Theory Seminar, RWTH Aachen

June 18, 2018

Exact Renormalization group for quantum spin systems

Peter Kopietz, Frankfurt

- A. Werth, PK, O. Tsyplyatyev, PRB 97, 180403(R) 2018
- J. Krieg, PK, PRB 99, 060403(R) (2019)
- D. Tarasevych, J. Krieg, PK, PRB 98, 235133 (2018)
- R. Goll, D. Tarasevych, J. Krieg, PK, to appear July 2019

1.) Methods for quantum spin models

2.) Functional Renormalization Group

3.) Spin-FRG

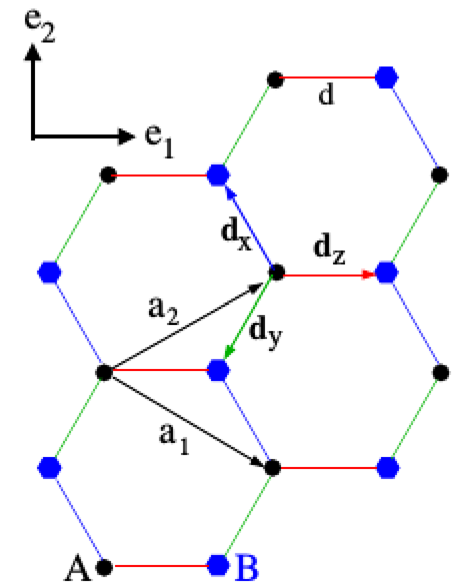
4.) Applications

1.) Motivation: describe frustrated magnets without magnetic order

- Kitaev model on honeycomb lattice (Kitaev, 2006)

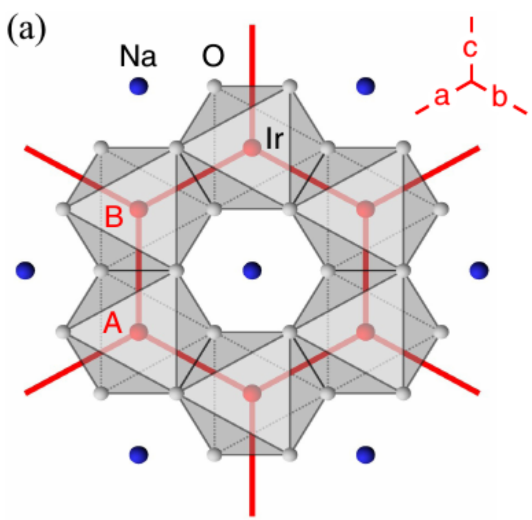
$$\mathcal{H}_{\text{Kitaev}} = K \sum_{\alpha=x,y,z} \sum_{\langle ij \rangle_\alpha} S_i^\alpha S_j^\alpha$$

completely solvable 2D quantum spin model, spin liquid ground state, Majorana fermions, topological order



- more realistic model for honeycomb iridates:

$$\mathcal{H} = J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + K \sum_{\alpha=x,y,z} \sum_{\langle ij \rangle_\alpha} S_i^\alpha S_j^\alpha$$
$$+ \sum_{\alpha\beta\gamma=x,y,z} \Gamma^\alpha_{\beta\gamma} \sum_{\langle ij \rangle_\alpha} S_i^\beta S_j^\gamma - \sum_i \boldsymbol{h}_i \cdot \boldsymbol{S}_i$$

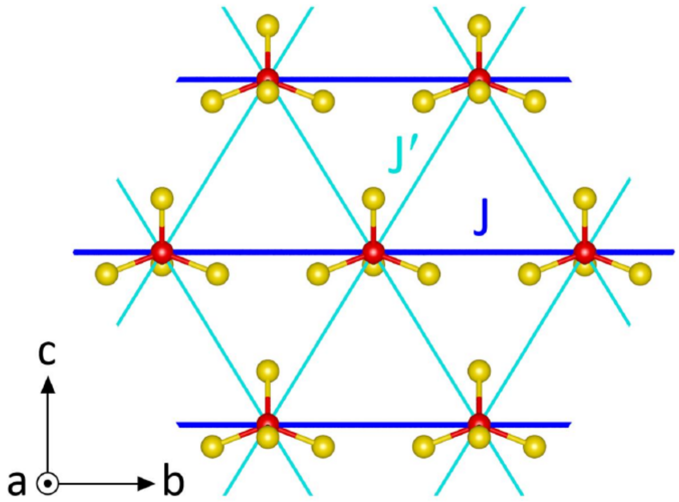


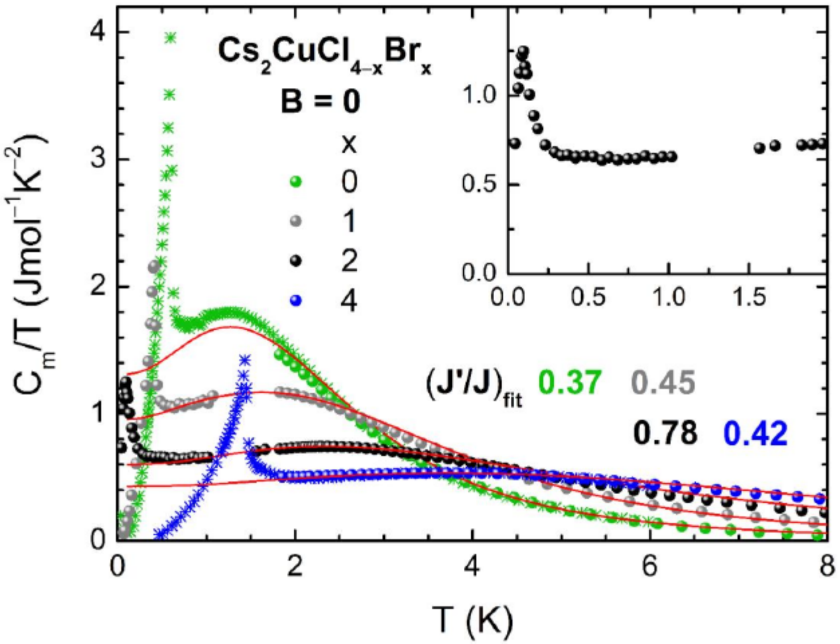
(Chaloupka, Khaliulin, PRB 2015)

...more motivation: layered triangular lattice antiferromagnets

- Frustration induced dimensional crossover in $Cs_2CuCl_{4-x}Br_x$

(Tutsch, Tsyplyatyev,...,PK, Lang, 2018 in preparation)





- urgently needed: analytic methods for quantum spin systems without magnetic order

spin algebra

- spin operators at different sites commute (like bosons), at same site satisfy SU(2)-algebra $[S_i^\alpha, S_j^\beta] = i\delta_{ij}\epsilon^{\alpha\beta\gamma} S_i^\gamma$

- single spin in magnetic field: Brillouin function:

$$\langle S^z \rangle = \frac{\mathrm{Tr}\left[e^{\beta h S^z} S^z\right]}{\mathrm{Tr}\left[e^{\beta h S^z}\right]} = \left(S + \frac{1}{2}\right) \coth\left[(S + \frac{1}{2})\beta h\right] - \frac{1}{2} \coth\left(\frac{\beta h}{2}\right) = b(\beta h)$$

- large S: Bose function: $b(y) \sim S + \frac{1}{2} - \frac{1}{2} \coth\left(\frac{y}{2}\right) = S - \frac{1}{e^y - 1}, \quad S \to \infty.$

- S=1/2: Fermi function: $b(y) = \frac{1}{2} \tanh\left(\frac{y}{2}\right) = \frac{1}{2} - \frac{1}{e^y + 1}, \quad S = 1/2$

$S^+S^- + S^-S^+ = 1, \quad S = 1/2$ similar to $cc^\dagger + c^\dagger c = 1$

- short-wavelength excitations in S=1/2 spin systems can have fermionic character

Methods for quantum spin models

1.) Numerical methods (exact diagonalization, Monte-Carlo, density-matrix renormalization group)

2.) 1/S-expansion; formally: Holstein-Primakoff transformation: (1940)

$$S_i^z = S - a_i^\dagger a_i \qquad [a_i, a_i^\dagger] = 1$$

$$S_i^+ = \sqrt{2S}\sqrt{1 - \frac{a_i^\dagger a_i}{2S}}\, a_i = \sqrt{2S} a_i - \frac{1}{2\sqrt{2S}} a_i^\dagger a_i a_i + \dots$$

$$S_i^- = \sqrt{2S} a_i^\dagger \sqrt{1 - \frac{a_i^\dagger a_i}{2S}} = \sqrt{2S} a_i^\dagger - \frac{1}{2\sqrt{2S}} a_i^\dagger a_i^\dagger a_i + \dots$$

- extremely successful for spin-waves in ordered magnets, even for S=1/2
- fails for disordered magnets

3.) Schwinger-bosons (Schwinger 1965)

$$S_i^z = \frac{1}{2}(a_i^\dagger a_i - b_i^\dagger b_i)$$

$$S_i^+ = a_i^\dagger b_i$$

$$S_i^- = b_i^\dagger a_i$$

constraint:

$$2S = a_i^\dagger a_i + b_i^\dagger b_i$$

Arovas, Auerbach 1988: useful for quantum magnets in low-dimensions

from preprint distributed by regular mail (1988):

Acknowledgements

We are grateful for conversations with N. E. Bickers, S. M. Girvin, F. D. M. Haldane, S. Kivelson, and A. Luther. This work was supported by several recent bank robberies in the Chicago area.

...methods for S=1/2 spin models

4.) Abrikosov pseudofermions for S=1/2 (Abrikosov 1965)

$$\boldsymbol{S}_i = (c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger)\frac{\boldsymbol{\sigma}}{2}\begin{pmatrix} c_{i\uparrow} \\ c_{i\downarrow} \end{pmatrix} \qquad c_{i\sigma}c_{j\sigma'}^\dagger + c_{j\sigma'}^\dagger c_{i\sigma} = \delta_{ij}\delta_{\sigma\sigma'}$$

- widely used for single-spin impurity models (Kondo)
- two states per site unphysical
- formally exact projection via imaginary chemical potential (Popov, Fedotov 1988)
- basis of functional RG for S=1/2 spin systems (Reuther, Wölfle 2010)

5.) Majorana fermions (Martin 1959; Mattis 1965; Coleman, Tsvelik 1990s)

$$S_i^x = -i\eta_i^y\eta_i^z, \quad S_i^y = -i\eta_i^z\eta_i^x, \quad S_i^z = -i\eta_i^x\eta_i^y \qquad \eta_i^\alpha\eta_j^\beta + \eta_j^\beta\eta_i^\alpha = \delta_{ij}\delta^{\alpha\beta}$$

- no unphysical states but redundancy in Hilbert space
- could be used to construct alternative FRG for S=1/2 systems

Spin-Hartree-Fock

(Werth, PK, Tsyplyatyev, PRB 2018)

- Main idea: work directly with physical spin-S operators, no unphysical states, no projections, no redundancy in Hilbert space
- Spin-Hartree-Fock theory for S=1/2: operator identity $S_i^z = S_i^+ S_i^- - \frac{1}{2}$

$$H = B\sum_{\mathbf{r}} S^z_{\mathbf{r}} + \frac{J}{2}\sum_{\mathbf{r},\delta} \mathbf{S}_{\mathbf{r}}\cdot\mathbf{S}_{\mathbf{r}+\delta}$$

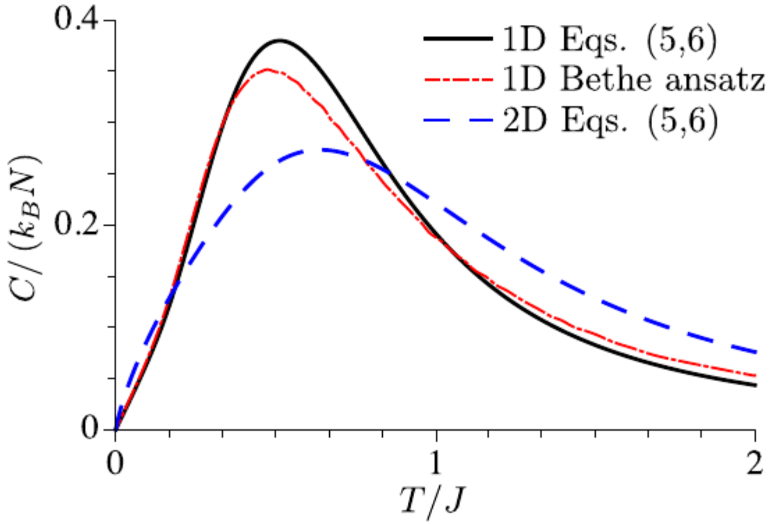
$$= \sum_{\mathbf{k}} (B - DJ + \varepsilon_{\mathbf{k}}) S^+_{\mathbf{k}} S^-_{\mathbf{k}} + \frac{1}{N}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \delta_{\mathbf{k}_1+\mathbf{k}_3,\mathbf{k}_2+\mathbf{k}_4}\varepsilon_{\mathbf{k}_3-\mathbf{k}_4} S^+_{\mathbf{k}_1} S^-_{\mathbf{k}_2} S^+_{\mathbf{k}_3} S^-_{\mathbf{k}_4}$$

$$\langle S^+_{\mathbf{k}_1} S^-_{\mathbf{k}_2} S^+_{\mathbf{k}_3} S^-_{\mathbf{k}_4}\rangle \approx$$

$$m_{\mathbf{k}_1} m_{\mathbf{k}_3}\delta_{\mathbf{k}_1,\mathbf{k}_2}\delta_{\mathbf{k}_3,\mathbf{k}_4} + m_{\mathbf{k}_1}(1 - m_{\mathbf{k}_2})\delta_{\mathbf{k}_1,\mathbf{k}_4}\delta_{\mathbf{k}_2,\mathbf{k}_3}$$

$$\langle S^+_{\mathbf{k}} S^-_{\mathbf{k}}\rangle = m_{\mathbf{k}}$$

$$= \frac{1}{e^{\beta(B - DJ + \varepsilon_{\mathbf{k}} - \frac{2}{N}\sum_{\mathbf{k}'}\varepsilon_{\mathbf{k}-\mathbf{k}'} m_{\mathbf{k}'})} + 1}$$



Spin diagram technique

Basic idea: Vaks, Larkin, Pikin 1968:

work directly with physical spin-S operators, no unphysical states, no projections, no redundancy in Hilbert space

VLP 1:
Wick theorem for spin operators, spin-diagram technique, thermodynamics

SOVIET PHYSICS JETP VOLUME 26, NUMBER 1 JANUARY, 1968

THERMODYNAMICS OF AN IDEAL FERROMAGNETIC SUBSTANCE

V. G. VAKS, A. I. LARKIN, and S. A. PIKIN

Submitted February 1, 1967

Zh. Eksp. Teor. Fiz. 53, 281–299 (July, 1967)

A diagram technique is proposed for a system of interacting spins which permits one to study the thermodynamics of a Heisenberg ferromagnet with arbitrary spin S at any temperature T or magnetic field strength H. The relevant high-temperature expansions are presented. Ex-

VLP 2:
spin waves, correlation functions

SOVIET PHYSICS JETP VOLUME 26, NUMBER 3 MARCH, 1968

SPIN WAVES AND CORRELATION FUNCTIONS IN A FERROMAGNETIC

V. G. VAKS, A. I. LARKIN, and S. A. PIKIN

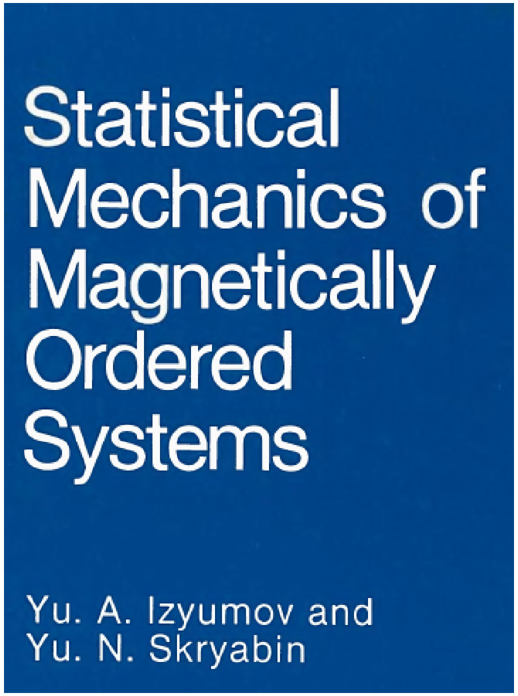
Submitted April 6, 1967

Zh. Eksp. Teor. Fiz. (U.S.S.R.) 53, 1089–1106 (September, 1967)

We consider the spin waves and correlation functions in a Heisenberg ferromagnet in the complete temperature range below the transition temperature T_c. We find the damping of the spin waves and

Time-ordered connected spin correlation functions

gentle introduction:



Statistical Mechanics of Magnetically Ordered Systems
by Yu. A. Izyumov, Yu.N. Skryabin
transl. by Roger Cooke
Hardcover: 295 pages
Publisher: Springer, September 30, 1988
Language: English (translated from Russian)
ISBN-10: 0306110156
ISBN-13: 978-0306110153

strategy: calculate directly imaginary-time-ordered, connected spin-correlation functions:

$$G_i^\alpha(\tau) = \langle S_i^\alpha(\tau)\rangle$$

$$G_{ij}^{\alpha\beta}(\tau,\tau') = \langle \mathcal{T}\left[S_i^\alpha(\tau) S_j^\beta(\tau')\right]\rangle - \langle S_i^\alpha(\tau)\rangle\langle S_j^\beta(\tau')\rangle$$

$$G_{i_1 i_2 \ldots i_n}^{\alpha_1\alpha_2\ldots\alpha_n}(\tau_1,\tau_2,\ldots,\tau_n) = \langle \mathcal{T}[S_{i_1}^{\alpha_1}(\tau_1) S_{i_2}^{\alpha_2}(\tau_2)\cdots S_{i_n}^{\alpha_n}(\tau_n)]\rangle_{\text{connected}}$$

satisfy bosonic Kubo-Martin-Schwinger boundary conditions:

$$G_{ij}^{\alpha\alpha'}(\beta,\tau') = G_{ij}^{\alpha\alpha'}(0,\tau'), \quad 0 < \tau' < \beta$$

$$G_{ij}^{\alpha\alpha'}(\tau,\beta) = G_{ij}^{\alpha\alpha'}(\tau,0), \quad 0 < \tau < \beta$$

Generalized blocks

- for $J_{ij} = 0$ time-ordered connected spin correlation functions are site-diagonal but non-trivial
- encode SU(2) spin algebra at given site
- building blocks of spin-diagram technique
- generating functional: $e^{\mathcal{G}[\boldsymbol{h}]} = \mathrm{Tr}\left[e^{\beta h \sum_i S_i^z} \mathcal{T} e^{\int_0^\beta d\tau \sum_i \boldsymbol{h}_i(\tau)\cdot \boldsymbol{S}_i(\tau)}\right]$

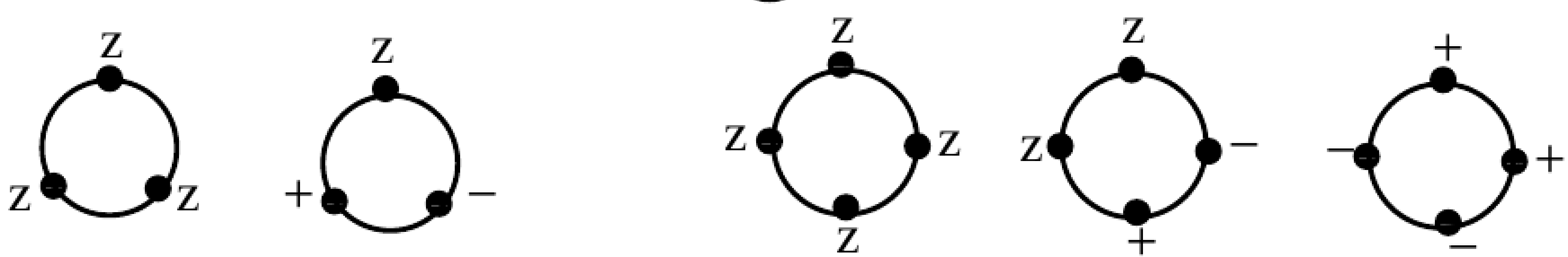
$$G^{\alpha_1\ldots\alpha_n}(\omega_1,\ldots,\omega_n) = \int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n e^{i(\omega_1\tau_1+\ldots+\omega_n\tau_n)} G^{\alpha_1\ldots\alpha_n}(\tau_1,\ldots,\tau_n)$$

$$G^{zz}(\omega,\omega') = \delta(\omega)\delta(\omega')b' =$$

$$G^{+-}(\omega,\omega') = \delta(\omega+\omega')\frac{b}{h-i\omega} =$$

Brillouin function

$$b = (S+\frac{1}{2})\coth\left[(S+\frac{1}{2})\beta h\right] - \frac{1}{2}\coth\left[\frac{\beta h}{2}\right]$$



Diagrammatics for quantum spin systems

- strategy: expand in powers of J_{ij}
- generalized Wick theorem for spin operators requires several type of vertices:

(from book by Izyumov and Skryabin, 1988)

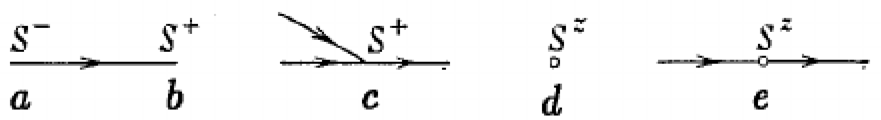
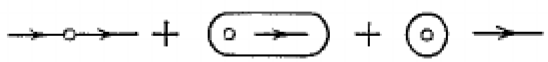
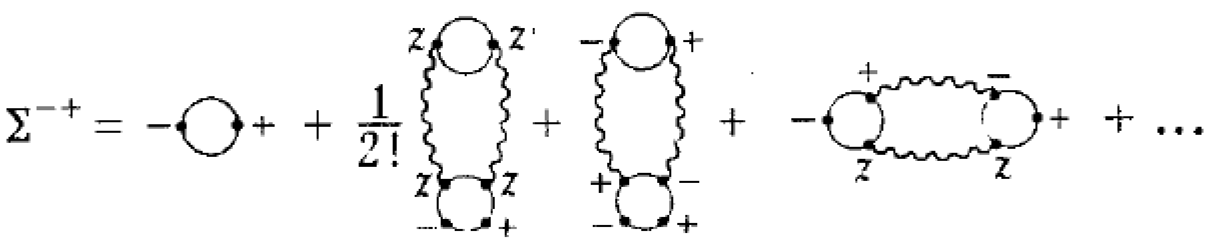


Fig. 2.1. Types of vertices for the exchange Hamiltonian. (For convenience vertices of types *d* and *e* are denoted by hollow bullets.)

$\langle T(S^- S^+ S^z)\rangle =$ 

- expansion of irreducible part of 2-point function in powers of J_{ij}



- complicated diagrammatics ⟹ method not very popular
- reformulate in framework of FRG ⟹ SFRG (Spin-FRG)

2.) Functional renormalization group

also called “exact RG” or “non-perturbative RG”

- exact functional differential equation for Wilsonian effective action (Wegner+Houghton 1972)

PHYSICAL REVIEW A VOLUME 8, NUMBER 1 JULY 1973

Renormalization Group Equation for Critical Phenomena

Franz J. Wegner
Institut für Festkörperforschung, KFA Jülich, D517 Jülich, Germany

Anthony Houghton*
Department of Physics, Brown University, Providence, Rhode Island 02912
(Received 27 October 1972)

An exact renormalization equation is derived by making an infinitesimal change in the cutoff in momentum space. From this equation the expansion for critical exponents around dimensionality 4 and the limit $n = \infty$ of the n-vector model are calculated. We obtain agreement with the results of Wilson and Fisher, and with the spherical model.

FRG literature

- alternative formulation for Legendre transform (average effective action) (Wetterich 1993)

Physics Letters B 301 (1993) 90–94
North-Holland

PHYSICS LETTERS B

$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2} \mathrm{STr} \left\{ \left[\mathbf{\Gamma}''_\Lambda[\Phi] + \mathbf{R}_\Lambda \right]^{-1} \partial_\Lambda \mathbf{R}_\Lambda \right\}$$

Exact evolution equation for the effective potential

Christof Wetterich
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, W-6900 Heidelberg, FRG

Received 15 November 1992; revised manuscript received 17 December 1992

We derive a new exact evolution equation for the scale dependence of an effective action. The corresponding equation for the effective potential permits a useful truncation. This allows one to deal with the infrared problems of theories with massless modes in less than four dimensions which are relevant for the high temperature phase transition in particle physics or the computation of critical exponents in statistical mechanics.

Reviews:

- Berges, Tetradis, Wetterich, Phys. Rept. 2002
- Pawlowski, Phys. Rept. 2007
- PK, Bartosch, Schütz, Introduction to the FRG, Springer, 2010
- Metzner, Salmhofer, Honerkamp, Meden, Schönhammer, Rev. Mod. Phys, 2012

what is "functional" about FRG?

- **main idea:** Wilsonian mode elimination can be expressed in terms of formally exact functional differential equation for generating functionals

- **generating functional of Green functions**

Green functions:

$$G^{(n)}_{\alpha_1\ldots\alpha_n} = \langle \Phi_{\alpha_n} \ldots \Phi_{\alpha_1} \rangle = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]} \Phi_{\alpha_n} \ldots \Phi_{\alpha_1}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}}$$

generating functional:

$$\boxed{\mathcal{G}[J] = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]+(J,\Phi)}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}}} \;, \qquad G^{(n)}_{\alpha_1\ldots\alpha_n} = \left. \frac{\delta^n \mathcal{G}[J]}{\delta J_{\alpha_n} \ldots \delta J_{\alpha_1}} \right|_{J=0}$$

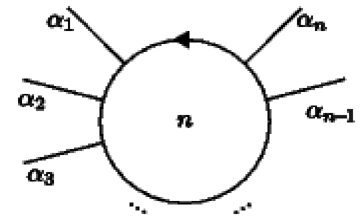
example: two-point function of Ising model at critical point:

$$G^{(2)}_{\boldsymbol{r}',\boldsymbol{r}} = \langle \varphi(\boldsymbol{r}) \varphi(\boldsymbol{r}') \rangle \propto \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^{D-2+\eta}} \qquad G^{(2)}(\boldsymbol{k}) \propto \frac{1}{|\boldsymbol{k}|^{2-\eta}}$$

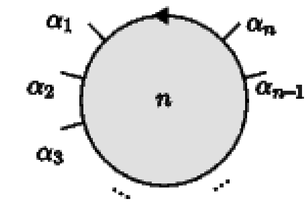
$\eta \approx 0.036$ anomalous dimension in D=3.

exact FRG flow equations

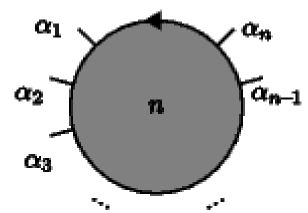
- different types of Green/vertex functions
  - connected Green functions $\mathcal{G}_c[J] = \ln\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\mathcal{G}[J]\right)$
    (Wegner-Houghton equation, 1972)



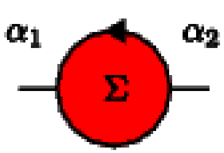
  - amputated connected Green functions
    (Polchinski equation, 1984)

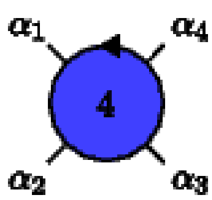


  - one-particle irreducible vertices
    (Wetterich equation, 1993)



$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2}\mathrm{STr}\left\{\left[\left(\frac{\delta}{\delta\Phi}\otimes\frac{\delta}{\delta\Phi}\right)^T \Gamma_\Lambda[\Phi] + \mathbf{R}_\Lambda\right]^{-1} \partial_\Lambda \mathbf{R}_\Lambda\right\}$$





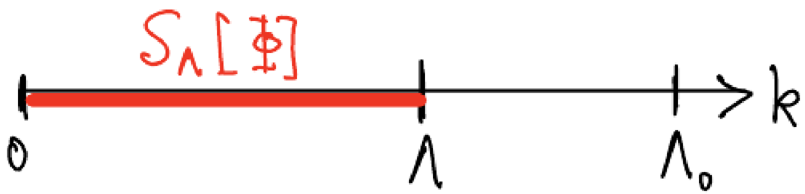
second functional derivative

regulator function

Wilson-Polchinski versus Wetterich equation

Wilson-Polchinski:

- RG flow of effective action for slow modes (fast modes are integrated)



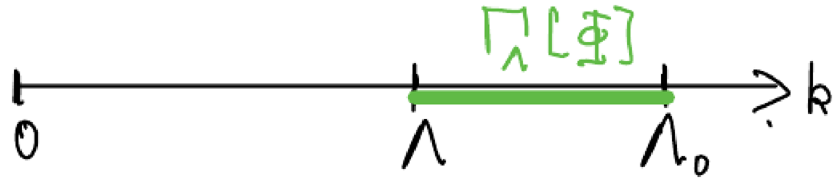
- technical complications for sharp cutoff:

$$\delta(x)\Theta(x) = \frac{1}{2}\delta(x) \qquad \delta(x)\Theta^2(x) = ?$$

$$\delta(x) f(\Theta(x)) = \delta(x) \int_0^1 dt f(t)$$ (Morris, 1994)

Wetterich:

- RG flow of effective action for fast modes:



- realized via Legende transformation

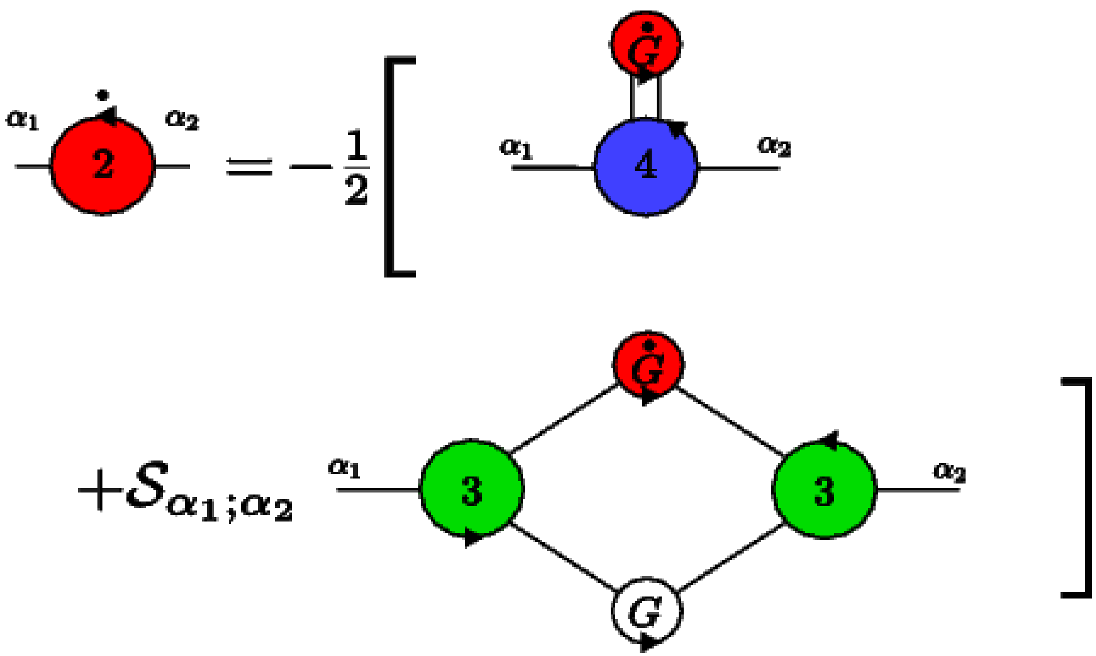
- no ambiguities even for sharp cutoff

vertex expansion 1

- functional Taylor expansion $\Gamma[\bar{\Phi}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \dots \int_{\alpha_n} \Gamma^{(n)}_{\alpha_1 \dots \alpha_n} \bar{\Phi}_{\alpha_1} \dots \bar{\Phi}_{\alpha_n}$

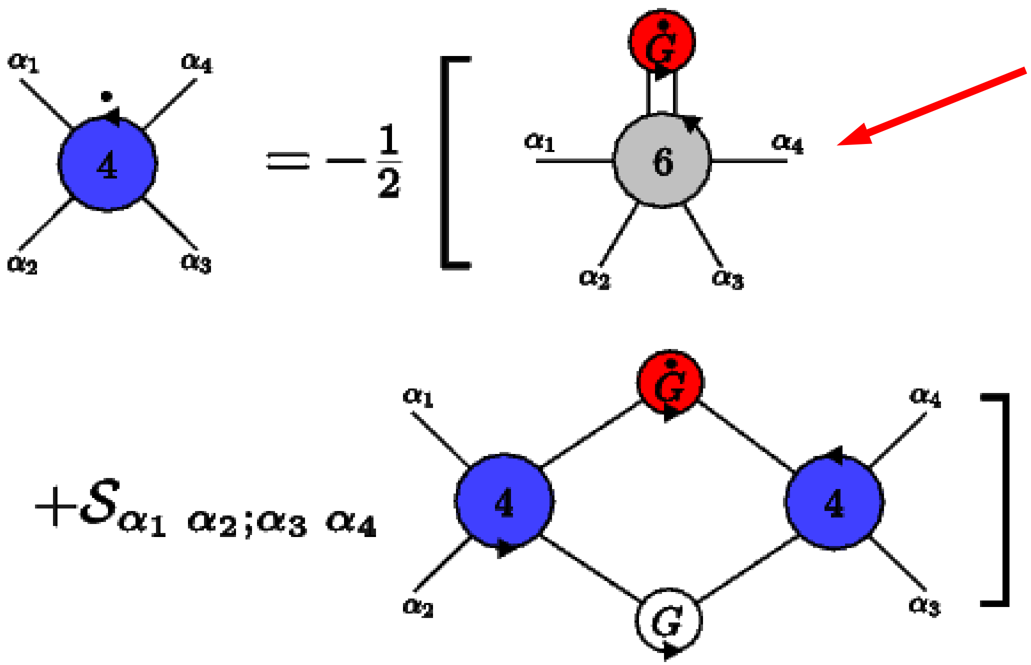
→ Wetterich equation reduces to infinite hierarchy of integro-differential equations for irreducible vertices

- exact flow equation for irreducible self-energy



vertex expansion 2

- exact flow equation for effective interaction



Three-body vertex renormalizes two-body interaction

- needed: controlled truncation strategies!

3.) Spin-Functional Renormalization Group (SFRG)

- Development of FRG formalism for bosons and fermions relies on path integral

- How about spin systems?

  Path integral not available and not needed!

J. Krieg and PK, PRB 99, 060403(R) (2019)

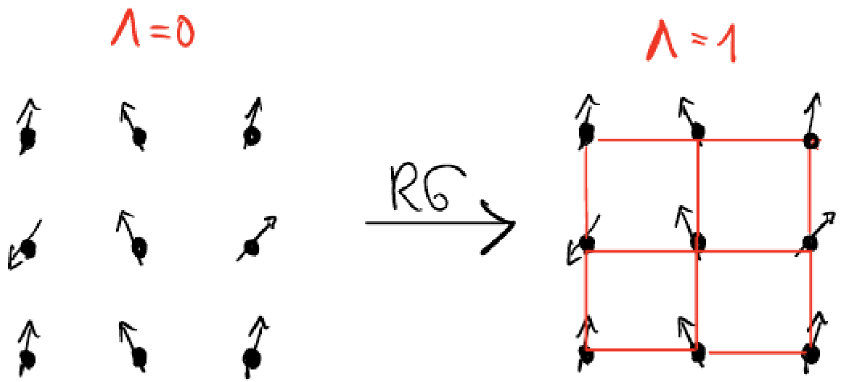
related work for classical spin model and hard-core bosons by Machado, Rancon, Dupuis, 2010-2014

SFRG: cutoff (deformation) schemes

- Heisenberg model: $\mathcal{H} = \frac{1}{2}\sum_{ij} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - h_0 \sum_i S_i^z$
- introduce continuous deformation of exchange interactions

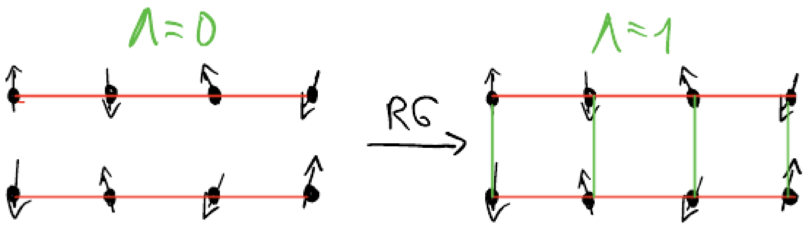
$J_{ij} \to J_{ij}^{\Lambda} = J_{ij} + R_{ij}^{\Lambda}$ deformation parameter $\Lambda \in [0, 1]$

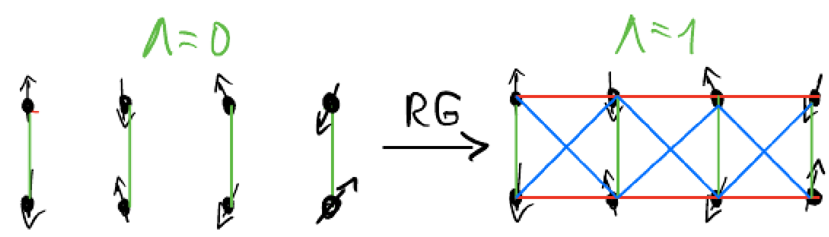
- different deformation schemes possible



$$J_{ij}^{\Lambda} = \Lambda J_{ij}$$

$$R_{ij}^{\Lambda} = (\Lambda - 1) J_{ij}$$





strategy

- calculate evolution of imaginary-time-ordered, connected spin-correlation functions when deformation parameter is changed

$$G_i^{\alpha}(\tau) = \langle S_i^{\alpha}(\tau) \rangle$$

$$G_{ij}^{\alpha\beta}(\tau,\tau') = \langle \mathcal{T}\big[S_i^{\alpha}(\tau) S_j^{\beta}(\tau')\big]\rangle - \langle S_i^{\alpha}(\tau)\rangle \langle S_j^{\beta}(\tau')\rangle$$

$$G_{i_1 i_2 \ldots i_n}^{\alpha_1 \alpha_2 \ldots \alpha_n}(\tau_1, \tau_2, \ldots, \tau_n) = \langle \mathcal{T}[S_{i_1}^{\alpha_1}(\tau_1) S_{i_2}^{\alpha_2}(\tau_2) \cdots S_{i_n}^{\alpha_n}(\tau_n)]\rangle_{\text{connected}}$$

- dual role of deformation (cutoff) parameter Λ

  1.) Regularizes divergencies (if there are any)

  2.) Keeps track of continuous deformation of model

- choose initial condition such that model can be solved in a controlled way

Flow for generating functional of connected spin correlation functions

- deformed hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}_\Lambda \quad \mathcal{H}_0 = -h_0 \sum_i S_i^z \qquad \mathcal{V}_\Lambda = \frac{1}{2} \sum_{ij} J_{ij}^\Lambda \boldsymbol{S}_i \cdot \boldsymbol{S}_j$
- generating functional of connected, imaginary-time-ordered spin correlation functions

$$e^{\mathcal{G}_\Lambda[\boldsymbol{h}]} = \mathrm{Tr}\left[e^{-\beta \mathcal{H}_0} \mathcal{T} e^{\int_0^\beta d\tau [\sum_i \boldsymbol{h}_i(\tau)\cdot \tilde{\boldsymbol{S}}_i(\tau) - \tilde{\mathcal{V}}_\Lambda(\tau)]} \right]$$

interaction picture $\tilde{\boldsymbol{S}}_i(\tau) = e^{\mathcal{H}_0 \tau} \boldsymbol{S}_i e^{-\mathcal{H}_0 \tau} \qquad \tilde{\mathcal{V}}_\Lambda(\tau) = e^{\mathcal{H}_0 \tau} \mathcal{V}_\Lambda e^{-\mathcal{H}_0 \tau}$

local magnetization $\langle \boldsymbol{S}_i(\tau) \rangle = \left. \frac{\delta \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta \boldsymbol{h}_i(\tau)} \right|_{\boldsymbol{h}=0}$

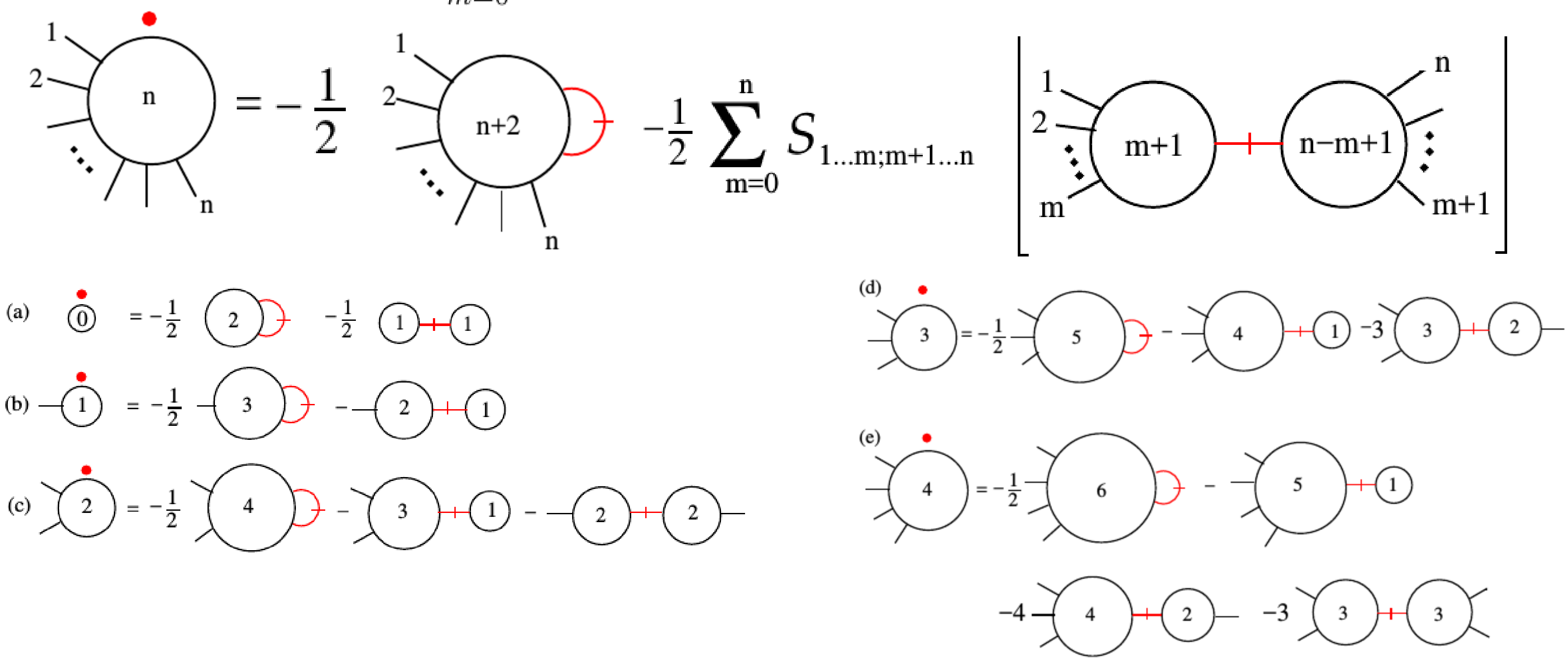
connected two -point function $G_{ij}^{\alpha\beta}(\tau,\tau') = \langle \mathcal{T}[S_i^\alpha(\tau) S_j^\beta(\tau')] \rangle - \langle S_i^\alpha(\tau) \rangle \langle S_j^\beta(\tau') \rangle = \left. \frac{\delta^2 \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_i^\alpha(\tau) \delta h_j^\beta(\tau')} \right|_{\boldsymbol{h}=0}$

- exact flow equation:

$$\partial_\Lambda \mathcal{G}_\Lambda[\boldsymbol{h}] = -\frac{1}{2} \int_0^\beta d\tau \sum_{ij} (\partial_\Lambda J_{ij}^\Lambda) \sum_\alpha \left[\frac{\delta^2 \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_i^\alpha(\tau) \delta h_j^\alpha(\tau)} + \frac{\delta \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_i^\alpha(\tau)} \frac{\delta \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_j^\alpha(\tau)} \right]$$

Hierarchy for connected spin-correlation functions

$$\partial_\Lambda G^{\alpha_1\ldots\alpha_n}_{i_1\ldots i_n}(\tau_1,\ldots,\tau_n) = -\frac{1}{2}\int_0^\beta d\tau \sum_{ij}(\partial_\Lambda J^\Lambda_{ij}) \sum_\alpha \Big[G^{\alpha_1\ldots\alpha_n\alpha\alpha}_{i_1\ldots i_n ij}(\tau_1,\ldots,\tau_n,\tau,\tau) + \sum_{m=0}^{n} \mathcal{S}_{1,\ldots,m;m+1,\ldots,n}\left\{G^{\alpha_1\ldots\alpha_m\alpha}_{i_1\ldots i_m i}(\tau_1,\ldots,\tau_m,\tau)\, G^{\alpha_{m+1}\ldots\alpha_n\alpha}_{i_{m+1}\ldots i_n j}(\tau_{m+1},\ldots,\tau_n,\tau)\right\}\Big]$$



efficient algorithm to generate expansion of spin correlation functions in powers of J_{ij} (alternative to method by Izyumov et al. 2002)

Flow of generating functional of irreducible spin-vertices

- subtracted Legendre transform:

$$\Gamma_\Lambda[\boldsymbol{M}] = \int_0^\beta d\tau \sum_i \boldsymbol{h}_i(\tau)\cdot\boldsymbol{M}_i(\tau) - \mathcal{G}_\Lambda[\boldsymbol{h}] \; - \frac{1}{2}\int_0^\beta d\tau \sum_{ij} R_{ij}^\Lambda \boldsymbol{M}_i(\tau)\cdot\boldsymbol{M}_j(\tau)$$

$$\partial_\Lambda \Gamma_\Lambda[\boldsymbol{M}] = \frac{1}{2}\int_0^\beta d\tau \sum_{ij}(\partial_\Lambda R_{ij}^\Lambda)\sum_\alpha \frac{\delta^2 \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_i^\alpha(\tau)\delta h_j^\alpha(\tau)}$$

$$\frac{\delta^2 \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_i^\alpha(\tau)\delta h_j^\alpha(\tau)} = [\mathbf{\Gamma}_\Lambda''[\boldsymbol{M}] + \mathbf{R}_\Lambda]^{-1}_{i\tau\alpha,j\tau\alpha} \qquad \left[\mathbf{\Gamma}_\Lambda''[\boldsymbol{M}]\right]_{i\tau\alpha,j\tau'\alpha'} = \frac{\delta^2\Gamma_\Lambda[\boldsymbol{M}]}{\delta M_i^\alpha(\tau)\delta M_j^{\alpha'}(\tau')}$$

$$\left[\mathbf{R}_\Lambda\right]_{i\tau\alpha,j\tau'\alpha'} = R_{ij}^\Lambda \delta_{\alpha\alpha'}\delta(\tau-\tau')$$

$$\boxed{\partial_\Lambda\Gamma_\Lambda[\boldsymbol{M}] = \frac{1}{2}\mathrm{Tr}\left\{\left(\mathbf{\Gamma}_\Lambda''[\boldsymbol{M}] + \mathbf{R}_\Lambda\right)^{-1}\partial_\Lambda \mathbf{R}_\Lambda\right\}}$$

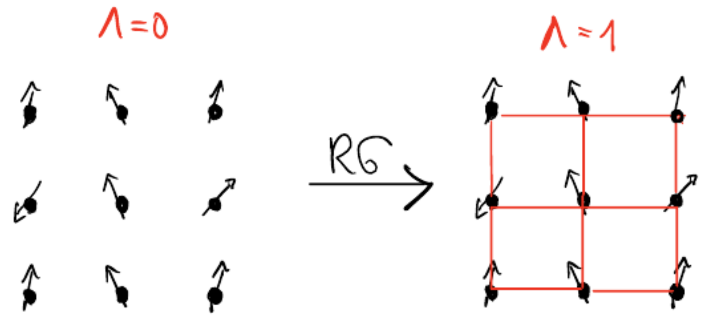
- bosonic Wetterich equation also vaild for quantum spin systems
- where is spin-algebra? initial conditions!

4.) First application: Tc of spin-S Ising models

- classical model (everything commutes); test deformation scheme

$$J_{ij}^{\Lambda} = \Lambda J_{ij}$$

$$R_{ij}^{\Lambda} = (\Lambda - 1) J_{ij}$$



$$\mathcal{V}_{\Lambda} = -\Lambda J \sum_{\langle ij \rangle} S_i^z S_j^z$$

- initial condition: decoupled sites $\mathcal{G}_0[h] = \sum_i B(\beta(h_0 + h_i))$

Brillouin function

$$B'(y) = b(y) = (S + \frac{1}{2}) \coth\left[(S+\frac{1}{2})y\right] - \frac{1}{2}\coth\left[\frac{y}{2}\right] \qquad B(y) = \ln\left[\frac{\sinh\left((S+\frac{1}{2})y\right)}{\sinh(y/2)}\right]$$

- vertex expansion of Legendre transform:

$$\Gamma_{\Lambda}[M] = \sum_{n=0}^{\infty} \frac{1}{n! N^{n-1}} \sum_{\boldsymbol{k}_1 \ldots \boldsymbol{k}_n} \delta_{\boldsymbol{k}_1 + \ldots + \boldsymbol{k}_n,} \Gamma_{\Lambda}^{(n)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) M_{\boldsymbol{k}_1} \ldots M_{\boldsymbol{k}_n}$$

all initial vertices $\Gamma_0^{(n)}$ are finite

$$b' = S(S+1)/3$$

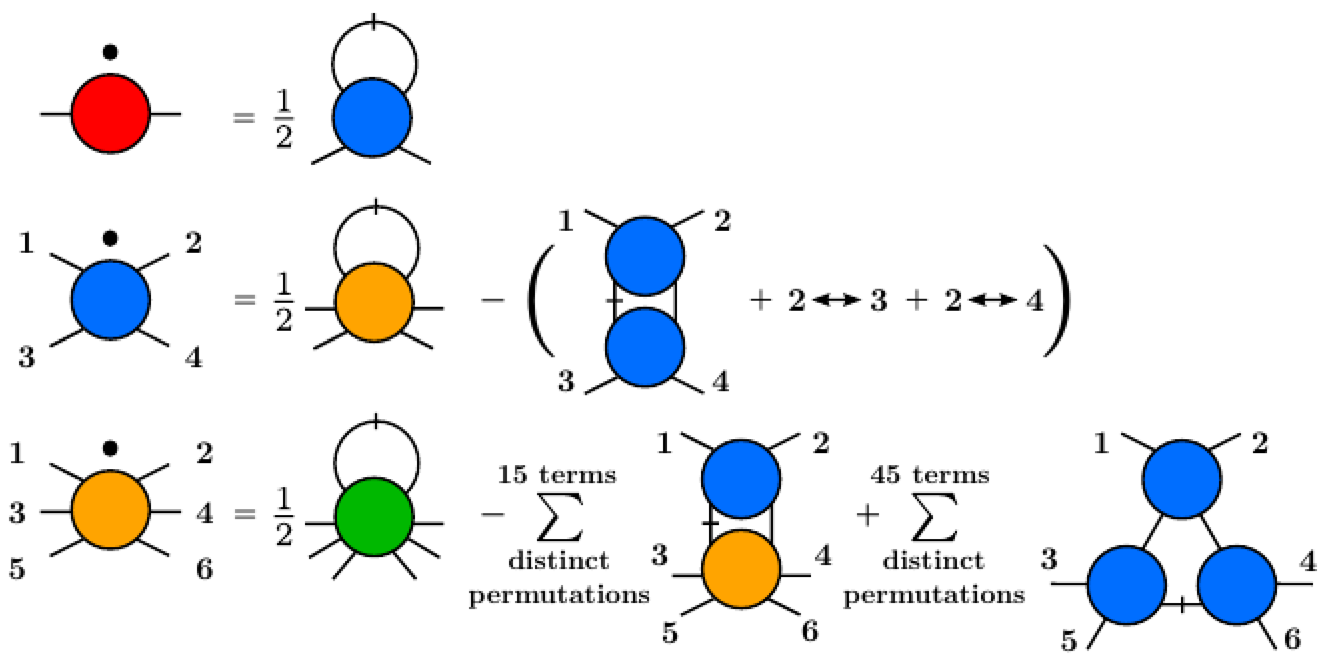
$$\Gamma_0^{(4)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = -b'''/(b')^4 \equiv u_0 > 0 \qquad b''' = [1 - (2S+1)^4]/120$$

FRG calculation of Tc

- condition for critical temperature $\Gamma^{(2)}_{\Lambda=1}(0) = 0$
- mean-field theory: initial condition of FRG: $\Gamma^{(2)}_{0}(0) = 1/b' - \beta V_0$

$$V_{\boldsymbol{k}} = 2DJ\gamma_{\boldsymbol{k}} \qquad \gamma_{\boldsymbol{k}} = D^{-1}\sum_{\mu=1}^{D}\cos k_\mu \qquad T_{c0} = 2DJS(S+1)/3$$

- beyond mean-field: integrate FRG flow:



Truncation and results

- truncation of hierarchy of flow equations

$$\Gamma_\Lambda^{(2)}(\boldsymbol{k}) \to \Gamma_0^{(2)}(\boldsymbol{k}) + \Delta_\Lambda = 1/b' - \beta V_{\boldsymbol{k}} + \Delta_\Lambda$$

$$\Gamma_\Lambda^{(4)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_2, \boldsymbol{k}_4) \to u_\Lambda \qquad \Gamma_\Lambda^{(6)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_6) \to \Gamma_0^{(6)} \qquad \Gamma_\Lambda^{(n)} = 0 \quad \text{for } n \geq 8.$$

- results for S=1/2 for different dimensions D:

D	T_c/T_{c0} for $S=1/2$			relative error in %	
	SFRG	$\mathcal{O}(D^{-1})$	exact	SFRG	$\mathcal{O}(D^{-1})$
1	0	0	0	0	0
2	0	0.50	0.57	-	12
3	0.744	0.79	0.752	1	5
4	0.839	0.85	0.835	0.5	2
5	0.880	0.89	0.878	0.3	1
6	0.904	0.908	0.903	0.2	0.6
7	0.920	0.923	0.919	0.1	0.4

(in D=3: similar accuracy as lattice NPRG by Machado+Dupuis 2010)

Inverse dimension expansion of Tc

- SFRG flow equations can be used to generate systematic expansion of Tc in powers of 1/D for any spin S
- leading correction to mean-field result: expand $\Gamma^{(2)}_{\Lambda=1}(0)$ to order 1/D and solve self-consistently for Tc:

$$\frac{T_c}{T_{c0}} = \frac{1}{2}\left[1 + \sqrt{1 - \frac{u_0 (b')^2}{D}}\right]$$

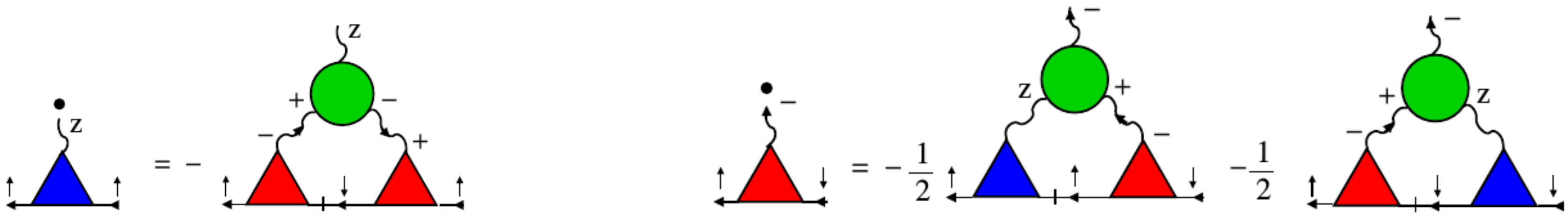
D	T_c/T_{c0} for $S = 1/2$			relative error in %	
	$\mathcal{O}(D^{-1})$	$\mathcal{O}(D^{-3})$	exact	$\mathcal{O}(D^{-1})$	$\mathcal{O}(D^{-3})$
1	0	0	0	0	0
2	0.50	0.50	0.57	12	12
3	0.79	0.740	0.752	5	2
4	0.85	0.832	0.835	2	0.4
5	0.89	0.8782	0.8778	1	0.04
6	0.908	0.9032	0.9029	0.6	0.03
7	0.923	0.9193	0.9192	0.4	0.01
8	0.933	0.9308	0.9307	0.3	0.01
9	0.941	0.93931	0.93926	0.2	0.005
10	0.9472	0.94595	0.94593	0.1	0.002

- (results with similar accuracy for Heisenberg models, FM or AFM, Jan Krieg, Dissertation)

Many applications (work in progress, partially completed)

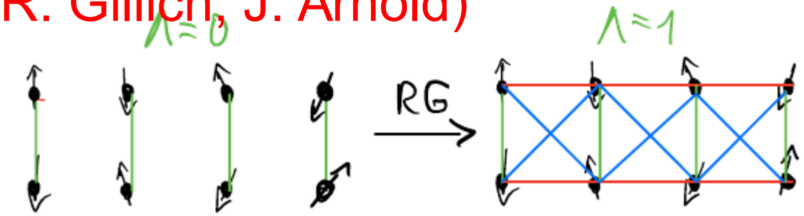
- anisotropic spin-S Kondo model: (Tarasevych, Krieg, PK, PRB 2018)

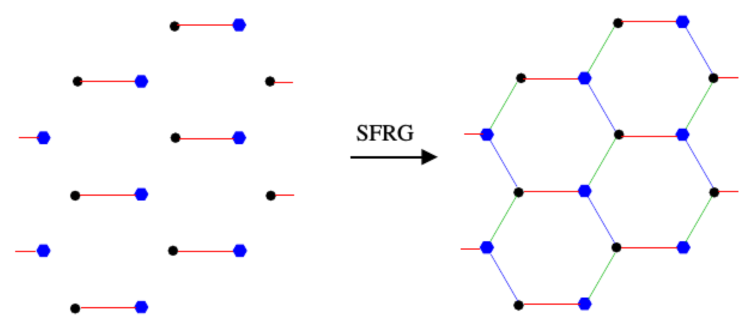
  “A rich man` s derivation of scaling laws for the Kondo problem” $-\Lambda\partial_\Lambda J^x_\Lambda = 2\rho_0 J^y_\Lambda J^z_\Lambda$



- spin-gap in in dimerized spin systems

  (initial conditions for FRG: BA theses R. Gillich, J. Arnold)



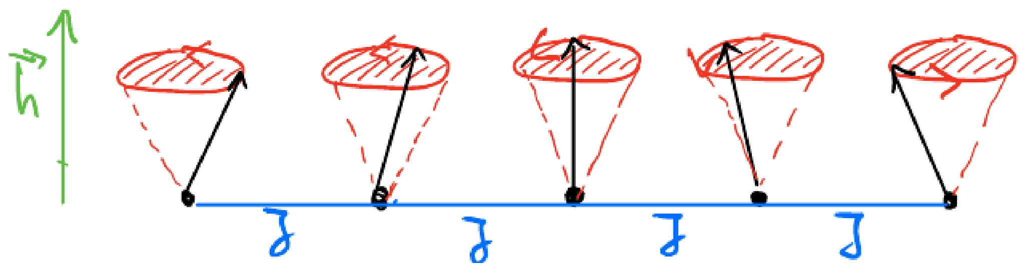


- Magnetization and magnon damping in 2D quantum ferromagnets

  (with R. Goll, D. Tarasevych, J. Krieg, preprint to appear July 2019)

Renormalized spin-waves in 2D quantum ferromagnets

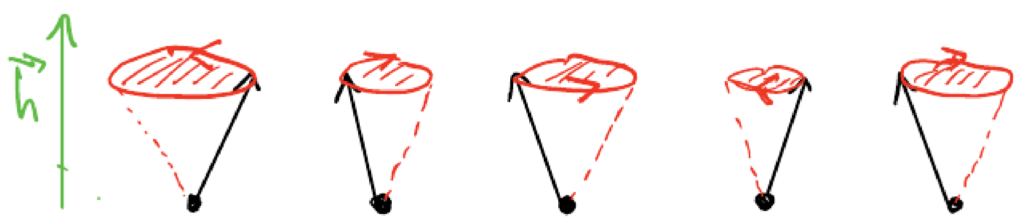
- intuitive picture: spin-waves=coherent precession of spins



$$G^{+-}(\boldsymbol{k}, i\omega_n) = \frac{Z}{h + \epsilon_{\boldsymbol{k}} - i\omega_n}$$

$$\epsilon_{\boldsymbol{k}} = S(J_0 - J_{\boldsymbol{k}}) \approx \rho_s \boldsymbol{k}^2$$

- isolated sites: $J_{ij} = 0$ precession is incoherent



$$G^{+-}(i\omega_n) = \frac{b}{h - i\omega_n}$$

$$G^{zz}(i\omega_n) = \beta \delta_{n,0} b'$$

$$G^{zz}(\tau, \tau') = \langle (S^z)^2 \rangle - \langle S^z \rangle^2 = b'$$

- deformation scheme where initially sites are decoupled should work
- problem: Legendre transform of initial $\mathcal{G}_0[\boldsymbol{h}]$ does not exist due to absence of longitudinal dynamics for isolated sites
- solution: use different type of generating functional

its time for the grim moment...

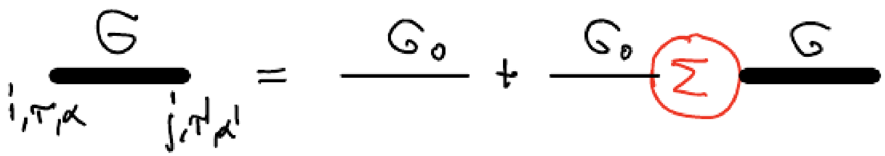
D.Mermin, Physics Today,
November 1992, page 9:

"It is absolutely impossible to give too elementary a physics talk. Every talk I have even attended in four decades of lecture going has been too hard. There is therfore no point In advising you to make your talk clear and comprehensible. You should merely strive to place as far as possible from the beginning the grim moment when more than 90% of your audience is able to make sense of less than 10% of anything you say."

FRG flow of amputated connected spin correlation functions

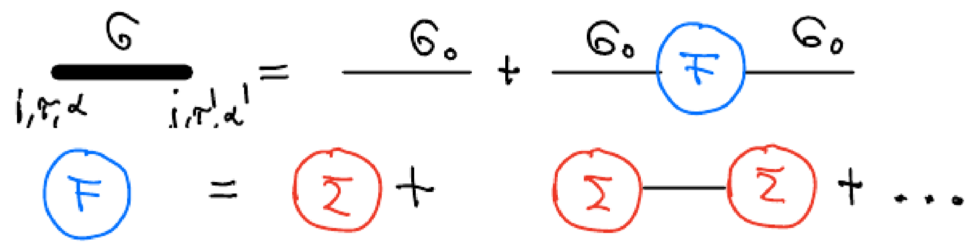
connected

$$\mathcal{G}_\Lambda[\boldsymbol{h}] = \ln \mathrm{Tr}\left[e^{-\beta \mathcal{H}_0} \mathcal{T} e^{\int_0^\beta d\tau [\sum_i \boldsymbol{h}_i(\tau)\cdot \tilde{\boldsymbol{S}}_i(\tau) - \tilde{\mathcal{V}}_\Lambda(\tau)]} \right]$$



amputated connected

$$\mathcal{F}_\Lambda[\boldsymbol{M}] = \mathcal{G}_\Lambda[\boldsymbol{h}_i \to -\sum_j J^\Lambda_{ij} \boldsymbol{M}_j] - \frac{1}{2}\int_0^\beta d\tau \sum_{ij} J^\Lambda_{ij} \boldsymbol{M}_i \cdot \boldsymbol{M}_j$$



exact flow equations

$$\partial_\Lambda \mathcal{G}_\Lambda[\boldsymbol{h}] = -\frac{1}{2}\int_0^\beta d\tau \sum_{ij,\alpha} (\partial_\Lambda J^\Lambda_{ij}) \left[\frac{\delta^2 \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_i^\alpha(\tau) \delta h_j^\alpha(\tau)} + \frac{\delta \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_i^\alpha(\tau)} \frac{\delta \mathcal{G}_\Lambda[\boldsymbol{h}]}{\delta h_j^\alpha(\tau)} \right]$$

$$\partial_\Lambda \mathcal{F}_\Lambda[\boldsymbol{M}] = \frac{1}{2}\int_0^\beta d\tau \sum_{ij,\alpha} (\partial_\Lambda \mathbf{J}_\Lambda^{-1})_{ij} \left[\frac{\delta^2 \mathcal{F}_\Lambda[\boldsymbol{M}]}{\delta M_i^\alpha(\tau) \delta M_j^\alpha(\tau)} + \frac{\delta \mathcal{F}_\Lambda[\boldsymbol{M}]}{\delta M_i^\alpha(\tau)} \frac{\delta \mathcal{F}_\Lambda[\boldsymbol{M}]}{\delta M_j^\alpha(\tau)} \right] - \partial_\Lambda \ln Z_{0,\Lambda}$$

$$Z_{0,\Lambda} = \det(-\beta \mathbf{J}_\Lambda)^{-1/2}$$

subtracted Legendre transformation

$$\Gamma_\Lambda[\boldsymbol{M}] = \int_0^\beta d\tau \sum_i \boldsymbol{h}_i(\tau)\cdot \boldsymbol{M}_i(\tau) - \mathcal{G}_\Lambda[\boldsymbol{h}] - \frac{1}{2}\int_0^\beta d\tau \sum_{ij} R^\Lambda_{ij} \boldsymbol{M}_i(\tau)\cdot \boldsymbol{M}_j(\tau) \qquad J^\Lambda_{ij} = J_{ij} + R^\Lambda_{ij}$$

$$\Phi_\Lambda[\boldsymbol{h}] = \int_0^\beta d\tau \sum_i \boldsymbol{M}_i(\tau)\cdot \boldsymbol{h}_i(\tau) - \mathcal{F}_\Lambda[\boldsymbol{M}[\boldsymbol{h}]] - \frac{1}{2}\int_0^\beta d\tau \sum_{ij} \tilde{R}^\Lambda_{ij} \boldsymbol{h}_i(\tau)\cdot \boldsymbol{h}_j(\tau) \qquad \mathbf{J}_\Lambda^{-1} = \mathbf{J}^{-1} - \tilde{\mathbf{R}}_\Lambda$$

Wetterich equation

$$\partial_\Lambda \Gamma_\Lambda[\boldsymbol{M}] = \frac{1}{2}\mathrm{Tr}\left\{ (\boldsymbol{\Gamma}''_\Lambda[\boldsymbol{M}] + \mathbf{R}_\Lambda)^{-1} \partial_\Lambda \mathbf{R}_\Lambda \right\}$$

$$\partial_\Lambda \Phi_\Lambda[\boldsymbol{h}] = \frac{1}{2}\mathrm{Tr}\left\{ \left[\boldsymbol{\Phi}''_\Lambda[\boldsymbol{h}] + \tilde{\mathbf{R}}_\Lambda \right]^{-1} \partial_\Lambda \tilde{\mathbf{R}}_\Lambda \right\} + \partial_\Lambda \ln \mathcal{Z}_{0,\Lambda}$$

in practice: hybrid functional

R. Goll, D. Tarasevych, J. Krieg, PK, preprint to appear July 2019

- idea: only partially amputated connected: irreducible in transverse fluctuations, amputated in connected longitudinal

$$\mathcal{F}_\Lambda[\boldsymbol{h}^\perp, s] = \mathcal{G}_\Lambda\left[\boldsymbol{h}_i^\perp, h_i^z \to -\sum_j J^z_{\Lambda,ij} s_j\right] \; - \frac{1}{2}\int_0^\beta d\tau \sum_{ij} J^z_{\Lambda,ij} s_i(\tau) s_j(\tau)$$

$$\Gamma_\Lambda[\boldsymbol{m}, \phi] = \int_0^\beta d\tau \sum_i (\boldsymbol{m}_i \cdot \boldsymbol{h}_i^\perp + \phi_i s_i) - \mathcal{F}_\Lambda[\boldsymbol{h}^\perp, s] - \frac{1}{2}\int_0^\beta d\tau \sum_{ij} \left(R^\perp_{\Lambda,ij} \boldsymbol{m}_i \cdot \boldsymbol{m}_j + R^\phi_{\Lambda,ij} \phi_i \phi_j \right)$$

- transverse and longitudinal regulator

$$R^\perp_{\Lambda,ij} = J^\perp_{\Lambda,ij} - J^\perp_{ij}, \qquad R^\phi_{\Lambda,ij} = -[\mathbb{J}^z_\Lambda]^{-1}_{ij} + [\mathbb{J}^z]^{-1}_{ij}$$

- Wetterich equation

$$\partial_\Lambda \Gamma_\Lambda[\boldsymbol{m}, \phi] = \frac{1}{2}\mathrm{Tr}\left\{ \left[(\boldsymbol{\Gamma}''_\Lambda[\boldsymbol{m}, \phi] + \mathbf{R}_\Lambda)^{-1} + \mathbf{J}^z_\Lambda \right] \partial_\Lambda \mathbf{R}_\Lambda \right\}$$

Order parameter flow and relation to Vaks-Larkin-Pikin

- take possibility of spontaneous symmetry breaking into account (symmetry restoration in 2D as result of fluctuations)

  Legendre transform can be extremal for finite field: $\left.\frac{\delta\Gamma_\Lambda[\boldsymbol{m}=0,\phi]}{\delta\phi_i(\tau)}\right|_{\phi=\phi_\Lambda}=0$

  expand around scale-dependent extremum:

$$\tilde{\Gamma}_\Lambda[\boldsymbol{m},\varphi]=\Gamma_\Lambda[\boldsymbol{m},\phi_\Lambda+\varphi]$$

additional terms in flow equations related to flow of magnetization

$$\partial_\Lambda\tilde{\Gamma}_\Lambda[\boldsymbol{m},\varphi]=\partial_\Lambda\Gamma_\Lambda[\boldsymbol{m},\phi]|_{\phi\to\phi_\Lambda+\varphi}+\int_0^\beta d\tau\sum_i\frac{\delta\tilde{\Gamma}_\Lambda[\boldsymbol{m},\varphi]}{\delta\varphi_i(\tau)}\partial_\Lambda\phi_\Lambda$$

- relation to Vaks-Larkin-Pikin approach

$$\Gamma_\Lambda^{+-}(K)=J^\perp(\boldsymbol{k})+\frac{H+\phi_0-i\omega}{M_0}+\Sigma_\Lambda(K)$$

$$\Gamma_\Lambda^{zz}(K)=-\frac{1}{J^z(\boldsymbol{k})}-\Pi_\Lambda(K)$$

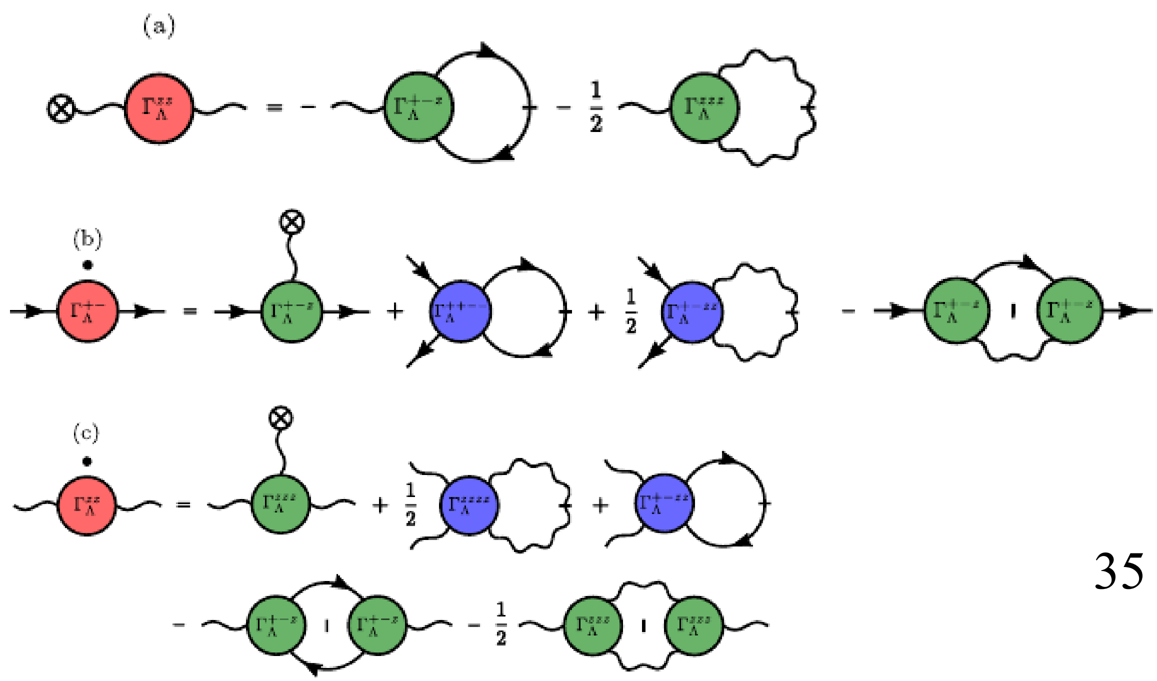
$$G_\Lambda(K)=\frac{1}{\frac{H+\phi_0+M_0J_\Lambda^\perp(\boldsymbol{k})-i\omega}{M_0}+\Sigma_\Lambda(K)}$$

$$G_\Lambda^{zz}(K)=\frac{\Pi_\Lambda(K)}{1+J_\Lambda^z(\boldsymbol{k})\Pi_\Lambda(K)}$$

vertex expansion and exact flow equations

$$
\begin{aligned}
\tilde{\Gamma}_\Lambda[\bar{\psi},\psi,\varphi] &= \beta N f_\Lambda + \int_K \Gamma_\Lambda^{+-}(K)\bar{\psi}_K\psi_K + \frac{1}{2!}\int_K \Gamma_\Lambda^{zz}(K)\varphi_{-K}\varphi_K \\
&+ \int_{K_1}\int_{K_2}\int_{K_3} \delta(K_1+K_2+K_3)\Gamma_\Lambda^{+-z}(K_1,K_2,K_3)\bar{\psi}_{-K_1}\psi_{K_2}\varphi_{K_3} \\
&+ \frac{1}{3!}\int_{K_1}\int_{K_2}\int_{K_3} \delta(K_1+K_2+K_3)\Gamma_\Lambda^{zzz}(K_1,K_2,K_3)\varphi_{K_1}\varphi_{K_2}\varphi_{K_3} \\
&+ \int_{K_1}\int_{K_2}\int_{K_3}\int_{K_4} \delta(K_1+K_2+K_3+K_4)\Big\{ \frac{1}{(2!)^2}\Gamma_\Lambda^{++--}(K_1,K_2,K_3,K_4)\bar{\psi}_{-K_1}\bar{\psi}_{-K_2}\psi_{K_3}\psi_{K_4} \\
&\quad + \frac{1}{2!}\Gamma_\Lambda^{+-zz}(K_1,K_2,K_3,K_4)\bar{\psi}_{-K_1}\psi_{K_2}\varphi_{K_3}\varphi_{K_4} + \frac{1}{4!}\Gamma_\Lambda^{zzzz}(K_1,K_2,K_3,K_4)\varphi_{K_1}\varphi_{K_2}\varphi_{K_3}\varphi_{K_4}\Big\} + \ldots
\end{aligned}
$$

- order parameter flow
- transverse self-energy
- longitudinal polarization



initial conditions: mean-field magnetization and generalized blocks

- initial magnetization: self-consistent mean-field

$$M_0 = b(\beta(h_0 + V_0 M_0)) \qquad b(y) = (S + \frac{1}{2}) \coth\left((S + \frac{1}{2})y\right) - \frac{1}{2} \coth\left(\frac{y}{2}\right)$$

- higher order vertices reflect non-trivial on-site spin dynamics:

$$\Gamma_0^{zz}(\omega) = -\delta(\omega) b',$$
$$\Gamma_0^{zzz}(\omega_1, \omega_2, \omega_3) = -\delta(\omega_1)\delta(\omega_2) b'',$$
$$\Gamma_0^{zzzz}(\omega_1, \omega_2, \omega_3, \omega_4) = -\delta(\omega_1)\delta(\omega_2)\delta(\omega_3) b'''$$

$$\Gamma_0^{+-}(\omega) = \frac{H - i\omega}{b} = G_0^{-1}(\omega)$$
$$b\Gamma_0^{+-z}(\omega_1, \omega_2, \omega_3) = 1 - G_0^{-1}(\omega_2)\delta(\omega_3) b'$$

$$b^2 \Gamma_0^{++--}(\omega_1, \omega_2, \omega_3, \omega_4) = G_0^{-1}(\omega_3) + G_0^{-1}(\omega_4) - [\delta(\omega_1 + \omega_3) + \delta(\omega_1 + \omega_4)] b' G_0^{-1}(\omega_3) G_0^{-1}(\omega_4)$$

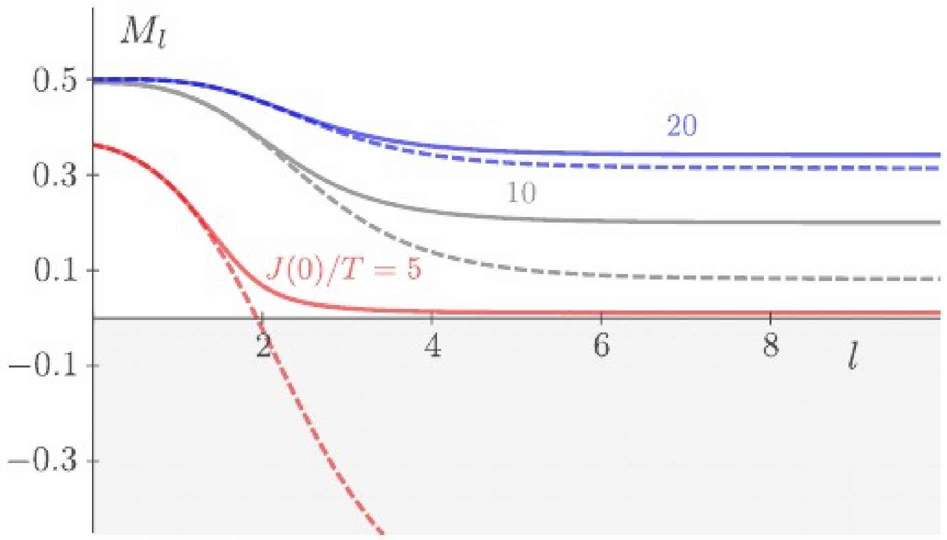
$$b^2 \Gamma_0^{+-zz}(\omega_1, \omega_2, \omega_3, \omega_4) = -[\delta(\omega_3) + \delta(\omega_4)] b' + G_0^{-1}(\omega_2)\delta(\omega_3)\delta(\omega_4)[2(b')^2 - bb''].$$

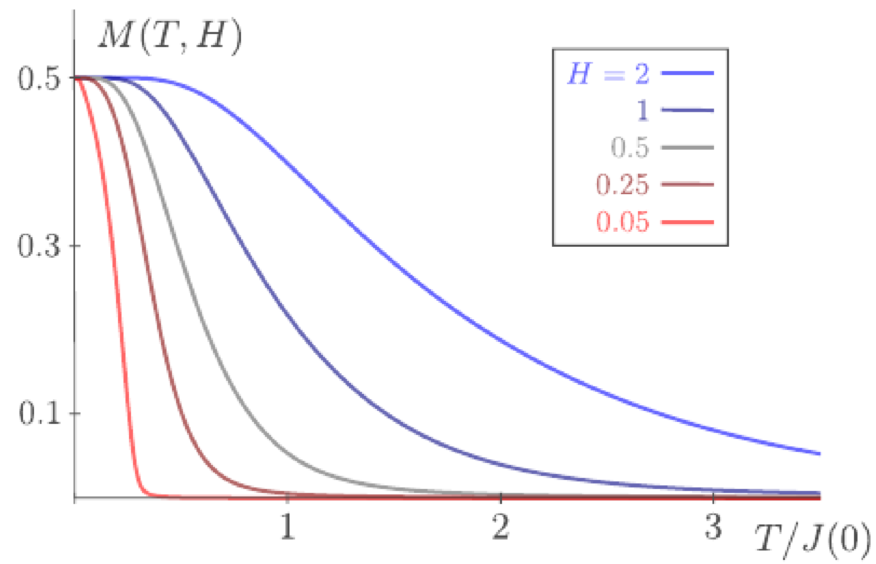
- truncation: use tree-approximation for higher-order vertices
- Keep Goldstone-mode gapless without fine-tuning: Ward identity!
- in 2D: symmetry restoration at low-energies (Mermin-Wagner)

Magnetic equation of state in 2D ferromagnets

- magnetization flows to zero logarithmically for scales

$$\Lambda \lesssim 1/\xi \propto \exp[-2\pi J S^2/T]$$





- in progress: magnon damping due to coupling to longitudinal spin fluctuations

conclusions + outlook

- SFRG: a new RG approach to quantum spin systems
- work directly with generating functionals of time-ordered spin correlation functions
- reformulation of spin-diagram technique via FRG
- Wetterich equation for quantum spin systems, SU(2) algebra in initial conditions
- accurate results for Tc of Ising and Heisenberg models
- many applications in quantum magnetism
- extension: Hubbard X-operators