delta v_y + \frac{\partial}{\partial y}\delta p = -\delta\rho g &(58)\\
&\rightarrow \quad -i\rho_0(\omega - kv_0)\delta v_y + \frac{\partial}{\partial y}\left[i\frac{(\omega - kv_0)}{k^2}\rho_0\frac{\partial}{\partial y}\delta v_y + i\frac{\rho_0}{k}\delta v_y \frac{\partial}{\partial y}v_0\right] \\
&\quad\quad = \frac{ig}{\omega - kv_0}\delta v_y \frac{\partial \rho_0}{\partial y} \quad \text{(using eq. (53) and (57))} &(59)
\end{aligned}$$

We multiply Eq. (59) by a factor of k^2/i to obtain that

$$\begin{aligned}
&-\rho_0 k^2(\omega - kv_0)\delta v_y + \frac{\partial}{\partial y}\left[(\omega - kv_0)\rho_0\frac{\partial}{\partial y}\delta v_y + \rho_0 k\delta v_y \frac{\partial v_0}{\partial y}\right] \\
&= \frac{gk^2}{\omega - kv_0}\delta v_y \frac{\partial \rho_0}{\partial y}. &(60)
\end{aligned}$$

Next we consider boundary condition for this problem. Since at the region of $y \neq 0$, $\partial\rho_0/\partial y = \partial v_0/\partial y = 0$, the Eq (60) can be expressed as

$$\begin{aligned}
(\omega - kv_0)\rho_0\frac{\partial^2}{\partial y^2}\delta v_y - \rho_0 k^2(\omega - kv_0)\delta v_y = 0 &\quad (61)\\
\rightarrow \quad [(\omega - kv_0)\rho_0]\left(\frac{\partial^2}{\partial y^2} - k^2\right)\delta v_y = 0 &\quad (62)
\end{aligned}$$

The perturbation in y-direction becomes small far from contact surface. Thus the perturbed velocity in y-direction can be given by

$$\delta v_y = A\exp(-k|y|). \tag{63}$$

At the contact surface ($y = 0$), perturbation gives change of surface. We introduce changing profile $Y = \eta(x,t)$. This surface should move with fluid motion. It means that

$$\delta v_y = \frac{DY}{Dt} = \left\{ \frac{\partial}{\partial t} + (v_0 + \delta v_x)\frac{\partial}{\partial x} \right\} \eta. \tag{64}$$

δy follows Fourier mode. Therefore $\eta \propto e^{i(kx-\omega t)}$. Since the amplitude is small, we can lininalize the eq (64),

$$\delta v_y = [-i\omega + (v_0 \delta v_x) ik]\eta = -i(\omega - kv_0)\eta. \tag{65}$$

The ratio between upper region of the contact surface and lower region of the contact surface is shown as

$$\frac{\delta v_y^{(1)}}{\delta v_y^{(2)}} = \frac{\omega - kv_0^{(1)}}{\omega - kv_0^{(2)}}. \tag{66}$$

Adding eq (66) in eq (63), we can obtain δv_y in $y \neq 0$ region,

$$\begin{cases} \delta v_y^{(1)} = (\omega - kv_0^{(1)})e^{-ky} \\ \delta v_y^{(2)} = (\omega - kv_0^{(2)})e^{-ky} \end{cases} \tag{67}$$

Next we consider contact surface ($y = 0$) region. Here we introduce $\Delta_s(f)$ which is integrated in small region between the upper and lower regions of contact surface, $[-\epsilon, \epsilon]$,

$$\Delta_s(f) = \lim_{\epsilon \to 0} \int_{0-\epsilon}^{0+\epsilon} \frac{\partial f}{\partial y} dy = \lim_{\epsilon \to 0} [f(\epsilon) - f(-\epsilon)]. \tag{68}$$

We integrate eq (60) in small region between the upper and lower regions of contact surface, $[-\epsilon, \epsilon]$,

$$\lim \int_{0-\epsilon}^{0+\epsilon} \rho_0 k^2 (\omega - kv_0) \delta v_y dy \to 0, \tag{69}$$

$$\lim \int_{0-\epsilon}^{0+\epsilon} \frac{\partial}{\partial y} \left[(\omega - kv_0)\rho_0 \frac{\partial}{\partial y} \delta v_y + \rho_0 k \delta v_y \frac{\partial v_0}{\partial y} \right] \tag{70}$$

$$= \Delta_s \left((\omega - kv_0)\rho_0 \frac{\partial}{\partial y} \delta v_y \right) + \Delta_s \left(\rho_0 k \delta v_y \frac{\partial v_0}{\partial y} \right) \tag{71}$$

$$= \Delta_s \left((\omega - kv_0)\rho_0 \frac{\partial}{\partial y} \delta v_y \right) \quad (\partial v_0 / \partial y = 0 \text{ at } y \neq 0), \tag{72}$$

$$\lim \int_{0-\epsilon}^{0+\epsilon} \frac{\delta v_y}{\omega - kv_0} \frac{\partial \rho_0}{\partial y} dy = \lim \int_{0-\epsilon}^{0+\epsilon} e^{-k|y|} \frac{\partial \rho_0}{\partial y} dy \quad \text{(using eq. (67))} \tag{73}$$

$$= e^{-k|y|} \Delta_s(\rho_0) - \lim \int_{0-\epsilon}^{0+\epsilon} \rho_0 \frac{\partial}{\partial y} e^{-k|y|} dy \tag{74}$$

$$= \frac{\delta v_y}{\omega - kv_0} \Delta_s(\rho_0). \tag{75}$$

Therefore the integrated Eq. (66) can be written as

$$\Delta_s \left[(\omega - kv_0)\rho_0 \frac{\partial}{\partial y} \delta v_y \right] = k^2 g \frac{\delta v_y}{\omega - kv_0} \Delta_s(\rho_0). \tag{76}$$

Eq (67) is inserted to eq (76) then

$$\begin{aligned} \Delta_s \left[(\omega - kv_0)\rho_0 \frac{\partial}{\partial y} \delta v_y \right] &= (\omega - kv_0^{(1)})\rho_0^{(1)}(\omega - kv_0^{(1)})(-k)e^{-ky} \\ &\quad -(\omega - kv_0^{(2)})\rho_0^{(2)}(\omega - kv_0^{(2)})(k)e^{ky} \qquad (77) \\ &\to -k[(\omega - kv_0^{(1)})^2\rho_0^{(1)} + (\omega - kv_0^{(2)})^2\rho_0^{(2)}] \quad (y \to 0) \qquad (78) \\ \Delta_s(\rho_0) &\Rightarrow \rho_0^{(1)} - \rho_0^{(2)}. \qquad (79) \end{aligned}$$

Therefore from eq (76) we can obtain

$$-k[(\omega - kv_0^{(1)})^2\rho_0^{(1)} + (\omega - kv_0^{(2)})^2\rho_0^{(2)}] = k^2 g(\rho_0^{(1)} - \rho_0^{(2)}) \tag{80}$$

$$\begin{aligned} \to \quad & (\rho_0^{(1)} + \rho_0^{(2)})\omega^2 - 2k(\rho_0^{(1)}v_0^{(1)} + \rho_0^{(2)}v_0^{(2)})\omega \\ & + k^2(\rho_0^{(1)}{v_0^{(1)}}^2 + \rho_0^{(2)}{v_0^{(2)}}^2) + kg(\rho_0^{(1)} - \rho_0^{(2)}) = 0. \qquad (81) \end{aligned}$$

This is dispersion relation for this problem. The solution is given by

$$\frac{\omega}{k} = \alpha_1 v_0^{(1)} + \alpha_2 v_0^{(2)} \pm \sqrt{-\alpha_1\alpha_2(v_0^{(1)} - v_0^{(2)})^2 - \frac{g}{k}(\alpha_1 - \alpha_2)}, \tag{82}$$

where $\alpha_1 = \rho_0^{(1)}/(\rho_0^{(1)} + \rho_0^{(2)})$ and $\alpha_2 = \rho_0^{(2)}/(\rho_0^{(1)} + \rho_0^{(2)})$. From this equation, the system is unstable when the inside of the root becomes negative. Therefore the system becomes unstable when

$$k > \frac{g({\rho_0^{(2)}}^2 + {\rho_0^{(1)}}^2)}{\rho_0^{(1)}\rho_0^{(2)}(v_0^{(1)} - v_0^{(2)})^2}. \tag{83}$$

This is stability condition of Kelvin-Helmholtz instability with gravity. From this criterion, we see that gravity stabilizes the KH instability at long-wavelengths.