Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture X

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Lecture X, Exercise 1.

The Rankine-Hugoniot conditions are express the conservation of rest mass, energy and momentum across a shock wave

$$[\![\rho u^\mu]\!] n_\mu = 0, \tag{1}$$
$$[\![T^{\mu\nu}]\!] n_\nu = 0. \tag{2}$$

For simplicity, we assume the flow is one-dimensional and the space-time is flat, then $n_\mu = (0, 1, 0, 0)$. Evaluating the equations (1) and (2) in the shock-front rest frame, they are written as

$$\rho_a v_a^x = \rho_b v_b^x, \quad T_a^{xx} = T_b^{xx}, \quad T_a^{tx} = T_b^{tx}, \tag{3}$$

or equivalently,

$$J := \rho_a W_a v_a \quad = \quad \rho_b W_b v_b, \tag{4}$$
$$\rho_a h_a W_a^2 v_a^2 + p_a \quad = \quad \rho_b h_b W_b^2 v_b^2 + p_b, \tag{5}$$
$$\rho_a h_a W_a^2 v_a \quad = \quad \rho_b h_b W_b^2 v_b, \tag{6}$$

where J is referred to the relativistic mass flux. From equations (4) and (5), the condition of the relativistic mass flux across the shock front is given by

$$[\![J]\!] = 0, \quad J^2 = \frac{[\![p]\!]}{[\![h/\rho]\!]}. \tag{7}$$

Similarly, using equations (4) we can rewrite the conservation of momentum as

$$[\![hW]\!] = 0. \tag{8}$$

Multiplying eq (7) by $(h_a/\rho_a + h_b/\rho_b)$ and combining it with eq (4) we obtain

$$(h_b W_b v_b)^2 - (h_a W_a v_a)^2 = -\left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right) [\![p]\!]. \tag{9}$$

We then take the square of eq (8) and subtract it from eq (9) to obtain

$$[\![h^2]\!] = \left(\frac{h_a}{\rho_a} + \frac{h_b}{\rho_b}\right) [\![p]\!]. \tag{10}$$

This is called Taub adiabat which represents the relativistic generalization of the classical Hugoniot adiabat for Newtonian shock fronts.

Next we consider the Newtonian limit of the Taub adiabat. In the Newtonian limit $h = 1 + \epsilon + p/\rho \approx 1$ and that $[\![h^2]\!] \approx 2[\![\epsilon + p/\rho]\!]$. Therefore from eq (10), the Newtonian limit of the Taub adiabat is given by

$$\left[\!\!\left[\epsilon + \frac{p}{\rho}\right]\!\!\right] = \frac{1}{2}\left(\frac{1}{\rho_a} + \frac{1}{\rho_b}\right)[\![p]\!]. \tag{11}$$

This is equivalent with the classical Hugoniot adiabat for Newtonian shock fronts.

Lecture X, Exercise 2.

The junction conditions can be expressed in terms of the velocities on the each side of the shock front in terms of the physical states there, i.e.,

$$v_a^2 = \frac{(p_a - p_b)(e_b + p_a)}{(e_a - e_b)(e_a + p_b)}. \tag{12}$$

Here we assume a highly relativistic shock, a cold fluid ahead of the shock, and an ultra relativistic one behind the shock, i.e.,

$$W_a \gg 1, \quad p_a \approx 0, \quad e_a \approx \rho_a, \quad p_b = \frac{e_b}{3}. \tag{13}$$

Then the eq (12) can be written as

$$v_a^2 = \frac{-e_b \cdot e_b/3}{(e_a - e_b)(e_a + e_b/3)}. \tag{14}$$

The denominator of eq (14) can be expanded as

$$\begin{aligned}
\frac{1}{3}(e_a - e_b)(3e_a + e_b) &= \frac{1}{3}(3e_a^2 + e_a e_b - 3e_a e_b - e_b^2) && (15)\\
&\simeq \frac{1}{3}(-2e_a e_b - e_b^2) && (16)\\
&= -\frac{e_b^2}{3}(2\frac{e_a}{e_b} + 1). && (17)
\end{aligned}$$

where we have used $e_a^2 \ll e_b^2,\ e_a^2 \ll e_a e_b$. Using eq (18), eq (14) can be expressed as

$$v_a^2 \simeq \frac{-e_b^2/3}{-e_b^2/3(2e_a/e_b + 1)} = \frac{1}{2e_a/e_b + 1}. \tag{18}$$

This equation can be rewritten as

$$
\begin{aligned}
(2e_a/e_b+1)v_a^2 &= 1 & (19)\\
2e_a v_a^2 + e_b v_a^2 &= e_b & (20)\\
e_b(1-v_a^2) &= 2e_a v_a^2 & (21)\\
e_b W_a^{-2} &= 2e_a v_a^2 & (22)\\
e_b &= 2e_a v_a^2 W_a^2 & (23)\\
&= 2e_a(W_a^2-1) & (24)\\
&\simeq 2e_a W_a^2 & (25)
\end{aligned}
$$

where we have used that $W_a^2 \gg 1$. In this case, the shock is ultra-relativistic relative to the fluid ahead and because the latter is cold, the shock front is also ultra-relativistic in the Eulerian frame, i.e.,

$$W_a^2 \sim W_s^2 \gg 1 \to e_b = 2e_a W_a^2 \sim e_a W_s^2. \tag{26}$$

Lecture X, Exercise 3.

The junction condition is given by

$$J := \rho_a W_a W_s (V_s - v_a) = \rho_b W_b W_s (V_s - v_b), \tag{27}$$

where V_s and W_s are the shock velocity and the shock Lorentz factor respectively. The exercises imposes that $V_s = 2v_a$, so that the mass flux is written as

$$J = \rho_a W_a W_s v_a. \tag{28}$$

The term $W_a W_s$ in eq (28) is given by

$$
\begin{aligned}
W_a W_s &= \left(\frac{1}{(1-v_a^2)(1-4v_a^2)}\right)^{1/2}, & (29)\\
&= \left(\frac{1}{1-4v_a^2-v_a^2+4v_a^4}\right)^{1/2}, & (30)\\
&= \left(\frac{1}{1-5v_a^2+4v_a^4}\right)^{1/2} \geq 1 \text{ for } v_a \in [0,1]. & (31)
\end{aligned}
$$

On the other hand, the Newtonian mass flux is obtained by

$$J_N = \rho_a (V_s - v_a) = \rho_a v_a \tag{32}$$

for $V_s = 2v_a$). Hence, the ratio between the Newtonian and the relativistic mass fluxes are given by

$$\frac{J_N}{J} = \frac{\rho_a v_a}{\rho_a v_a W_a W_s} = \frac{1}{W_a W_s} \leq 1. \tag{33}$$

In other words, the Newtonian mass flux is smaller than the relativistic one for shocks moving at the same speed.