Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture XIII

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Lecture XIII, Exercise 1.

Since the magnetic field does not produce a work on the particle, particle energy in a static magnetic field is conserved. However, if magnetic field is time-dependent, there must be an accompanying electric field

$$\vec{\nabla} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}. \tag{1}$$

Clearly the electric field can not be uniform in the space and so we must expect that electric field will change the particle's energy. Here we focus on the motion normal to the magnetic field. The perpendicular component of particle kinetic energy is given by

$$U_\perp = \frac{1}{2} m v_\perp^2. \tag{2}$$

Taking a time derivative and expressing the acceleration of the particle in terms of the electric fiedl, i.e.,

$$m \frac{d\vec{\boldsymbol{v}}_\perp}{dt} = q\vec{\boldsymbol{E}}. \tag{3}$$

we obtain that

$$\frac{dU_\perp}{dt} = q\vec{\boldsymbol{v}}_\perp \cdot \vec{\boldsymbol{E}}. \tag{4}$$

Expressing the perpendicular velocity as

$$\vec{\boldsymbol{v}}_\perp =: \frac{d\vec{\boldsymbol{X}}}{dt}, \tag{5}$$

where $\vec{\boldsymbol{X}}(t)$ denotes the trajectory of the particle, we can rewrite eq (12) as

$$\frac{dU_\perp}{dt} = q\frac{d\vec{\boldsymbol{X}}}{dt} \cdot \vec{\boldsymbol{E}}. \tag{6}$$

The total change in $U_\perp$ over the one cycle of the orbital motion is given by

$$\Delta U_\perp = \int_0^P q\frac{d\vec{\boldsymbol{X}}}{dt} \cdot \vec{\boldsymbol{E}} dt, \tag{7}$$

where P is period of the motion. The time variation of the magnetic field implies a time variation in both the gyro radius and the gyro period. Therefore the orbit will not be closed. However, if we assume that the change in the magnetic field during one period of the circular motion is small compared to the magnitude of magnetic field, i.e., if

$$P\left|\frac{d\vec{\boldsymbol{B}}}{dt}\right| = \frac{2\pi}{\omega_c}\left|\frac{d\vec{\boldsymbol{B}}}{dt}\right| \ll |\vec{\boldsymbol{B}}|. \tag{8}$$

then the time integral of eq (15) can be replaced by a line integral taken over a fictitious circular orbit of the particle

$$\Delta U_\perp = \oint q\vec{\boldsymbol{E}}dl. \tag{9}$$

By using Stokes' theorem, this can be expressed as

$$\Delta U_\perp = -q\int(\vec{\vec{\nabla}} \times \vec{\boldsymbol{E}})ds, \tag{10}$$

which shows that the surface integral follows the particle motion as for our assumption. Using eq (10), we rewrite eq (18) as

$$\Delta U_\perp = |q|\int\frac{\partial\vec{\boldsymbol{B}}}{\partial t}ds, \tag{11}$$

where the change in energy is in fact independent of the sign of the charge. If we assume that the magnetic field is uniform, the surface integral is expressed as πr_L^2, then eq (20) yields

$$\Delta U_\perp = |q|\pi r_L^2\frac{d\vec{\boldsymbol{B}}}{dt}. \tag{12}$$

As a result, the rate of change of energy per one gyration period is given by

$$\frac{dU_\perp}{dt} = \frac{\Delta U_\perp}{P} = \frac{1}{2}|q|\omega_c r_L^2\frac{d\vec{\boldsymbol{B}}}{dt}. \tag{13}$$

Lecture XIII, Exercise 2.

Here we focus on charged particle motion in a non-uniform magnetic field. When particles move into a weaker field, the Larmor radius increases. It decreases again as particle moves back into strong field. Different Larmor radii generates a drift motion of particle. Now qe consider the most simple case

$$\vec{\boldsymbol{E}} = 0, \quad \vec{\boldsymbol{B}} = (0, 0, B^z(y)). \tag{14}$$

We assume Larmor radius is much smaller than the lengthscale of the variation of magnetic field, i.e., $r_L/L \ll 1$. The orbit theory also assume the velocity can be decomposed into components with a small drift velocity

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_D + \vec{\boldsymbol{v}}_\perp, \tag{15}$$

where we assune the drift velocity to be much smaller than the other components of the velocity

$$v_D \ll v_\perp. \tag{16}$$

The equation of motion is written as

$$m\frac{d\vec{\boldsymbol{v}}}{dt} = q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}). \tag{17}$$

The components of the Lorentz force are given by

$$F_x = qv_y B^z, \tag{18}$$
$$F_y = -qv_z B^z, \tag{19}$$
$$F_z = 0. \tag{20}$$

The magnetic field has the z-component only which is a function of y. the gradient of magnetic field is

$$\frac{dB^z}{dy} \sim \frac{B^z}{L} \ll \frac{B^z}{r_L}. \tag{21}$$

Therefore the gradient of magnetic field is small,

$$r_L\frac{dB^z}{dy} \ll B^z. \tag{22}$$

From this condition, we take the Taylor expansion of the magnetic field

$$B^z(y) = B_0 + yB'_z + O(y^2), \tag{23}$$

where $B'_z = dB^z/dy$. Using this expression of magnetic field, the Lorentz force can be written as

$$F_x = qv_y(B_0 + yB'_z), \tag{24}$$
$$F_y = -qv_x(B_0 + yB'_z). \tag{25}$$

In the uniform magnetic field, the particle performs a circular motion. Therefore the position is given by

$$x = r_L \sin(\omega_c t), \tag{26}$$
$$y = r_L \cos(\omega_c t). \tag{27}$$

The velocity for the circular motion is expressed as

$$v_x = -v_\perp \cos(\omega_c t), \tag{28}$$
$$v_y = \pm v_\perp \sin(\omega_c t). \tag{29}$$

Using eqs (13) -(16), the various components of the Lorentz force can be obtained as

$$F_x = -qv_\perp \sin(\omega_c t)[B_0 \pm r_L \cos(\omega_c t)B'_z], \tag{30}$$
$$F_y = -qv_\perp \cos(\omega_c t)[B_0 \pm r_L \cos(\omega_c t)B'_z]. \tag{31}$$

We take a time average of them over the one period of the circular motion

$$\langle \psi \rangle := \frac{1}{\Delta t} \int_0^P \psi dt, \tag{32}$$

Using the above definition, the each components of Lorentz force are

$$\langle \dot{F}_x \rangle = -qv_\perp [B_0 \langle \sin(\omega_c t) \rangle \pm r_L \langle \sin(\omega_c t) \cos(\omega_c t) \rangle B'_z] \tag{33}$$

$$\langle \dot{F}_y \rangle = -qv_\perp [B_0 \langle \cos(\omega_c t) \rangle \pm r_L \langle \cos^2(\omega_c t) \rangle B'_z], \tag{34}$$

where $\langle \sin(\omega_c t) \rangle$, $\langle \cos(\omega_c t) \rangle$, and $\langle \sin(\omega_c t) \cos(\omega_c t) \rangle$ are zero, and $\langle \cos^2(\omega_c t) \rangle = 1/2$. Therefore

$$\langle \dot{F}_x \rangle = 0, \tag{35}$$

$$\langle \dot{F}_y \rangle = \pm \frac{qv_\perp r_L}{2} B'_z. \tag{36}$$

The drift velocity of the general forces is defined by

$$\vec{\boldsymbol{v}}_F = \frac{1}{q} \frac{\vec{\boldsymbol{F}} \times \vec{\boldsymbol{B}}}{B^2}. \tag{37}$$

Therefore in this case the drift velocity is given by

$$\vec{\boldsymbol{v}}_F = \frac{1}{q} \frac{\langle \dot{F}_y \rangle \hat{\boldsymbol{y}} \times B^z \hat{\boldsymbol{z}}}{B_z^2} \tag{38}$$

$$= \mp \frac{v_\perp r_L}{2B^z} \frac{dB^z}{dy} \hat{\boldsymbol{x}}. \tag{39}$$

In three-dimensions, this results can be generalized to

$$\vec{\boldsymbol{v}}_{\vec{\nabla} B} = \pm \frac{1}{2} v_\perp r_L \frac{\vec{\boldsymbol{B}} \times \vec{\nabla} |\vec{\boldsymbol{B}}|}{B^2}. \tag{40}$$

This drift is the so-called "grad-B" drift.

Lecture XIII, Exercise 3.

(a) The earth's magnetic field has a dipole field which contains gradient and curvature. Therefore we need to consider two drift motion, magnetic field gradient (grad-B) drift and curvature drift. The grad-B drift is given by

$$\vec{\boldsymbol{v}}_{\vec{\nabla} B} = \frac{mv_\perp^2}{2qB^3} (\vec{\boldsymbol{B}} \times \vec{\nabla} |\vec{\boldsymbol{B}}|). \tag{41}$$

And the curvature drift is expressed as

$$\vec{\boldsymbol{v}}_c = \frac{mv_\parallel^2}{qB^4} \vec{\boldsymbol{B}} \times [(\vec{\boldsymbol{B}} \cdot \vec{\nabla}) \vec{\boldsymbol{B}}]. \tag{42}$$

From the vector identity,

$$(\vec{\boldsymbol{B}}\cdot\vec{\nabla})\vec{\boldsymbol{B}}-\vec{\nabla}B^2/2 = (\vec{\nabla}\times\vec{\boldsymbol{B}})\times\vec{\boldsymbol{B}}=\mu_0\vec{\boldsymbol{J}}\times\vec{\boldsymbol{B}} \quad (43)$$

$$= 0 \quad (44)$$

because of the absence of currents. Using this equation, the curvature drift can be written as

$$\vec{\boldsymbol{v}}_c = \frac{mv_\parallel^2}{qB^3}(\vec{\boldsymbol{B}}\times\vec{\nabla}|\vec{\boldsymbol{B}}|). \quad (45)$$

Therefore the drift velocity in this case is given by

$$\vec{\boldsymbol{v}}_d = \vec{\boldsymbol{v}}_{\vec{\nabla}B}+\vec{\boldsymbol{v}}_c = \frac{m}{qB}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{(\vec{\boldsymbol{B}}\times\vec{\nabla}|\vec{\boldsymbol{B}}|)}{B^2} \quad (46)$$

$$= \frac{m}{qB}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)\frac{\vec{\nabla}|\vec{\boldsymbol{B}}|}{B}. \quad (47)$$

Here the earth's magnetic field in the equatorial plane is expressed as $B=k/r^3$. The gradient of earth's magnetic field in the equatorial plane is obtained as

$$\left|\vec{\nabla}|\vec{\boldsymbol{B}}|\right| := \left(\vec{\nabla}|\vec{\boldsymbol{B}}|\right)\cdot\vec{\boldsymbol{e_r}} = \frac{\partial B}{\partial r} = -\frac{3k}{r^4}. \quad (48)$$

Therefore

$$\left|\vec{\nabla}|\vec{\boldsymbol{B}}|\right| = -\frac{3}{r}. \quad (49)$$

The particle velocity is obtained by the thermal velocity, which in three-dimension is given by

$$v_{th}^2 = \frac{3k_BT}{m} = v_x^2+v_y^2+v_z^2. \quad (50)$$

If we consider the magnetic field is on the z-direction, the velocity component parallel and perpendicular of the magnetic field is expressed as

$$v_\parallel^2 = v_z^2 = \frac{k_BT}{m}, \quad (51)$$

$$v_\perp^2 = v_x^2+v_y^2 = \frac{2k_BT}{m}. \quad (52)$$

Therefore

$$v_\parallel^2+\frac{v_\perp^2}{2} = \frac{2k_BT}{m}. \quad (53)$$

Using eqs (36) and (40), the drift velocity is obtained

$$\vec{\boldsymbol{v}}_d = \pm\frac{m(2k_BT/m)(-3/r)}{eB} = \mp\frac{6k_BT}{eBr}. \quad (54)$$

We now consider particles that at are the about five Earth's radii, $r/R_E = 5$. The elections have an energy of 30 keV and the protons an energy of 1 eV. Using the eq (41), the drift velocity of electrons and protons are

$$v_{d,e} = 2.4 \times 10^6 \text{ cm/s}. \tag{55}$$

$$v_{d,p} = 79 \text{ cm/s}, \tag{56}$$

where we have used that 1eV$= 1.16 \times 10^4 K$. The drift motion makes circular motion around the Earth which is on the equatorial plane, with the electrons and protons moving in opposite directions.

(b) Because the drift motion of electrons and protons is in opposite directions, it leads to a ring current given by

$$\vec{j} = ne(\vec{v}_{d,p} - \vec{v}_{d,e}) = 3.8 \times 10^{-15} \text{ A/cm}^2. \tag{57}$$

(c) The drift time around the Earth is calculated by

$$t_d = 2\pi r / v_d. \tag{58}$$

Using $r = 5R_E$, the drift time of elections and protons are

$$\text{electrons:} \quad 8.5 \times 10^4 \text{ s}, \sim 24\,\text{h} \tag{59}$$

$$\text{protons:} \quad 2.8 \times 10^8 \text{ s} \sim 8.8\,\text{yr}. \tag{60}$$