Hydrodynamics and Magnetohydrodynamics: Solutions of the exercises in Lecture I

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Lecture I, Exercise 1.

Prove the Newtonian H-theorem, that is,

$$\frac{\partial f_0}{\partial t} = \Gamma(f_0) = 0, \tag{1}$$

where f_0 is the equilibrium distribution function. Condition (1) is fully equivalent to the condition

$$f_0(\vec{\boldsymbol{u}'}_2) f_0(\vec{\boldsymbol{u}'}_1) - f_0(\vec{\boldsymbol{u}}_2) f_0(\vec{\boldsymbol{u}}_1) = 0, \tag{2}$$

where $f_{1,2} := f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{u}}_{1,2})$, $f'_{1,2} := f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{u}}'_{1,2})$ are the distribution functions before and after the collision at time t and position $\vec{\boldsymbol{x}}$.

Here we introduce Boltzmann's H function as

$$H(t) = \int f(t, \vec{\boldsymbol{u}}) \ln(f(t, \vec{\boldsymbol{u}})) d^3u. \tag{3}$$

Taking a time derivative gives

$$\frac{dH(t)}{dt} = \int \frac{\partial f(t, \vec{\boldsymbol{u}})}{\partial t} [1 + \ln f(t, \vec{\boldsymbol{u}})] d^3u. \tag{4}$$

If $\partial f / \partial t = 0$, $dH/dt = 0$. So $dH/dt = 0$ is necessary condition for $\partial f / \partial t = 0$.

Next, we consider binary collisions, which gives

$$\frac{\partial f}{\partial t} = \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_1 - \vec{\boldsymbol{u}}_2| [f(\vec{\boldsymbol{u}}'_2) f(\vec{\boldsymbol{u}}'_1) - f(\vec{\boldsymbol{u}}_2) f(\vec{\boldsymbol{u}}_1)] = 0. \tag{5}$$

By adding Eq. (5) in Eq. (4) we obtain

$$\frac{dH(t)}{dt} = \int d^3u_1 \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_1 - \vec{\boldsymbol{u}}_2| (f'_2 f'_1 - f_2 f_1)[1 + \ln f_1] = 0, \tag{6}$$

which is equivalent to

$$\frac{dH(t)}{dt} = \int d^3u_1 \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1| (f'_2 f'_1 - f_2 f_1)[1 + \ln f_2] = 0, \tag{7}$$

because the cross section $\sigma(\Omega)$ is invariant under the swapping of u_1 with u_2. Thus we can add the two equations to obtain

$$\frac{dH(t)}{dt} = \frac{1}{2} \int d^3u_1 \int d^3u_2 \int d\Omega \sigma(\Omega) |\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1| (f'_2 f'_1 - f_2 f_1)[2 + \ln(f_1 f_2)] = 0. \tag{8}$$

Since for each collision there is an inverse collision with the same cross section, the integral (8) is invariant under change of $\vec{\boldsymbol{u}}_1$, $\vec{\boldsymbol{u}}_2$ with $\vec{\boldsymbol{u}}'_1$, $\vec{\boldsymbol{u}}'_2$. Similarly f_2, f_1 and f'_2, f'_1, i.e.

$$\frac{dH(t)}{dt} = \frac{1}{2}\int d^3u'_1 \int d^3u'_2 \int d\Omega\sigma'(\Omega)|\vec{\boldsymbol{u}}'_2 - \vec{\boldsymbol{u}}'_1|(f_2f_1 - f'_2f'_1)[2 + \ln(f'_1f'_2)] = 0. \tag{9}$$

By adding together Eq. (8) and Eq. (9) using $d^3u'_1d^3u'_2 = d^3u_1d^3u_2$, $|\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1| = |\vec{\boldsymbol{u}}'_2 - \vec{\boldsymbol{u}}'_1|$, and $\sigma(\Omega) = \sigma'(\Omega)$ we obtain

$$\frac{dH(t)}{dt} = \frac{1}{4}\int d^3u_1 \int d^3u_2 \int d\Omega\sigma(\Omega)|\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1|(f'_2f'_1 - f_2f_1)[\ln(f_1f_2) - \ln(f'_1f'_2)] = 0. \tag{10}$$

Using $x = (f_1f_2)/(f'_1f'_2)$, this is changed to

$$\frac{dH(t)}{dt} = \frac{1}{4}\int d^3u_1 \int d^3u_2 \int d\Omega\sigma(\Omega)|\vec{\boldsymbol{u}}_2 - \vec{\boldsymbol{u}}_1|(f'_2f'_1)[(1-x)\ln x] = 0. \tag{11}$$

The integrand of Eq. (11) is never positive for $x \geq 0$, which implies that

$$\frac{dH}{dt} \leq 0. \tag{12}$$

As a result, $dH/dt = 0$ only when

$$(f'_2f'_1 - f_2f_1) = 0. \tag{13}$$

Lecture I, Exercise 2.

The transport equation is

$$\frac{\partial(n\langle\psi\rangle)}{\partial t} + \frac{\partial(n\langle u_i\psi\rangle)}{\partial x_i} - n\left\langle u_i\frac{\partial\psi}{\partial x_i}\right\rangle - \frac{n}{m}\left\langle F_i\frac{\partial\psi}{\partial u_i}\right\rangle - \frac{n}{m}\left\langle\frac{\partial F_i}{\partial u_i}\psi\right\rangle = 0. \tag{14}$$

For the *first moment*, we use as collisional invariant $\psi = m$ in Eq. (14). Let's consider each terms as follows. First term is

$$\partial_t(n\langle\psi\rangle) = \partial_t(n\langle m\rangle) = \partial_t(nm) = \partial_t\rho, \tag{15}$$

where $nm = \rho$. The second term is

$$\partial_i(n\langle u_i\psi\rangle) = \partial_i(n\langle u_im\rangle) = \partial_i(nm\langle u_i\rangle) = \partial_i(\rho v_i). \tag{16}$$

The third term is

$$-n\langle u_i\partial_i\psi\rangle = -n\langle\partial_im\rangle = 0. \tag{17}$$

The fourth term is

$$-\frac{n}{m}\left\langle F_i\frac{\partial\psi}{\partial u_i}\right\rangle = -\frac{n}{m}\left\langle F_i\frac{\partial m}{\partial u_i}\right\rangle = 0. \tag{18}$$

The fifth term is

$$-\frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i}\psi \right\rangle = -\frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i} m \right\rangle = 0. \tag{19}$$

because $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(\vec{\boldsymbol{x}})$. Thus, the *first moment equation* becomes

$$\partial_t \rho + \partial_i(\rho v_i) = 0 \tag{20}$$

This is the *mass conservation equation* (continuity equation).

For the *second moment*, we use as collisional invariant $\psi = m u_j$ in Eq. (14). Let's consider each terms as follows. First term is

$$\partial_t(n\langle\psi\rangle) = \partial_t(n\langle m u_j\rangle) = \partial_t(nm\langle u_j\rangle) = \partial_t(\rho v_j). \tag{21}$$

The second term is

$$\partial_i(n\langle u_i \psi\rangle) = \partial_i(n\langle m u_i u_j\rangle) = \partial_i(\rho\langle u_i u_j\rangle). \tag{22}$$

Here we introduce $P_{ij} = \rho\langle(u_i - v_i)(u_j - v_j)\rangle$, which is also called the "pressure tensor". We consider

$$\begin{aligned} P_{ij}/\rho &= \langle(u_i - v_i)(u_j - v_j)\rangle = \langle u_i u_j - u_i v_j - v_i u_j + v_i v_j\rangle & (23)\\ &= \langle u_i u_j\rangle - \langle u_i v_j\rangle - \langle v_i u_j\rangle + \langle v_i v_j\rangle & (24)\\ &= \langle u_i u_j\rangle - v_i v_j - v_i v_j + v_i v_j = \langle u_i u_j\rangle - v_i v_j & (25)\end{aligned}$$

Thus

$$\langle u_i u_j\rangle = \langle(u_i - v_i)(u_j - v_j)\rangle + v_i v_j \tag{26}$$

Using Eq. (26), Eq. (22) can be written as

$$\begin{aligned} \partial_i(\rho\langle u_i u_j\rangle) &= \partial_i(\rho v_i v_j) + \partial_i \rho\langle(u_i - v_i)(u_j - v_j)\rangle & (27)\\ &= \partial_i(\rho v_i v_j) + P_{ij}. & (28)\end{aligned}$$

The third term is

$$-n\langle u_i \partial_i \psi\rangle = -n\langle u_i \partial_i m u_j\rangle = -nm\langle u_i \partial_i u_j\rangle = 0, \tag{29}$$

because $\partial_i u_j = 0$. The fourth term is

$$-\frac{n}{m}\left\langle F_i \frac{\partial \psi}{\partial u_i}\right\rangle = -\frac{nm}{m}\left\langle F_i \frac{\partial u_j}{\partial u_i}\right\rangle = -\frac{\rho}{m}\langle F^i \delta^i_j\rangle = \frac{-\rho}{m} F_j. \tag{30}$$

The fifth term is

$$-\frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i}\psi \right\rangle = -\frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i} m u_j \right\rangle = 0. \tag{31}$$

Thus, *the second moment equation* is written as

$$\partial_t(\rho v_j) + \partial_i(\rho v_i v_j) + \partial_i P_{ij} - \frac{\rho}{m} F_j = 0. \tag{32}$$

This is the *momentum conservation equation*.

For *the third moment*, we use as collisional invariant $\Psi = \frac{1}{2}m|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2$ in Eq. (14). Let's consider each terms as follows. First term is

$$\partial_t(n\langle\psi\rangle) = \partial_t \left(n \left\langle \frac{1}{2}m|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 \right\rangle \right) = \partial_t \left(\frac{1}{2}nm\langle|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle \right) = \partial_t \left(\frac{1}{2}\rho\langle|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle \right). \tag{33}$$

The second term is

$$\partial_i n\langle\psi u_i\rangle = \partial_i n \left\langle \frac{1}{2}m|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 u_i \right\rangle = \partial_i \frac{1}{2}\rho\langle u_i|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle = \frac{1}{2}\rho\partial_i\langle u_i|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle. \tag{34}$$

The third term is

$$-n\langle u_i\partial_i\psi\rangle = -n \left\langle u_i\partial_i \left(\frac{1}{2}m|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 \right) \right\rangle = -\frac{1}{2}\rho\langle u_i\partial_i(|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2)\rangle. \tag{35}$$

The fourth term is

$$-\frac{n}{m} \left\langle F_i \frac{\partial\psi}{\partial u_i} \right\rangle = -\frac{n}{m} \left\langle F_i \frac{\partial}{\partial u_i} \left(\frac{1}{2}m|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 \right) \right\rangle = 0, \tag{36}$$

because the kinetic energy is a function of space only. The fifth term is

$$-\frac{n}{m} \left\langle \psi \frac{\partial F_i}{\partial u_i} \right\rangle = -\frac{n}{m} \left\langle \frac{1}{2}m|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 \frac{\partial F_i}{\partial u_i} \right\rangle = 0. \tag{37}$$

Therefore the *third moment equation* is

$$\partial_t \left(\frac{1}{2}\rho\langle|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle \right) + \frac{1}{2}\rho\partial_i\langle u_i|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle - \frac{1}{2}\rho\langle u_i\partial_i(|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2)\rangle. \tag{38}$$

Here, we introduce two quantities,

$$\epsilon = \frac{1}{2}\langle|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle \tag{39}$$

$$q_i = \frac{1}{2}\langle(u_i - v_i)|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle. \tag{40}$$

Using these quantities, first term of Eq. (38) is

$$\partial_t \left(\frac{1}{2}\rho\langle|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle \right) = \partial_t(\rho\epsilon). \tag{41}$$

The second term of Eq. (38) is

$$\rho\partial_i\langle u_i|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle = \partial_i\langle(u_i - v_i)|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2 + \rho v_i|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle \tag{42}$$

$$= \partial_i\langle(u_i - v_i)|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle + \partial_i\langle\rho v_i|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}|^2\rangle \tag{43}$$

$$= 2\partial_i q_i + 2\partial_i(\rho\epsilon v_i). \tag{44}$$

For the third term of Eq. (38), we use $\vec{u} - \vec{v} = \vec{A}$. Then it becomes

$$\begin{aligned}
\rho\langle u_i \partial_i (A^j A^k \delta_{jk})\rangle &= \rho\langle u_i [(\partial_i A^j) A^k \delta_{jk} + A^j (\partial_i A^k)\delta_{jk}]\rangle && (45)\\
&= 2\rho\langle u_i \partial_i A^j A_j\rangle && (46)\\
&= 2\rho\langle u_i [\partial_i (u_j - v_j)](u_j - v_j)\rangle = 2\rho\langle u_i [\partial_i u_j - \partial_i v_j](u_j - v_j)\rangle && (47)\\
&= -2\rho\langle u_i \partial_i v_j (u_j - v_j)\rangle = -2\partial_i v_j \langle u_i (u_j - v_j)\rangle && (48)
\end{aligned}$$

Next, we reconsider the pressure tensor,

$$\begin{aligned}
P_{ij} &= \rho\langle (u_i - v_i)(u_j - v_j)\rangle && (49)\\
&= \rho\langle u_i (u_j - v_j) - v_j (u_j - v_j)\rangle = \rho\langle u_i (u_j - v_j)\rangle - \rho\langle v_i (u_j - v_j)\rangle && (50)\\
&= \rho\langle u_i (u_j - v_j)\rangle - \rho[\langle v_i u_j\rangle - \langle v_i v_j\rangle] && (51)\\
&= \rho\langle u_i (u_j - v_j)\rangle - \rho[v_i \langle u_j\rangle - v_i v_j] = \rho\langle u_i (u_j - v_j)\rangle. && (52)
\end{aligned}$$

Using Eq. (52), the third term of Eq. (38) is written as

$$\rho\langle u_i \partial_i |\vec{u} - \vec{v}|^2\rangle = -2\rho\partial_i v_j \langle u_i (u_j - v_j)\rangle = -2P_{ij}\partial_i v_j. \quad (53)$$

Here we introduce $\partial_i v_j = A_{ij}$. This is a generic tensor. However, P_{ij} is a symmetric tensor. Hence A_{ij} must also be symmetric tensor.

$$\begin{aligned}
A_{ij} &= \frac{1}{2}(A_{ij} + A_{ji}) && (54)\\
&= \frac{1}{2}(\partial_i v_j + \partial_j v_i) = \frac{1}{2}\Lambda_{ij}. && (55)
\end{aligned}$$

Using Eq. (55), Eq. (53) can be changed as

$$-2P_{ij}\partial_i v_j = -P_{ij}\Lambda^{ij}. \quad (56)$$

Finally, we can obtain *the third moment equation*

$$\partial_t(\rho\epsilon) + \partial_i(\rho\epsilon v_i) + \partial_i q_i + P_{ij}\Lambda^{ij} = 0. \quad (57)$$

This is the *energy conservation equation*.

Lecture I, Exercise 3.

The mass conservation equation Eq. (20) can be written as

$$\partial_t \rho = -\partial_i(\rho v_i) = -\rho\partial_i v_i - v_i \partial_i \rho. \quad (58)$$

Next, we consider the momentum conservation equation Eq. (32). We expand the derivative in Eq. (32) and using Eq. (58),

$$\begin{aligned}
& \partial_t(\rho v_j) + \partial_i(\rho v_i v_j) + \partial_i P_{ij} - \frac{\rho}{m}F_j && (59)\\
=\ & \rho\partial_t v_j + v_j \partial_t \rho + \rho v_i \partial_i v_j + \rho v_j \partial_i v_i + v_i v_j \partial_i \rho + \partial_i P_{ij} - \frac{\rho}{m}F_j && (60)\\
=\ & \rho\partial_t v_j - \rho v_j \partial_i v_i - v_i v_j \partial_i \rho + \rho v_i \partial_i v_j + \rho v_j \partial_i v_i + v_i v_j \partial_i \rho + \partial_i P_{ij} - \frac{\rho}{m}F_j && (61)\\
=\ & \rho\partial_t v_j + \rho v_i \partial_i v_j + \partial_i P_{ij} - \frac{\rho}{m}F_j = 0. && (62)
\end{aligned}$$

Then it is divided by ρ, we obtain

$$\partial_t v_j + v_i \partial_i v_j + \frac{1}{\rho}\partial_i P_{ij} - \frac{1}{m}F_j = 0. \tag{63}$$

Next, we consider the energy conservation equation Eq. (57). We expand the derivative in Eq. (57) and using Eq. (58),

$$\begin{aligned}
& \partial_t(\rho\epsilon) + \partial_i(\rho\epsilon v_i) + \partial_i q_i + P_{ij}\Lambda^{ij} && (64)\\
= \; & \rho\partial_t\epsilon + \epsilon\partial_t\rho + \rho\epsilon\partial_i v_i + \rho v_i\partial_i\epsilon + \epsilon v_i\partial_i\rho + \partial_i q_i + P_{ij}\Lambda^{ij} && (65)\\
= \; & \rho\partial_t\epsilon - \rho\epsilon\partial_i v_i - \epsilon v_i\partial_i\rho + \rho\epsilon\partial_i v_i + \rho v_i\partial_i\epsilon + \epsilon v_i\partial_i\rho + \partial_i q_i + P_{ij}\Lambda^{ij} && (66)\\
= \; & \rho\partial_t\epsilon + \rho v_i\partial_i\epsilon + \partial_i q_i + P_{ij}\Lambda^{ij} = 0. && (67)
\end{aligned}$$

Finally, dividing by ρ we obtain

$$\partial_t\epsilon + v_i\partial_i\epsilon + \frac{1}{\rho}\partial_i q_i + \frac{1}{\rho}P_{ij}\Lambda^{ij} = 0. \tag{68}$$