Exercise 5 (Spin-1 Ising model):
Consider the following spin-1 generalization of the Ising model (Blume-Capel model), in which the values of the spin variables may be either 0 or ±1,

\[ H = -J \sum_{ij} s_i s_j + \Delta \sum_i s_i^2 - h \sum_i s_i. \]

The new parameter \( \Delta \), in the model Hamiltonian, determines the average number of spins that are parallel to the magnetization axis (i.e., with \( |s_i| = 1 \)). Such a model can be viewed either as a classical analog of a quantum spin-1 magnet, or as an Ising model doped with vacancies, where the vacancies correspond to the \( s_i = 0 \) states of the spin variables. In the latter case, \( \Delta \) plays the role of a chemical potential for the vacancies.

a) Find out in which limit the above model reduces to the standard Ising model, with only \( s_i = \pm 1 \) spins.

b) Show that at \( T = h = 0 \) the model exhibits a first-order phase transition for \( \Delta = zJ/2 \), where \( z = 2d \) is the coordination number in a \( d \)-dimensional hypercubic lattice.

c) Evaluate the partition function of the model within the mean-field approximation and derive the corresponding Landau free energy \( f_{MF}(T, \Delta, h; m) \), where the variational parameter \( m \) represents the average magnetization as in the case of the standard Ising model.

d) Expanding \( f_{MF}(T, \Delta, h; m) \) in powers of the order parameter \( m \) up to order \( m^6 \), show that there is a point \( (T^*_c, \Delta^*_c) \) in the \( h = 0 \) phase diagram where the coefficients of \( m^2 \) and \( m^4 \) are both vanishing. This is an example of a tricritical point and corresponds to the point in which a second-order transition line (characterized by a positive coefficient of \( m^4 \)) turns into a first-order transition line.

e) Sketch qualitatively the line of phase transitions \( T_c(\Delta) \) in the \( h = 0 \) phase diagram and evaluate the mean-field values of the critical exponents at the tricritical point.