

Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture II

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Lecture II, Exercise 1.

The Maxwell-Boltzmann (equilibrium) distribution function is considered in the following general form

$$\ln(f_0(\vec{u}_0)) = -A(\vec{u} - \vec{u}_0)^2 + \ln C, \quad (1)$$

$$f_0 = C \exp(-A(\vec{u} - \vec{u}_0)^2). \quad (2)$$

We recall the following integral identities.

$$\int_{-\infty}^{\infty} e^{-A\vec{u}^2} d^3u = \frac{\pi^{3/2}}{A^{3/2}}, \quad \int_{-\infty}^{\infty} \vec{u} e^{-A\vec{u}^2} d^3u = 0, \quad \int_{-\infty}^{\infty} \vec{u}^2 e^{-A\vec{u}^2} d^3u = \frac{3}{2} \frac{\pi^{3/2}}{A^{5/2}}. \quad (3)$$

First we consider the number density n . Using Eq.(2),

$$n = \int f d^3u = \int_{-\infty}^{\infty} C e^{-A(\vec{u}-\vec{u}_0)^2} d^3u. \quad (4)$$

Using $\vec{u} = (\vec{u} - \vec{u}_0)$ and $d\vec{u} = du$, Eq. (3) becomes

$$\int_{-\infty}^{\infty} C e^{-A\vec{u}^2} d^3\vec{u} = C \left(\frac{\pi^{3/2}}{A^{3/2}} \right). \quad (5)$$

Thus, the constant C can be written as

$$C = n \left(\frac{A}{\pi} \right)^{3/2}. \quad (6)$$

Next, we consider the specific internal energy ϵ . Using Eq. (2),

$$\epsilon = \frac{1}{2}(|\vec{u} - \vec{u}_0|^2) = \frac{1}{2n} \int |\vec{u} - \vec{u}_0|^2 f d^3u \quad (7)$$

$$= \frac{1}{2n} \int_{-\infty}^{\infty} |\vec{u} - \vec{u}_0|^2 C e^{-A(\vec{u}-\vec{u}_0)^2} d^3u. \quad (8)$$

Using $\vec{u} = (\vec{u} - \vec{u}_0)$ and $d\vec{u} = du$, Eq. (8) becomes

$$\frac{C}{2n} \int_{-\infty}^{\infty} \vec{u}^2 e^{-A\vec{u}^2} d^3\vec{u} = \frac{3C}{4n\pi} \left(\frac{\pi}{A} \right)^{5/2}. \quad (9)$$

Using Eq. (6), the internal energy can be written as

$$\epsilon = \frac{3}{4n\pi} n \left(\frac{A}{\pi} \right)^{3/2} \left(\frac{\pi}{A} \right)^{5/2} = \frac{3}{4A}. \quad (10)$$

Therefore the constant A can be found to be

$$A = \frac{3}{4\epsilon}. \quad (11)$$

Using Eq. (11), the constant C is also written as

$$C = n \left(\frac{3}{4\pi\epsilon} \right)^{3/2}. \quad (12)$$

Lecture II, Exercise 2.

The specific internal energy is given by

$$\epsilon = \frac{3}{2} \frac{k_B T}{m}. \quad (13)$$

Using Eq. (13), the constant A and C can be obtained

$$A = \frac{3}{4} \cdot \frac{2m}{3k_B T} = \frac{m}{2k_B T}, \quad (14)$$

$$C = n \left(\frac{3}{4\pi} \cdot \frac{2m}{3k_B T} \right)^{3/2} = n \left(\frac{m}{2\pi k_B T} \right)^{3/2}. \quad (15)$$

Next, we put these two quantities into the general form of the Maxwell-Boltzmann (equilibrium) distribution function Eq. (2). In this way we obtain

$$f_0(\vec{u}) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{m(\vec{u} - \vec{u}_0)^2}{2k_B T} \right). \quad (16)$$

Lecture II, Exercise 3.

Using a set of spherical coordinates (r, θ, ϕ) in the velocity space, any distribution function can be written as $(du_x du_y du_z = u^2 \sin \theta du d\theta d\phi)$

$$\int_{-\infty}^{\infty} f(u) d^3 u = \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^{\infty} u^2 f(u) du = \int_0^{\infty} 4\pi u^2 f(u) du. \quad (17)$$

Next, we use the Maxwell-Boltzmann distribution function with zero macroscopic velocity (i.e., $\vec{v} = 0$), which is given by

$$f = 4\pi n u^2 \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{m u^2}{2k_B T} \right). \quad (18)$$

The average speed v can be calculated as

$$v = \langle u \rangle = \frac{1}{n} \int_{-\infty}^{\infty} \vec{u} f d^3u \quad (19)$$

$$= \frac{1}{n} \int_0^{\infty} 4\pi u^2 n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(\frac{-mu^2}{2k_B T} \right) u du. \quad (20)$$

$$= 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} \int_0^{\infty} u^3 \exp\left(\frac{-mu^2}{2k_B T} \right) du. \quad (21)$$

Now we introduce a new parameter $x = u^2$, so that taking a derivative we obtain

$$\frac{d}{du} u^2 = \frac{dx}{du}, \quad (22)$$

$$du = \frac{1}{2u} dx. \quad (23)$$

The integral part of Eq. (19) becomes

$$\int_0^{\infty} u^3 \exp\left(\frac{-mu^2}{2k_B T} \right) du = \frac{1}{2} \int_0^{\infty} x \exp\left(-\frac{mx}{2k_B T} \right) dx \quad (24)$$

$$= \frac{1}{2} \int_0^{\infty} x \left[-\frac{2k_B T}{m} \exp\left(-\frac{2k_B T}{m} \right)' \right] dx \quad (25)$$

$$= \frac{1}{2} \left[x \left\{ -\frac{2k_B T}{m} \exp\left(-\frac{mx}{2k_B T} \right) \right\} \right]_0^{\infty} - \frac{1}{2} \int_0^{\infty} (x)' \left[-\frac{2k_B T}{m} \exp\left(-\frac{mx}{2k_B T} \right) \right] dx \quad (26)$$

$$= -\frac{k_B T}{m} \left[x \exp\left(-\frac{mx}{2k_B T} \right) \right]_0^{\infty} + \frac{k_B T}{m} \int_0^{\infty} \exp\left(-\frac{mx}{2k_B T} \right) dx \quad (27)$$

$$= 0 + \frac{k_B T}{m} \left[-\frac{2k_B T}{m} \exp\left(-\frac{mx}{2k_B T} \right) \right]_0^{\infty} \quad (28)$$

$$= -\frac{2k_B^2 T^2}{m^2} (0 - 1) = \frac{2k_B^2 T^2}{m^2}. \quad (29)$$

Therefore the average velocity is obtained to be

$$v = 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} \cdot \frac{2k_B^2 T^2}{m^2} \quad (30)$$

$$= \frac{4k_B T}{m} \sqrt{\frac{m}{2\pi k_B T}} = \sqrt{\frac{8k_B T}{\pi m}}. \quad (31)$$

Lecture II, Exercise 4.

The most probable speed is the maximum of Maxwell-Boltzmann distribution function, that is, the one for which $df/du = 0$. We use the Maxwell-Boltzmann distribution function with zero macroscopic velocity in spherical coordinates (i.e., Eq. (18)).

Taking a derivative of Eq (18)

$$\frac{df}{du} = \left[4\pi n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \right] \left[2u \exp \left(-\frac{mu^2}{2k_B T} \right) + u^2 \left(-\frac{2mu}{2k_B T} \right) \exp \left(-\frac{mu^2}{2k_B T} \right) \right] = 0 \quad (32)$$

Considering only the second bracket, i.e.,

$$\exp \left(-\frac{mu^2}{2k_B T} \right) u \left(2 - \frac{mu^2}{k_B T} \right) = 0. \quad (33)$$

and because the solution is $u \neq 0$, the only relevant term is the second one, i.e.,

$$2 - \frac{mu^2}{k_B T} = 0. \quad (34)$$

Therefore the most probable speed is

$$u_p = \sqrt{\frac{2k_B T}{m}} \quad (35)$$