Exercise 16 (Universality and the correlation length of the one-dimensional Ising model with next-to-nearest neighbor interactions):

Consider the classical infinite Ising chain with nearest and next-to-nearest neighbor interactions ($J_1 > 0$, $J_2 > 0$):

$$H = -\sum_i \left( J_1 s_i s_{i+1} + J_2 s_i s_{i+2} \right).$$

a) Let $\Delta$ be the energy cost to create a domain wall between a state with all spins up and a state with all spins down. Find the value of $\Delta$.

b) Write down the partition function for $H$ as a transfer matrix product. The transfer matrix will be a $4 \times 4$ matrix and it will “transfer” the spin configuration by 2 sites. Alternatively, think of it in terms of a model of “superspins” with 4 states, where each superspin represents the possible states of a pair of nearest neighbor Ising spins.

c) Determine the correlation length $\xi$ in the limit of large $K_1 \equiv \beta J_1$ and large $K_2 \equiv \beta J_2$. Show that $\xi = (1/2)e^{\beta \Delta}$.

d) We will now show that the relationship $\xi = (1/2)e^{\beta \Delta}$ holds quite generally. First argue that for large $\beta \Delta$, the density of domain walls is given by $\rho = e^{-\beta \Delta}$. You need therefore to establish that $\xi = (1/2\rho)$. Assume that the positions of the domain walls are statistically uncorrelated from each other. Consider then a long chain of length $L \gg \xi$ with $M = \rho L$ domain walls in it. The probability that any given domain wall is between positions 0 and $x$ (with $x > 0$) is $q = x/L$. Now use the statistical independence of the domain wall positions to argue that

$$\langle s_0 s_x \rangle = \sum_{j=0}^{M} (-1)^j q^j (1-q)^{M-j} \frac{M!}{j!(M-j)!}.$$

Evaluate the above expression in the limit $M, L \to \infty$, with $\rho = M/L$ fixed, in order to establish the desired result.

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