Introduction to the Renormalization Group

Hands-on course to the basics of the RG
(based on: “Introduction to the Functional Renormalization Group”
by P. Kopietz, L. Bartosch, and F. Schütz)

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Outline

1. History of the RG
2. Phase Transitions and scaling
3. Mean-Field Theory
4. Wilsonian RG
5. Functional (exact) RG
6. Applications
Introduction

- What is the renormalization group?
  “All renormalization group studies have in common the idea of re-expressing the parameters which define a problem in terms of some other, perhaps simpler set, while keeping unchanged those physical aspects of a problem which are of interest.” (John Cardy, 1996)

- “Meta-Theory” about theories

- Make the problem as simple as possible, but not simpler.

- Describe general properties qualitatively, but not necessarily quantitatively.
Quantum Field Theory

- Problem: Perturbation theory in quantum electrodynamics gives rise to infinite terms.

- Solution: all infinities can be absorbed in redefinition (=renormalization) of the parameters which have to be fixed by the experiment.

- Renormalizable theories: finite number $n$ of parameters sufficient, $n$ experiments needed to fix them, predict all other experiments.
Renormalization of QED

- 3 divergent diagrams → 3 parameters
- Electron self energy
  \[ \Sigma(P) = \frac{e^2}{8\pi^2\epsilon}(4m - P) \]
  \[ m = \frac{Z_m m_r}{Z_2^2} \]
  electron mass renormalization

- Photon self energy
  \[ \Pi^{\mu\nu}(P) = \frac{e^2}{6\pi^2\epsilon} \left( K^\mu K^\nu - g^{\mu\nu} K^2 \right) \]
  \[ Z_3 = 1 - \frac{e_r^2}{6\pi^2\epsilon} \]
  field renormalization

- Vertex correction
  \[ \Lambda(P, P + Q, Q)^\mu = \frac{e^2}{8\pi^2\epsilon} \gamma^\mu \]
  \[ e_r^2 = \frac{e^2}{1 - \frac{e^2}{6\pi^2} \ln \mu / \mu_0} \]
  charge renormalization
History

• Kenneth Wilson (1971/1972) calculation of critical exponents which are universal for a class of models
  \[ C'(t) = |t|^{-\alpha} \] specific heat
  \[ m(t) \sim (-t)^{\beta} \] magnetization
  \[ t = \frac{T - T_c}{T_c} \]

• new formulation of the RG idea (Wilsonian RG)

• Nobel Prize in Physics 1982:
  "...for his theory of critical phenomena in connection with phase transitions..."
Phase transitions and scaling hypothesis

- Phase transitions: examples
- Paramagnet-Ferromagnet Transition

ordering parameter: magnetization

\[ m = - \lim_{h \to 0} \frac{\partial f}{\partial h} \propto (T_c - T)^\beta \]

critical exponent: general for systems characterized by symmetry and dimensionality
Phase transitions and scaling hypothesis

- Liquid-Gas Transition

order parameter: density

\[ n - n_c \propto (T - T_c)^\beta \]

same symmetry class as Ising model (classical spins in a magnetic field)

\[ H = -J \sum_{ij} s_i s_j - h \sum_i s_i \]

\[ s_i = \pm 1 \]
Universality classes

- **Ising model, gas-liquid transition**
  \[ H = -J \sum_{ij} s_i s_j \]
  \[ Z_2 : s_i \rightarrow -s_i \]

- **$XY_3$, Bose gas, (magnets in magnetic fields)**
  \[ H = -J \sum_{ij} \left[ S_i^x S_j^x + S_i^y S_j^y + (1 + \lambda) S_i^z S_j^z \right] \]
  \[ O(2) : \vec{S} \rightarrow \vec{S}' = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{S} \]

- **Heisenberg**
  \[ H = -J \sum_{ij} \vec{S}_i \cdot \vec{S}_j \]
  \[ O(3) : \vec{S} \rightarrow \vec{S}' \]
Critical exponents

- specific heat \( C(t) \propto |t|^{-\alpha} \)
- spontaneous magnetization \( m(t) \propto (-t)^{\beta} \)
- magnetic susceptibility \( \chi(t) \propto |t|^{-\gamma} \)
- critical isotherm \( m(h) \propto |h|^{1/\delta} \text{sgn}(h) \)
- correlation length \( G(\vec{r}) \propto \frac{e^{-|\vec{r}|/\xi}}{\sqrt{\xi^{D-3}|r|^{D-1}}} \)
- anomalous dimension \( G(\vec{k}) \sim |\vec{k}|^{-2+\eta} \quad T = T_c \)
Scaling Hypothesis

- only two of six exponents are independent
- consider free energy density
  \[ f(t, h) = f_{\text{sing}}(t, h) + f_{\text{reg}}(t, h) \]
- singular part satisfies homogeneity relation
  \[ f_{\text{sing}} = \left| t \right|^D/y_t \Phi_\pm \left( \frac{h}{\left| t \right| y_n/y_t} \right) \quad \Phi_\pm(x) = f_{\text{sing}}(\pm 1, x) \]
- critical exponents from derivatives
  \[ C = \left. \frac{1}{T_c} \frac{\partial^2 f}{\partial t^2} \right|_{h=0} \propto \left| t \right|^{-\alpha} \]
  \[ m = -\left. \frac{\partial f}{\partial h} \right|_{h=0} \propto (-t)^\beta \]
Scaling Hypothesis

- relations between exponents
  \[2 - \alpha = 2\beta + \gamma = \beta(\delta + 1)\]

- scaling hypothesis for correlation function delivers two additional relations
  \[2 - \alpha = D\nu \quad \gamma = (2 - \eta)\nu\]

- relation between thermodynamic exponents and correlation function exponents
  \[
  \begin{align*}
  \alpha & = 2 - D\nu \\
  \beta & = \frac{\nu}{2}(D - 2 + \eta) \\
  \gamma & = \nu(2 - \eta) \\
  \delta & = \frac{D + 2 - \eta}{D - 2 + \eta}
  \end{align*}
  \]
Exercise 1: van der Waals Gas

- equation of state
  \[
  \left( p + a \left( \frac{N}{V} \right)^2 \right) (V - Nb) = NT
  \]

- sketch of isotherms
Exercise 1: Critical properties

- Thermodynamics: calculate free energy from pressure

\[ F(T, V) = - \int_{V_0}^V p(V') dV' + \text{const.}(T) \]

\[ F(T, V)_{\text{ideal}} = N k_B T \ln \left( \frac{h^3}{(2\pi m k_B T)^{3/2} V} \right) + N k_B T \]

- obtain quantities from derivatives of the free energy

\[ C_v = -T \left( \frac{\partial^2 F}{\partial T^2} \right)_V \propto |t|^{-\alpha} \quad \text{specific heat} \]
Exercise 1: Critical exponents (continued)

- use equation of state

\[ p(V) = \frac{N k_B T}{V - V_c/3} - 3p_c \frac{V_c^2}{V^2} \]

susceptibility \( \rightarrow \) compressibility

\[ \kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T = -\frac{1}{V} \left( \frac{\partial p}{\partial V} \right)_T,V=V_c^{-1} \propto t^{-\gamma} \]

- rewrite at the critical temperature

\[ n = n_c + \Delta n \]

\[ p(\Delta n) = p_c + \text{const.} \Delta n^\delta \quad \Rightarrow (n - n_c) \propto (p - p_c)^{1/\delta} \]
Mean Field Theory

- Example: Ising model

\[ H = -J \sum_{ij} s_i s_j - h \sum_i s_i \]

- partition function

\[ Z(T, h) = \sum_{\{s_i\}} e^{-\beta H} \]

- magnetization

\[ m = \langle s_i \rangle = \frac{\sum_{\{s_i\}} s_i e^{-\beta H}}{Z} \]

- simplify term in Hamiltonian

\[ s_i s_j = (m + \delta s_i)(m + \delta s_j) = -m^2 + m(s_i + s_j) + \delta s_i + \delta s_j \]

- free spins in field

\[ H_{\text{MF}} = N \frac{zJ}{2} m^2 - \sum_i (h + zJm) s_i \]
Mean Field Theory

- calculate partition function and free energy
  \[ Z_{\text{MF}}(t, h) = e^{-\beta N Z J m^2/2} \left[ 2 \cosh[\beta(h + z J m)] \right]^N \]
  magnetization \( m(t, h) \)

- self consistency equation for magnetization
  \[ Z_{\text{MF}}(t, h) = e^{-\beta N \mathcal{L}} \]
  \[ \frac{\partial \mathcal{L}}{\partial m} = 0 \]
  \[ m_0 = \tanh[\beta(h + z J m_0)] \]

- free energy
  \[ f(T, h) = \mathcal{L}(T, h, m_0) \]
Mean Field Theory: critical exponents (only correct at D>4)

- minimum condition \[ \frac{\partial \mathcal{L}}{\partial m} = (T - T_c)m_0 + \frac{T_c}{3}m_0^3 - h = 0 \]

- magnetization \[ m_0 = \sqrt{\frac{T_c - T}{2T_c}} \propto (-t)^{1/2} \quad \beta = 1/2 \]

- susceptibility \[ \chi = \frac{\partial m_0}{\partial h} \bigg|_{h=0} \propto \frac{1}{T - T_c} \quad \gamma = 1 \]

- critical isotherm \[ m_0(h) \propto h^{1/3} \quad \delta = 3 \]

- specific heat \[ C = -T \frac{\partial^2 f(T, h)}{\partial T^2} \]
  \[ C \approx T_c \frac{\partial^2 (T \ln 2)}{\partial T^2} \quad T > T_c \]
  \[ C \approx T_c \left[ \frac{\partial^2 (T \ln 2)}{\partial T^2} - \frac{3(T - T_c)^2}{4T_c} \right] \quad T > T_c \]
  \[ \alpha = 0 \]
Wilsonian RG

- Basic idea: take into account interactions iteratively in small steps
- Formulation in terms of functional integrals: example Ising model (\(\varphi^4\) theory)
  \[
  S_{\Lambda_0}[\varphi] = \frac{1}{2} \int \left[ r_0 + c_0 \vec{k}^2 \right] \varphi(-\vec{k}) \varphi(\vec{k})
  \]
  free action: particle is characterized by the the values of two parameters
  \[
  S_1 = \frac{n_0}{4!} \int \cdots \int \delta(\vec{k}_1 + \ldots + \vec{k}_4) \varphi(\vec{k}_1) \varphi(\vec{k}_2) \varphi(\vec{k}_3) \varphi(\vec{k}_4)
  \]
  - cutoff \( |\vec{k}| < \Lambda_0 \)
    interaction between (scalar) particles: characterized by the coupling constant
    only particles allowed up to a momentum for example due to a lattice in a condensed matter system
- partition function \[ \mathcal{Z} = \int \mathcal{D}[\varphi] e^{S_{\Lambda_0} + S_1} \]
Step 1: Mode elimination

- Integrate out degrees of freedom associated with fluctuations at high energies

\[
\mathcal{Z} = \int \mathcal{D}[\varphi] e^{-S_{\Lambda_0} - S_1} = \int \mathcal{D}[\varphi^<] \int \mathcal{D}[\varphi^>] e^{-S_{\Lambda_0} - S_1}
\]

\[
e^{-S'_\Lambda + S'_1} = \int \mathcal{D}[\varphi^>] e^{-S_{\Lambda_0} - S_1}
\]

End up: theory with modified couplings due to interactions

\[
S'_\Lambda[\varphi] = \frac{1}{2} \int_{\vec{k}} \left[ r^< + c^< k^2 \right] \varphi(-\vec{k}) \varphi(\vec{k})
\]

\[
S'_1 = \frac{n^<}{4!} \int_{\vec{k}_1} \cdots \int_{\vec{k}_4} \delta(\vec{k}_1 + \cdots + \vec{k}_4) \varphi(\vec{k}_1) \varphi(\vec{k}_2) \varphi(\vec{k}_3) \varphi(\vec{k}_4)
\]
Step 2: Rescaling

- Fields: defined on reduced space
- Blow up again the momentum space
- Rescale wave vectors to get action with the same form as before (free and interaction part)

\[
\begin{align*}
\vec{k}' &= \Lambda_0 / \Lambda \vec{k} \\
\varphi' &= \zeta_b^{-1} \varphi < \\
\zeta_b &= b^{1+D/2} \sqrt{c_0/c} < \\
b &= \Lambda_0 / \Lambda
\end{align*}
\]
Step 3: Iterative Procedure

- get relations for mode elimination and rescaling (semi-group)
  \[ r'(r_0, n_0) = b^2 Z_b \left[ r_0 + \frac{n_0}{2} \int_{\Lambda}^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{r_0 + c_0 k^2} \right] \]
  \[ n'(r_0, n_0) = b^{4-D} Z_b^2 \left[ n_0 - \frac{3n_0^2}{2} \int_{\Lambda}^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{(r_0 + c_0 k^2)^2} \right] \]

- iteration in infinitesimal steps (differential equations: flow equations)
  \[ \Lambda = \Lambda_0 e^{-\delta l} \approx \Lambda_0 (1 + \delta l) \]

  \[ \partial_l r(l) = 2r(l) + \frac{1}{2} \frac{n(l)}{1 + r(l)} \]

  \[ \partial_l n(l) = (4 - D)n(l) + \frac{3}{2} \frac{n(l)^2}{(1 + r(l))^2} \]
Flow diagrams

- solve coupled differential equations for different initial conditions (parameters of real system)
- critical surface: critical systems determined by the values of the coupling constants

one fixed point: Gaussian fixed point (mean field)

two fixed points: Gaussian, Wilson-Fisher
RG fixed points and critical exponents

- RG fixed points: describe scale invariant system
- Critical fixed points
  - relevant / irrelevant directions
  - correlation length: infinite
  - critical manifold
    (surface describes system at critical point)
- Critical exponents
  - eigenvalues of linearised flow equations near to fixed point
Functional renormalization group

- basic idea: Express Wilsonian mode elimination in terms of formally exact functional differential equations
- generating functional of Green functions

\[ G^{(n)}_{\alpha_1 \ldots \alpha_n} = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]} \Phi_{\alpha_1} \cdots \Phi_{\alpha_n}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}} \]

\[ G[J] = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]+(J,\Phi)} \Phi_{\alpha_1} \cdots \Phi_{\alpha_n}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}} \]

\[ G^{(n)}_{\alpha_1 \ldots \alpha_n} = \frac{\delta^n G[J]}{\delta J_{\alpha_1} \cdots \delta J_{\alpha_n}} \]

example: two point function of Ising model

\[ G^{(2)}(\mathbf{k}) \propto \frac{1}{|\mathbf{k}|^{2-n}} \]
Exact renormalization group

- introduce cutoff: modify Gaussian propagator:
  allow only propagation of modes up to a certain momentum
  \[ S_0 = \frac{1}{2} \int (r_0 + c_0 \vec{k}^2) \varphi(-\vec{k})\varphi(\vec{k}) = \frac{1}{2} \int G_0^{-1}(\vec{k}) \varphi(-\vec{k})\varphi(\vec{k}) \]

  \[ G_0(\vec{k}) = \frac{1}{r_0 + c_0 \vec{k}^2} \rightarrow \frac{1 - \Theta_c(\|\vec{k}\| - \Lambda)}{r_0 + c_0 \vec{k}^2} \]

- take derivative of generating functional with respect to cutoff
  \[ \rightarrow \text{FRG flow equation} \]
Exact FRG flow equation

- Wetterich equation

\[
\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2} \text{Tr} \left[ \frac{\partial_\Lambda R_\Lambda}{\partial^2 \Gamma_\Lambda[\Phi]} + R_\Lambda \right]
\]

- flow equations for vertex functions (coupling constants)
Applications

- BCS-BEC crossover (electron gas with attractive interactions)
  - mean field theory (BCS-theory)

\[ H = \sum_{\vec{k}\sigma} \epsilon_{\vec{k}\sigma} c_{\vec{k}\sigma}^\dagger c_{\vec{k}\sigma} - \frac{g_0}{V} \sum_{\vec{k},\vec{k}',\vec{p}} c_{\vec{k}+\vec{p}}^\dagger c_{\vec{k}-\vec{k}'}^\dagger c_{\vec{k}'} c_{\vec{k}'} + \Sigma \]

- flow equations of order parameter
- FRG needs additionally Ward identities (relations between vertex functions
Applications

- **interacting fermions**

\[ H = \sum_{\vec{k},\vec{k}'\sigma} \epsilon_{\vec{k}\sigma} c_{\vec{k}\sigma}^\dagger c_{\vec{k}\sigma} + \frac{1}{2V} \sum_{\vec{k},\vec{k}',\vec{p},\vec{k}'} V_{\vec{k},\vec{k}',\vec{p}} c_{\vec{k}+\vec{p}}^\dagger c_{\vec{k}\sigma}^\dagger c_{-\vec{k}'} c_{\vec{k}'+\vec{p}} \]

\[ Z = \int \mathcal{D}[\psi, \bar{\psi}] e^{-S[\psi, \bar{\psi}]} S_1[\psi, \bar{\psi}] \]

\[ S_1[\psi, \bar{\psi}] = \frac{1}{2} \int_{k, k', q} V \bar{\psi} \psi \psi \psi \]

- **Hubbard-Stratonovich transformation**

\[ e^{-S_1[\psi, \bar{\psi}]} = \int \mathcal{D}[\phi, \phi^*] e^{-S_0[\phi, \phi^*] - S'[\psi, \bar{\psi}, \phi, \phi^*]} \]

\[ S_0[\phi, \phi^*] = \frac{1}{2} \int_k V^{-1} \phi_k^* \phi_k \]

\[ e^{-\frac{\chi^2}{2!}} = \int_{-\infty}^{\infty} dx \ e^{-\frac{x^2}{2} - ixy} \]
Hubbard Stratonovich transformation

- interacting fermions $\rightarrow$ bosons and fermions with Yukawa type interaction

$$S_1[\psi, \bar{\psi}] = \frac{1}{2} \int_{k,k',q} V \bar{\psi} \psi \bar{\psi} \psi$$

$$S'[\psi, \bar{\psi}, \phi, \phi^*] = i \int_k \int_q \psi_{k+q} \psi_k \phi_q$$

- FRG of coupled bosons and fermions
Summary

- phase transitions → critical exponents
  \[ m = - \lim_{h \to 0} \frac{\partial f}{\partial h} \propto (T_c - T)^\beta \]
- universality classes (same symmetry → same properties at critical point)
- mean field theory
- Wilsonian Renormalisation Group

- functional Renormalization group
Exercise 2: Real-space RG of the 1D Ising model

- model
  \[ H = -J \sum_{i=1}^{N} s_i s_{i+1} \]

- transfer matrix method to calculate partition function
  \[ Z = \text{Tr}[T^N] \quad T = \begin{pmatrix} e^g & e^{-g} \\ e^{-g} & e^g \end{pmatrix} \quad g = \beta J \]

- calculate trace in diagonal basis
  \[ T = U^\dagger \tilde{T} U \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]
Exercise 2: Real-space RG

- keep only every b’s spin and derive effective model with new coupling

\[ Z = \text{Tr}[T^N] = \text{Tr}[T^b] \frac{N}{b} \]

\[ T^b = T' = \begin{pmatrix} e^{g'} & e^{-g'} \\ e^{-g'} & e^{g'} \end{pmatrix} \]

- derive recursion relation (RG transformation)

\[ g'(g) = \text{Artanh}(\tanh^b(g)) \]
Exercise 2: Real-space RG

- variable transformation
  \[ y = e^{-2g} \quad y' = e^{-2g} \quad y'(y) = \frac{(1 + y)^b - (1 - y)^b}{(1 + y)^b + (1 - y)^b} \]

- infinitesimal transformation (differential equation)
  \[ b = e^{\delta l} \approx 1 + \delta l \quad \frac{dy}{dl} = \frac{1 - y^2}{2} \ln\left(\frac{1 + y}{1 - y}\right) \]

- fixed points and flow
  \[ \frac{dy}{dl} = 0 \]
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