Recently several authors presented conductance measurements in ultralow-disorder semiconductor quantum wires and suggested that an unusual feature in the range (0.5–0.7)×e^2/h of conductance can be explained in terms of spontaneous ferromagnetism. 1–3 At first sight this interpretation seems to contradict the Lieb-Mattis theorem, 4 which rules out magnetized ground states for electrons moving on a line, as well as for one-band lattice models in one dimension (1D) with nearest-neighbor hopping and interactions involving only densities. However, there is no fundamental principle that forbids ferromagnetic ground states in quasi-one-dimensional systems with finite width or one-band lattice models in 1D with more general hopping. Indeed, numerical studies 5 show that the ground state of the one-dimensional Hubbard model with hopping between nearest and next-nearest neighbors can be ferromagnetic in a substantial range of densities and on-site interactions U. Clearly, the precise form of the energy dispersion ε_k plays an important role in stabilizing ferromagnetism. 5–8 In principle, it should therefore be possible to design metallic systems with ferromagnetic ground states by properly adjusting the hopping integrals between the relevant orbitals. A promising class of one-dimensional materials where this might be achieved are certain types of organic polymers, 9 whose molecular structure can be designed in a controlled manner in the laboratory. Motivated by these new developments, in this work we shall use a combination of perturbation theory and bosonization to derive some physical properties of itinerant ferromagnets in 1D.

Let us briefly consider this problem from a renormalization group (RG) point of view. The usual RG approach to one-dimensional metals is based on the assumption that their long-wavelength and low-energy properties are determined by wave vectors k in the vicinity of the Fermi wave vectors ±k_F. Given a general energy dispersion ε_k, it therefore seems reasonable to expand for k close to k_F:

$$\epsilon_k = \epsilon_{k_F} + v_F(k-k_F) + \frac{(k-k_F)^2}{2m^*} + \frac{\lambda}{6}(k-k_F)^3 + \ldots \quad (1)$$

and similarly for k≈−k_F. By power counting, the Fermi velocity v_F is a marginal coupling, while the inverse effective mass 1/m* and the cubic parameter λ are irrelevant in the RG sense. In the field-theoretical formulation of the RG, 10 these irrelevant couplings are simply ignored. However, as shown below, the cubic term in Eq. (1) is crucial to stabilize a ferromagnetic ground state in 1D, so that a proper RG treatment of itinerant ferromagnetism should include also the irrelevant couplings associated with band curvature effects. Therefore methods that cannot properly handle these couplings, such as the field-theoretical RG (Ref. 10) or bosonization, lose much of their power. Nevertheless, as shown below, in certain regimes bosonization is still useful to obtain nonperturbative results for correlation functions.

We consider the following Hamiltonian describing interacting electrons on a 1D lattice with length L:

$$\hat{H} = \sum_{\kappa \sigma} \epsilon_{\kappa \sigma} \hat{c}_{\kappa \sigma}^\dagger \hat{c}_{\kappa \sigma} + \frac{1}{2L} \sum_{q,ij} f_{ij} \hat{\rho}_i(q) \hat{\rho}_j(-q), \quad (2)$$

where \(\hat{c}_{\kappa \sigma}^\dagger\) and \(\hat{c}_{\kappa \sigma}\) are creation and annihilation operators for electrons with momentum \(k\) and spin \(\sigma\). The labels \(i\) and \(j\) assume values in \(\{n,m\}\), where \(n\) corresponds to the charge density \(\hat{\rho}_n(q) = \sum_{\kappa \sigma} \epsilon_{\kappa \sigma} \hat{c}_{\kappa \sigma}^\dagger \hat{c}_{\kappa \sigma} + q\sigma\), and \(m\) denotes the spin density \(\hat{\rho}_m(q) = \sum_{\kappa \sigma} \epsilon_{\kappa \sigma} \hat{c}_{\kappa \sigma}^\dagger \hat{c}_{\kappa \sigma} + q\sigma\). To discuss spontaneous symmetry breaking we should start from a spin-rotationally invariant \(\hat{H}\), which constrains the bare \(f_{ij}\) to satisfy \(f_{nn} = f_{mm} = 0\) and precludes any momentum dependence of \(f_{nm}\). We also take \(f_{nn} = f_{mm}\) to be momentum-independent. 11

Our goal is to derive a theory of the ferromagnetic phase of \(\hat{H}\). As a first step, we describe the ferromagnetic instability within Hartree-Fock theory, although it is well known that it overestimates this instability (see, e.g., Ref. 8); however, here we take the occurrence of ferromagnetism for granted and are not concerned with the microscopic mechanisms leading to it. Adding and subtracting the counterterm \(\Delta_\sigma(m) = f_{nn} + \sigma f_{nm}\), where \(n = \langle \hat{\rho}_n(0) \rangle/L\) is the density and \(m = \langle \hat{\rho}_m(0) \rangle/L\) is the spin density, we may write \(\hat{H} = \hat{H}_0 + \hat{H}_1\), with:

$$\hat{H}_0 = \mu \hat{N} = \sum_{\kappa \sigma} \xi_{\kappa \sigma} \epsilon_{\kappa \sigma} \hat{c}_{\kappa \sigma}^\dagger \hat{c}_{\kappa \sigma} - \frac{L}{2} [f_{nn}^2 + f_{nm}^2], \quad (3)$$

and

$$\hat{H}_1 = (2L)^{-1} \sum_{q,\alpha,\beta} \delta_{\alpha \beta} \hat{\rho}_\alpha^\dagger(q) \hat{\rho}_\beta(q). \quad (4)$$

Here \(\xi_{\kappa \sigma} = \epsilon_{\kappa \sigma} - \mu + \Delta_\sigma(m)\) is the Hartree-Fock energy, \(\delta_{\alpha \beta} \hat{\rho}_\alpha^\dagger(q) \hat{\rho}_\beta(q) = \hat{\rho}_\sigma^\dagger(q) \hat{\rho}_\sigma(q)\), and \(\hat{N} = \hat{\rho}_n(0)\). In the ferromagnetic state the Fermi wave vectors \(k_\sigma\) and velocities \(v_\sigma\) are defined by...
$\epsilon_{k\sigma} - \mu + \Delta_{\sigma}(m) = 0$ and $v_\sigma = \partial \epsilon_{k\sigma} / \partial k_{|k\sigma|}$, while in the normal state $\epsilon_{k\sigma} - \mu + \Delta_{\sigma}(0) = 0$ and $v_F = \partial \epsilon_{k\sigma} / \partial k_{|k\sigma|}$. Hence $\epsilon_{k\sigma} - \epsilon_k + f_\sigma \delta_n = \sigma \Delta$ where $\Delta = -f_\sigma m$ and $\delta_n = n(m) - n(0)$. For convenience we keep the chemical potential $\mu$ constant, so that the density $n$ is a function of $m$. The two equations $\epsilon_{k\sigma} - \epsilon_k + f_\sigma \delta_n = \sigma \Delta$, $\sigma = \pm 1$, together with the self-consistency conditions $m = \pi^{-1}(k_{1\uparrow} - k_{1\downarrow})$ and $\delta n = \pi^{-1}(k_{1\downarrow} - k_{1\downarrow} - 2k_F)$ fix the four quantities $k_{1\uparrow}$, $k_{1\downarrow}$, $\delta n$, and $m$.

Throughout this work we shall assume $m \ll n$ (weak ferromagnetism). The low-energy properties are then determined by wave vectors in the vicinity of the Fermi surface, as discussed in the classic work by Dzyaloshinskii and Kondratenko.\textsuperscript{12} Hence we may expand $\epsilon_k$ around $\pm k_F$. To leading order, it is sufficient to truncate the expansion at the third order, see Eq. (1). Keeping in mind that $\pi m \ll k_F$ and defining $q_m = \Delta/v_F$ we obtain

$$k_{\sigma} - k_F = \alpha q_m = \frac{\lambda_1 q_m^2}{2(1 + F_0)} - 2\alpha q_m^3 + \cdots,$$

where $A = \frac{\pi}{2}[(\lambda_2 - 3\lambda_2^2)/(1 + F_0)]\frac{1}{2}$. with $\lambda_1 = 1/(m^2v_F)$, $\lambda_2 = \lambda/v_F$, and $F_0 = 2f_n/\pi v_F$. The Fermi velocities are

$$v_{\sigma}/v_F = 1 + \lambda \alpha q_m + \frac{1}{2} \left( \frac{\lambda_2}{1 + F_0} \right) q_m^2 + \cdots.$$ (5)

Substituting Eq. (4) into $m = \pi^{-1}(k_{1\uparrow} - k_{1\downarrow})$, it is easy to see that, besides the solution $m = 0$, there is a nontrivial solution $m_0 = \left[ 2(1 - 1)/(I_{0\sigma}^2) \right]^{1/2}$, provided the radicand is positive. Here $I_0 = -2f_n/\pi v_F$ is the dimensionless Stoner parameter.\textsuperscript{6} To see whether the solution $m_0$ is stable, we consider the energy change $\delta \Omega_0(m) = \Omega_0(m) - \Omega_0(0)$ due to a finite value of $m$, where $\Omega_0(m) = (H_0 - \mu \hat{N})$. We obtain

$$\delta \Omega_0(m) = \frac{L v_F}{4 \pi} \left[ -I_0(I_0 - 1)(\pi m)^2 + \frac{A}{4 I_0^2} (\pi m)^4 + \cdots \right].$$ (6)

Obviously, a necessary condition for $m_0$ to represent a minimum of $\Omega_0(m)$ is $A > 0$. In addition, the square root $[2(1 - 1)/(I_{0\sigma}^2)]^{1/2}$ is only real if either $I_0 < 0$ or $I_0 > 1$. For consistency, we should also require that $\pi m_0 \ll k_F$ and that the band structure is such that the higher-order corrections in Eq. (6) are small. For some special form of $\epsilon_k$ it should be possible to satisfy these conditions even for small negative $I_0$ provided $k^2 F_A \gg \pi^2|I_{0\sigma}|^{-3}$. Here we shall not further consider this case, but focus instead on the regime close to the Stoner threshold, where $I_0$ is slightly larger than unity. The distance from the critical point is then measured by the small parameter $\delta_0 = 2(1 - 1)/I_0$. Interestingly, the numerical results of Ref. 5 indeed show a critical $I_0$ of order unity for not too large densities, which suggests that even in 1D the Stoner criterion can be a reasonable estimate for the ferromagnetic instability.

For simplicity we now set $f_n = -f_m = f_0 > 0$, corresponding to a repulsive Hubbard on-site interaction.\textsuperscript{13} Note that close to the phase transition $I_0 = F_0 = 1 + O(\delta_0)$. Let us first consider the density-density ($\chi_{\sigma\sigma}$) and the longitudinal spin-spin ($\chi_{\sigma\sigma}$) correlation functions. Within the random-phase approximation (RPA) we obtain

$$\chi^{\text{RPA}}_{\sigma\sigma}(q, i\omega) = \frac{[\chi^{0}_{\sigma\sigma}(q, i\omega) - 4f_0 \chi^{0}_{\sigma\sigma}(0)]/D}, \quad (7a)$$

$$\chi^{\text{RPA}}_{\sigma\sigma}(q, i\omega) = \frac{[\chi^{0}_{\sigma\sigma}(q, i\omega) + 4f_0 \chi^{0}_{\sigma\sigma}(0)]/D}, \quad (7b)$$

where $D(q, i\omega) = 1 - 4f_0 \chi^{0}_{\sigma\sigma}(0)$. Introducing $\chi^{0}_{\sigma\sigma}(q, i\omega)$, we can then write as

$$\chi^{\text{RPA}}_{\sigma\sigma}(q, i\omega) = \frac{v_\sigma q}{\pi (v_\sigma)^2 + \omega^2}.$$ (9)

For $\omega > 0$ the dynamic structure factors $S^{\text{RPA}}_{\sigma\sigma}(q, i\omega) = \pi^{-1} \text{Im} \chi^{\text{RPA}}_{\sigma\sigma}(q, i\omega)$ can then be written as

$$S^{\text{RPA}}_{\sigma\sigma}(q, i\omega) = Z |q| \delta(\omega - v_\sigma |q|),$$ (10)

with $Z = [\pi^2 q(1 + F_0)]^{-1}$, $v_\sigma = v_F \sqrt{1 + F_0}$, and $Z_m = [\pi^2 q |\delta_0|^{-1}]^{-1}$, $v_m = v_F q^{1/2}$. Note that $S^{\text{RPA}}_{\sigma\sigma}(q, i\omega)$ satisfy the sum rules\textsuperscript{13} $2 \lim_{q \to 0} \omega(q, 0) S^{\text{RPA}}_{\sigma\sigma}(q, i\omega) = \chi^{0}_{\sigma\sigma}$. The latter is related to the Hartree-Fock energy (6) via $\chi^{0}_{\sigma\sigma} = L^{-\delta} \Omega_0(m)/\partial m^2_{|m_0|}$. We conclude that longitudinal spin fluctuations in 1D can propagate ballistically, with velocity $v_\sigma \ll v_F$. In contrast, in 3D itinerant ferromagnets the longitudinal spin mode can decay into particle-hole pairs and is therefore strongly Landau-damped.\textsuperscript{5}

Next, let us calculate the transverse spin-spin correlation function $\chi^{0}_{\sigma\sigma}(q, i\omega)$ within the ladder approximation shown in Fig. 1, which yields

$$\chi^{\text{LAD}}_{\sigma\sigma}(q, i\omega) = [\chi^{0}_{\sigma\sigma}(q, i\omega)]^{-1} - 2f_0^{-1}. \quad (11)$$

For $|q| < q_m$ and $|\omega| < \Delta$ we may expand

$$\chi^{0}_{\sigma\sigma}(q, i\omega) \approx \frac{m_0}{2\Delta} \left[ \frac{1+i\omega}{2\Delta} - Bq^2 \right],$$ (12)

with the nonuniversal constant\textsuperscript{14} $B = \frac{1}{2\pi} \left( \lambda_2 - \lambda_2^2 \right)$. Note that $B \gg A > 0$. Using $\Delta = f_\sigma m_0$ we obtain
\[ \chi_{11}^{LAD}(q,i\omega) = -\frac{m_0}{i\omega - bq^2}, \tag{13} \]

where \( b = 2\Delta B \) is the spin wave stiffness. This implies a \( \delta \)-function peak in the dynamic structure factor, \( S_{11}(q,\omega) = m_0\delta(\omega - bq^2) \), which exhausts the sum rule\( \int_0^\infty (d\omega / \omega) S_{11}(q,\omega) = m_0 / bq^2 \). The existence of well-defined transverse spin waves in the symmetry broken phase follows from general hydrodynamic arguments.\(^{13}\) However, in 1D it may well be that interactions lead to anomalous damping of spin waves and a breakdown of hydrodynamics. This problem deserves further attention.

Because the ferromagnetic instability is triggered by interactions with zero momentum transfer, we expect that at low energies the relevant interaction is dominated by forward scattering. Moreover, for \( m = 0 \) it is known that repulsive backscattering interactions are marginally irrelevant.\(^{10}\) We assume that this remains true in the ferromagnetic state and expect that this assumption can be verified using RG methods. Note also that weak ferromagnetism in 3D can be understood within the framework of Fermi liquid theory,\(^{12}\) so that it is natural to expect that Luttinger liquid theory is the corresponding low-energy theory in 1D, at least if the characteristic magnetic wave vector \( q_m \) is small compared with \( k_F \).

The leading long-distance behavior of correlation functions can then be obtained from a generalized Tomonaga-Luttinger model, where the energy dispersion is linearized around the Fermi points \( \pm k_\sigma \). Introducing a bandwidth cutoff \( \Lambda \) such that \( q_m \ll \Lambda \ll k_F \), and defining field operators \( \hat{\psi}_\sigma(q) = \sqrt{\mathcal{L}} \hat{c}_{a\kappa_q + q,\sigma} \), where \( \alpha = \pm 1 \) labels the Fermi points, the kinetic energy is represented by \( \Sigma_{\alpha\alpha'}\int_\Lambda (dq / 2\pi) \epsilon_{\kappa_q} \hat{\psi}_\alpha^\dagger(q) \hat{\psi}_\alpha(q) \). The interaction is formally identical to Eq. (2), but with an implicit momentum transfer cutoff \( 1 / r_0 \ll k_F \) and the density operators now given by \( \hat{\rho}_\sigma(q) = \Sigma_{\alpha\alpha'}\int_\Lambda (dq / 2\pi) \hat{\psi}_\alpha^\dagger(q) \hat{\psi}_\alpha^\dagger(q + q') \), and similarly for the spin-density operator \( \hat{\rho}_m(q) \). Moreover, the bare couplings \( f_{ij} \) in Eq. (2) should be replaced by renormalized low-energy couplings \( g_{ij} \), which characterize the Luttinger liquid fixed point.\(^{15}\) Note that for \( m \neq 0 \) the renormalized interaction is not spin-rotationally invariant, so that in general \( g_{nm} \neq 0 \). However, for \( m \ll n \) we expect that the generic behavior of correlation functions [with the possible exception of \( \chi_{11}(q,\omega) \) in the spin wave regime \( |q| \ll q_m \)] can be correctly obtained for the special case \( g_{nn} = 0 \) and \( g_{nm} = -g_{mn} = \hat{g} > 0 \).

Given the effective low-energy theory, the closed loop theorem\(^{16}\) guarantees that all corrections to the RPA for the density-density and longitudinal spin-spin correlation functions cancel for small \( q \) and \( \omega \). Hence Eqs. (7a) and (7b) are asymptotically exact if we replace the bare quantities \( f_0, I_0 \), and \( \delta_0 \) by the corresponding renormalized quantities \( g, I, \) and \( \delta \). In particular, the existence of a propagating longitudinal spin mode with velocity \( v_p \equiv v_p \sqrt{\delta} \ll v_F \) is a robust result, and not an artifact of the RPA.

Due to the linearized energy dispersion and the irrelevance of scattering processes with large momentum trans-
FIG. 3. Dispersion of the longitudinal and the transverse spin excitations. The dashed line indicates that only for $|q| \leq q_m$ we expect transverse spin waves to be well defined. The triangle touching the horizontal axis at $k_1 - k_1$ is the regime where the bosonization result (16) for $S_{11}(q, \omega)$ can be trusted and yields a finite weight. The intensity of the shading is proportional to the magnitude of $S_{11}(q, \omega)$.

(15) yields an accurate approximation for the transverse dynamic structure factor $S_{11}(q, \omega)$. For $\omega > 0$ we obtain

$$S_{11}(q, \omega) = C_m \Theta(\omega - v_m |q| k + k_1) \times \left\{ \frac{\omega}{v_m} (|q| - k_1 + k_1) \right\}^2 \eta_m^{-1} \times \left\{ \omega + v_m (|q| - k_1 + k_1) \right\}^2 \eta_m^{-1},$$

(16)

with $C_m = [4 \pi v_m \Gamma(2 \eta_m) \Gamma(2 + 2 \eta_m)]^{-1} (r_0/v_m)^4 \eta_m$. The region where $S_{11}(q, \omega)$ is finite represents the onedimensional Stoner continuum. The complete picture of low-energy spin excitations is depicted in Fig. 3 and is qualitatively quite similar to its three-dimensional counterpart. However, in 1D there is no Landau damping and the structure factor shows anomalous scaling associated with broken spin-rotational symmetry of a Luttinger liquid phase.

For an outlook from a renormalization-group perspective, we note that while the ferromagnetic ground state is stabilized by nonlinear terms in the energy dispersion close to the Fermi points, the flow of the corresponding irrelevant couplings is not accessible within the usual field-theoretical RG. However, using modern formulations of the RG (Ref. 20) based on Wilson’s idea of eliminating degrees of freedom and rescaling, it should be possible to examine the subtle role played by irrelevant couplings in stabilizing a ferromagnetic ground state in 1D.

In conclusion, we presented the effective low-energy theory of weakly ferromagnetic Luttinger liquids. Many of their properties only depend on the effective Stoner parameter $I$, i.e., on the distance $\delta = (I - 1)/I \ll 1$ from the ferromagnetic instability. Neutron scattering experiments should be able to test our predictions for spin-spin correlation functions. Furthermore the propagating longitudinal mode with small velocity $v_m \propto \delta^{1/2}$ and large residue $Z_m \propto \delta^{-1}$ dominates some thermodynamic quantities, for example, through the divergence of the uniform spin susceptibility, $\chi_m \propto Z_m \propto v_m \propto \delta^{-1}$. The discussed features of the weakly ferromagnetic regime should be accessible in specially designed organic polymers, for which the effective Stoner parameter $I$ can be controlled by adjusting the density via external gate voltages. Our predictions are also relevant to semiconductor quantum wires which are believed to show spontaneous ferromagnetism.

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11. In the Hubbard model $f_s = -f_m = Ua/2$, where $U$ is the on-site interaction and $a$ is the lattice spacing.
14. $B = [4 \Delta(k_1 - k_1)]^{-1} [v_1 + v_1 - \Delta^{-1} f_1 d k v_2^2]$ for a general dispersion $\epsilon_k$ with velocity $v_k = \partial \epsilon_k / \partial k$.