Schladming Winter School 2011:
Physics at all scales: the renormalization group

Present and future prospects of the (functional) renormalization group

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panel discussion March 2, 2011

1.) Vertex expansion versus derivative expansion

2.) Real frequency spectral functions

3.) Quantum impurity models

4.) Non-equilibrium FRG

1. Vertex expansion versus derivative expansion

- starting point of FRG: "Wetterich-equation"

$$\boxed{\partial_\Lambda \Gamma_\Lambda^{\mathrm{We}}[\bar{\Phi}] = \frac{1}{2}\mathrm{STr}\left[[\partial_\Lambda \mathbf{R}_\Lambda] \left(\mathbf{\Gamma}_\Lambda^{\mathrm{We}(2)}[\bar{\Phi}] + \mathbf{R}_\Lambda \right)^{-1} \right] .}$$

- two solution strategies:

1. derivative expansion: popular in high-energy community:

$$\Gamma_\Lambda^{\mathrm{We}}[\bar{\varphi}] = \int d^D r \left[U_\Lambda(\rho(\boldsymbol{r})) + \frac{c_0}{2} Z_\Lambda^{-1}(\rho(\boldsymbol{r})) \sum_{i=1}^{N} (\boldsymbol{\nabla}\bar{\varphi}_i(\boldsymbol{r}))^2 + \frac{c_0}{4} Y_\Lambda(\rho(\boldsymbol{r}))\,(\boldsymbol{\nabla}\rho(\boldsymbol{r}))^2 + \dots \right]$$

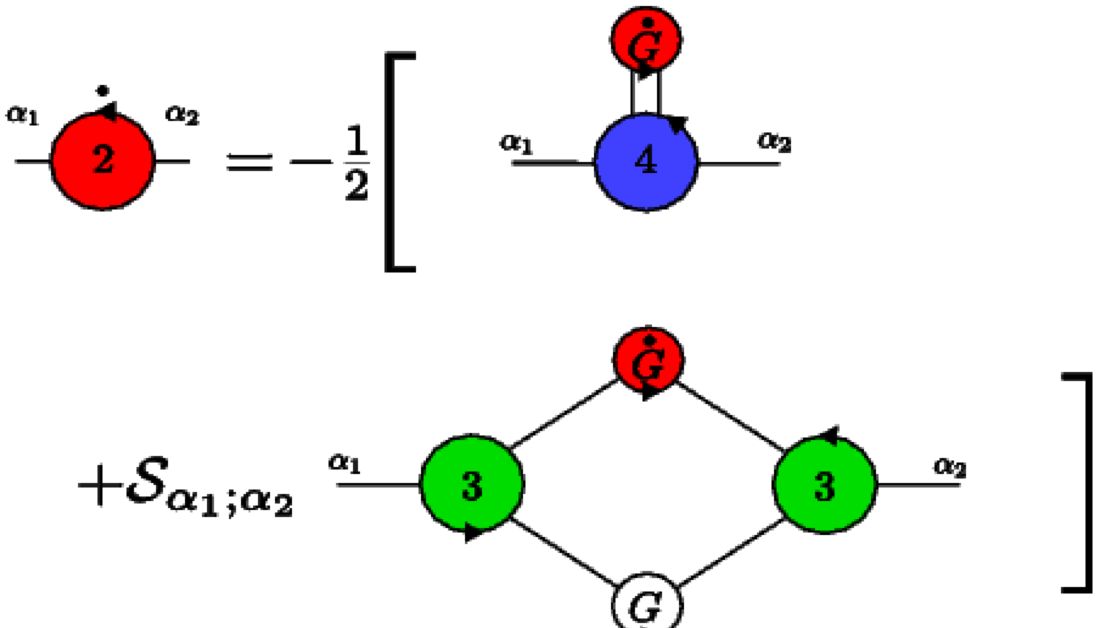
2. vertex expansion: popular in condensed matter community:

$$\Gamma[\bar{\Phi}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \cdots \int_{\alpha_n} \Gamma^{(n)}_{\alpha_1 \dots \alpha_n} \bar{\Phi}_{\alpha_1} \dots \bar{\Phi}_{\alpha_n}$$

vertex expansion: self-energy

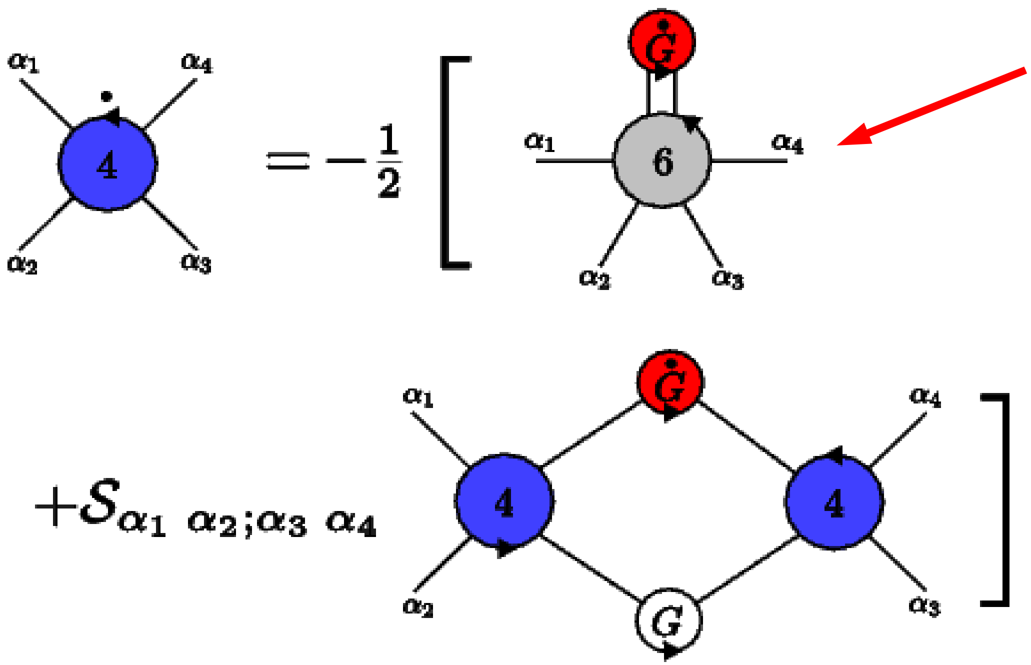
Wetterich equation reduces to infinite hierarchy of integro-differential equations for irreducible vertices

- exact flow equation for irreducible self-energy



vertex expansion:effective interaction

- exact flow equation for effective interaction



three-body renormalizes two-body interaction

- urgently needed: trunction strategies!

weak point of vertex expansion for fermions:

There are no truncation strategies which work in the strong coupling regime of strongly correlated electrons.

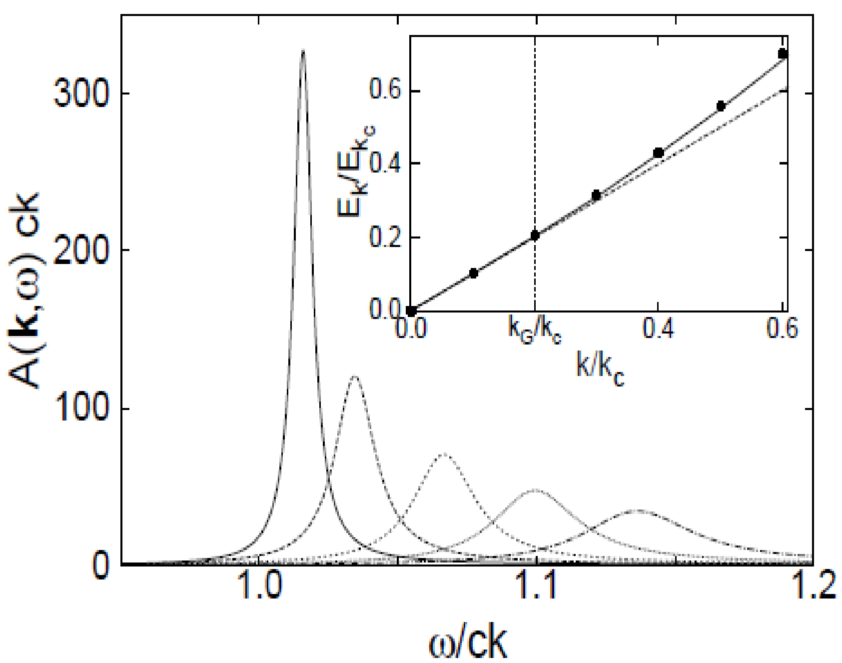
Ideas:

- fermionic FRG with partial bosonization in several competing channels
- closing hierarchy of vertex expansion via Ward identities
- multi-channel ansatz for freqeuncy-dependent effective interaction
- scale-dependent bosonization

2.FRG calculations of real frequency spectral functions

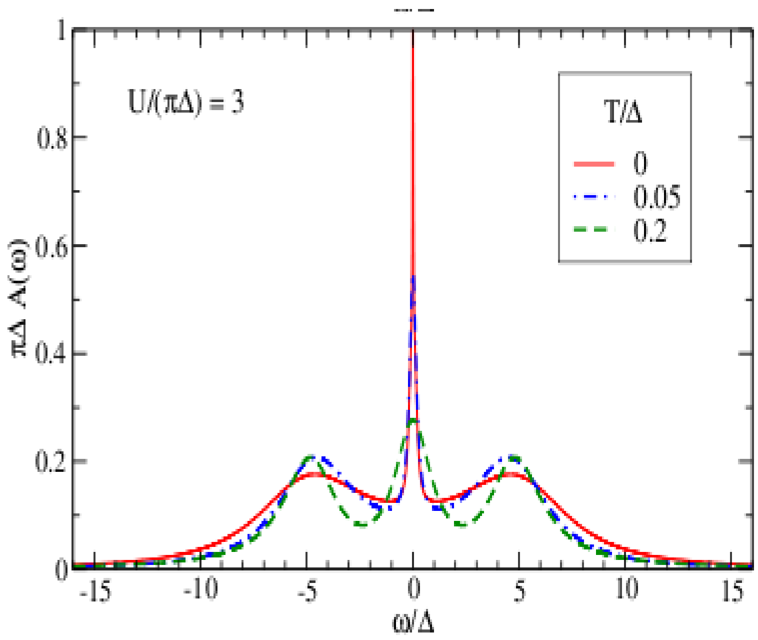
examples:

- interacting bosons in 2d



(Sinner, Hasselmann, PK, PRL 2009)

- Anderson impurity model



(Isidori, Rosen, Bartosch, Hofstetter, PK, PRB 2010)

... real frequency spectral functions

- need truncation of FRG which gives sufficiently accurate frequency-dependence of 2-point vertex
- high energy fluctuations must be included
- numerical analytic continuation to real frequencies

claim: vertex expansion is here better than derivative expansion!

2. Quantum impurity models

- Anderson impurity model:

$$\begin{aligned}\hat{H} &= \sum_{\boldsymbol{k}\sigma}(\epsilon_{\boldsymbol{k}} - \sigma h)\hat{c}^\dagger_{\boldsymbol{k}\sigma}\hat{c}_{\boldsymbol{k}\sigma} \\ &+ \sum_{\sigma}(E_d - \sigma h)\hat{d}^\dagger_\sigma \hat{d}_\sigma + U\hat{d}^\dagger_\uparrow \hat{d}_\uparrow \hat{d}^\dagger_\downarrow \hat{d}_\downarrow \\ &+ \sum_{\boldsymbol{k}\sigma}(V^*_{\boldsymbol{k}}\hat{d}^\dagger_\sigma \hat{c}_{\boldsymbol{k}\sigma} + V_{\boldsymbol{k}}\hat{c}^\dagger_{\boldsymbol{k}\sigma}\hat{d}_\sigma).\end{aligned}$$

- Kondo model:

$$\begin{aligned}\hat{H} &= \sum_{\boldsymbol{k}\sigma}\epsilon_{\boldsymbol{k}}\hat{c}^\dagger_{\boldsymbol{k}\sigma}\hat{c}_{\boldsymbol{k}\sigma} \\ &+ \frac{J}{2}\sum_{\boldsymbol{k}\boldsymbol{k}'}\sum_{\sigma\sigma'}\hat{c}^\dagger_{\boldsymbol{k}\sigma}(\vec{\sigma})_{\sigma\sigma'}\hat{c}_{\boldsymbol{k}'\sigma'}\cdot \boldsymbol{S}_d\end{aligned}$$

- want: Green function/spectral function of d-electrons

$$G_\sigma(i\omega) = \frac{1}{i\omega - E_d + \mu + \sigma h + i\Delta\mathrm{sgn}\omega - \Sigma_\sigma(i\omega)} \qquad A_\sigma(\omega) = -\frac{1}{\pi}\mathrm{Im}G_\sigma(\omega + i0)$$

- exact result for particle-hole symmetric point: (Bethe-Ansatz)

$$G(i\omega) = \frac{Z}{i\omega + i\tilde{\Delta}\mathrm{sgn}\omega}$$

at strong coupling: $Z \sim \sqrt{\frac{8U}{\pi^2\Delta}}e^{-\pi U/(8\Delta)}$

FRG for Anderson impurity model

- nobody has found a truncation of the FRG which reproduces the known strong-coupling behavior of Z

- playground for testing approximation strategies: at the end, you have to compare with exact results!
  (you don`t get away with bad approximations!)

- spectral function can be calculated numerically exactly using Wilson´s numerical RG

- FRG with partial bosonization reproduces spectral line-shape at finite temperatures

  (Isidori, Rosen, Bartosch, Hofstetter, PK, PRB 2010)

the art of Hubbard-Stratonovich transformations

$$n_\uparrow(\tau) = \bar{d}_\uparrow(\tau)d_\uparrow(\tau)$$
$$n_\downarrow(\tau) = \bar{d}_\downarrow(\tau)d_\downarrow(\tau)$$

$$s(\tau) = \bar{d}_\downarrow(\tau)d_\uparrow(\tau)$$

$$U n_\uparrow n_\downarrow = -U\bar{s}s$$

decoupling in spin-singlet particle-hole channel:

$$e^{-\int_0^\beta U n_\uparrow n_\downarrow} = \frac{\int \mathcal{D}[\bar{\chi},\chi] e^{-\int_0^\beta U^{-1}\bar{\chi}\chi + \bar{\chi}s + \bar{s}\chi}}{\int \mathcal{D}[\bar{\chi},\chi] e^{-\int_0^\beta U^{-1}\bar{\chi}\chi}}$$

alternative decouplings:

charge and spin-triplet particle hole:

$$n(\tau) = n_\uparrow(\tau) + n_\downarrow(\tau) \qquad m(\tau) = n_\uparrow(\tau) - n_\downarrow(\tau) \qquad U n_\uparrow n_\downarrow = \frac{U}{4}(n^2 - m^2)$$

add particle-particle channel:

$$p(\tau) = d_\downarrow(\tau)d_\uparrow(\tau) \qquad U n_\uparrow n_\downarrow = \frac{U_\parallel}{2}(n^2 - m^2) + U_\perp \bar{s}s + U_p \bar{p}p$$

$$2U_\parallel + U_\perp + U_p = U$$

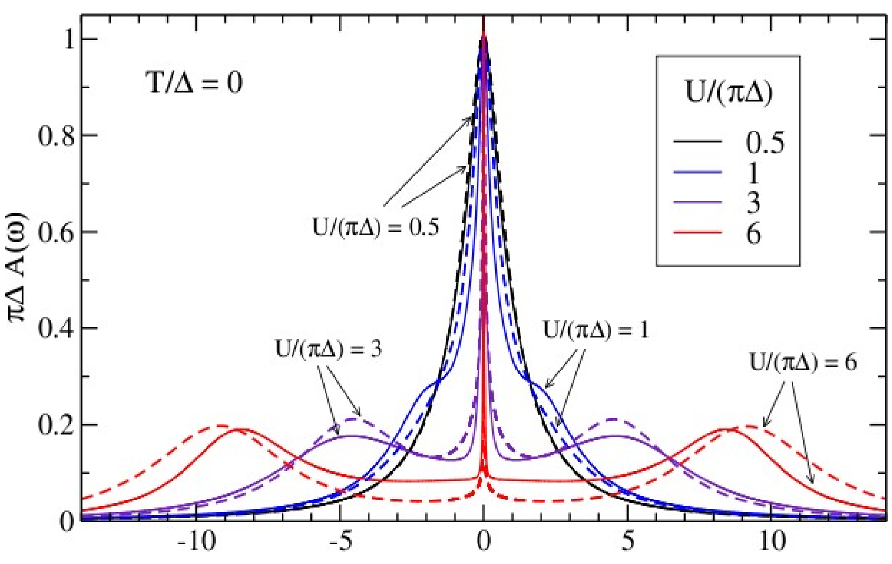
freedom leads to ambiguities!

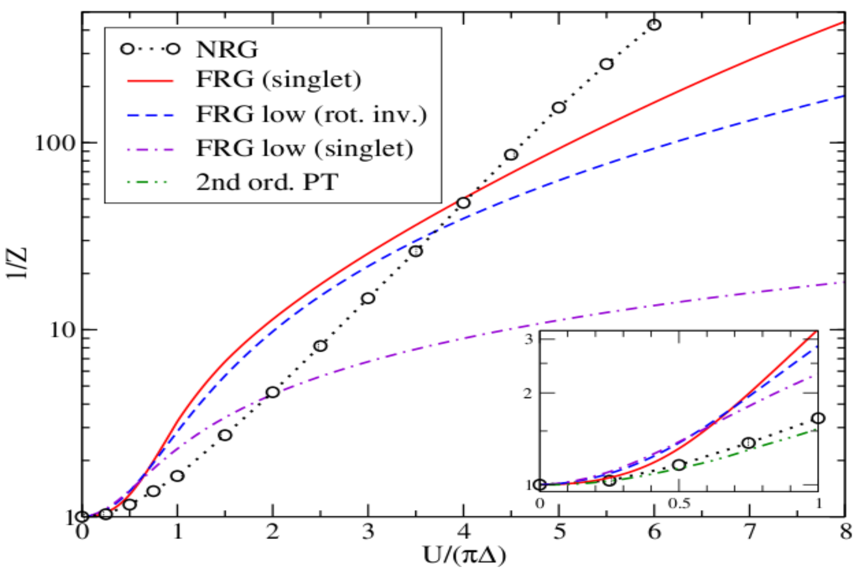
partial bosonization of AIM: spin-singlet particle-hole channel

- which decoupling is the best? depends on the problem!
- example: FRG for AIM (Isidori et al, PRB 2010)

$$\partial_\Lambda \Sigma_\Lambda(i\omega) = \int_{\bar{\omega}} \dot{F}_\Lambda(i\bar{\omega}) G_\Lambda(i\omega - i\bar{\omega}) \qquad G_\Lambda(i\omega) = \frac{1}{i\omega + i\Delta\,\mathrm{sign}\,\omega - \Sigma_\Lambda(i\omega)}$$

$$\Pi_\Lambda(i\bar{\omega}) = -\int_\omega G_\Lambda(i\omega) G_\Lambda(i\omega - i\bar{\omega}) \qquad F_\Lambda(i\bar{\omega}) = \frac{1}{U^{-1} + R_\Lambda(i\bar{\omega}) - \Pi_\Lambda(i\bar{\omega})}$$





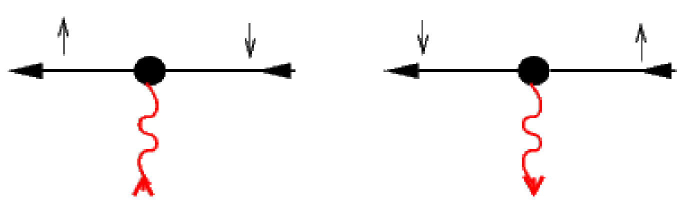
scale-dependent bosonization: does not help here!

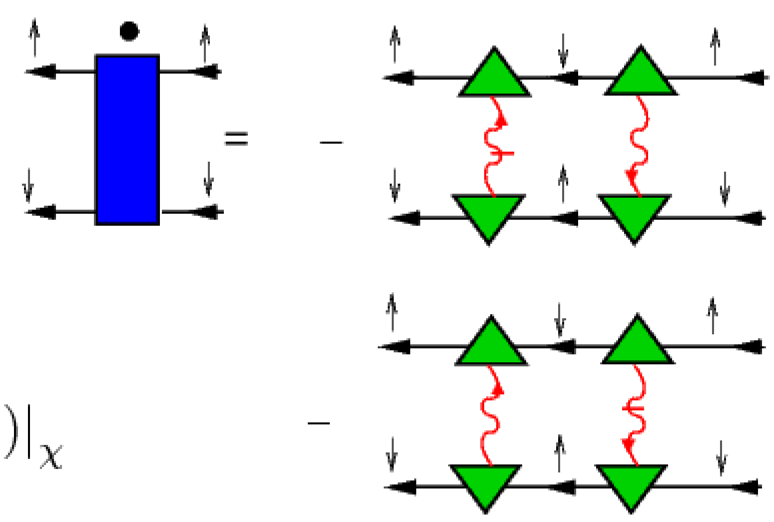
- idea: choose scale-dependent HS field such that RG does not re-generate fermionic four-point vertex

(Gies, Wetterich, 2002)

$$\partial_\Lambda \Gamma_\Lambda[\bar{d}, d, \bar{\chi}, \chi] = \partial_\Lambda \Gamma_\Lambda[\bar{d}, d, \bar{\chi}, \chi]\big|_\chi + \int_{\bar{\omega}} \left[\frac{\delta \Gamma_\Lambda}{\delta \chi_{\bar{\omega}}} \partial_\Lambda \chi_{\bar{\omega}} + \frac{\delta \Gamma_\Lambda}{\delta \bar{\chi}_{\bar{\omega}}} \partial_\Lambda \bar{\chi}_{\bar{\omega}} \right]$$

- for AIM with spin singlet particle-hole-decoupling:





$$\partial_\Lambda \chi_{\bar{\omega}} = s_{\bar{\omega}} \partial_\Lambda \alpha_\Lambda(i\bar{\omega})$$

$$\partial_\Lambda \alpha_\Lambda(i\bar{\omega}) = \frac{1}{2\gamma_\Lambda(i\bar{\omega})} \partial_\Lambda U_\Lambda(i\bar{\omega})\big|_\chi$$

does not help to recover Kondo scale!

4.Non-equilibrium FRG

• many numerical RG versions in condensed matter:

PHYSICAL REVIEW B **70**, 121302(R) (2004)

1.) **Nonequilibrium electron transport using the density matrix renormalization group method**

Peter Schmitteckert
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 76128 Karlsruhe, Germany
(Received 22 March 2004; revised manuscript received 10 May 2004; published 8 September 2004)

We extended the density matrix renormalization group method to study the real time dynamics of interacting one-dimensional spinless Fermi systems by applying the full time evolution operator to an initial state. As an

2.) PRL **95,** 196801 (2005) PHYSICAL REVIEW LETTERS week ending 4 NOVEMBER 2005

Real-Time Dynamics in Quantum-Impurity Systems: A Time-Dependent Numerical Renormalization-Group Approach

Frithjof B. Anders[1] and Avraham Schiller[2]
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(Received 29 April 2005; published 31 October 2005)

We develop a general approach to the nonequilibrium dynamics of quantum-impurity systems for arbitrary coupling strength. The numerical renormalization group is used to generate a complete basis set

versions of numerical non-equilibrum RG

3.)

PHYSICAL REVIEW B **80**, 045117 (2009)

Real-time renormalization group in frequency space: A two-loop analysis of the nonequilibrium anisotropic Kondo model at finite magnetic field

Herbert Schoeller and Frank Reininghaus
Institut für Theoretische Physik, Lehrstuhl A, RWTH Aachen, 52056 Aachen, Germany and JARA-Fundamentals of Future Information Technology
(Received 9 February 2009; revised manuscript received 25 May 2009; published 22 July 2009)

We apply a recently developed nonequilibrium real-time renormalization group (RG) method in frequency space to describe nonlinear quantum transport through a small fermionic quantum system coupled weakly to

4.)

Annals of Physics 324 (2009) 2146–2178

Real-time evolution for weak interaction quenches in quantum systems

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(flow equations for continuous unitary transfomations)

A R T I C L E I N F O

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A B S T R A C T

Motivated by recent experiments in ultracold atomic gases that explore the nonequilibrium dynamics of interacting quantum many-body systems, we investigate the nonequilibrium properties

non-equilibrium time-evolution: kinetic equations

- Keldysh component of non-equilibrium Dyson equation gives kinetic equation for distribution function:

$$\mathbf{G} = \begin{pmatrix} [\mathbf{G}]^{CC} & [\mathbf{G}]^{CQ} \\ [\mathbf{G}]^{QC} & 0 \end{pmatrix} = \begin{pmatrix} \hat{G}^K & \hat{G}^R \\ \hat{G}^A & 0 \end{pmatrix} \qquad \mathbf{\Sigma} = \begin{pmatrix} 0 & [\mathbf{\Sigma}]^{CQ} \\ [\mathbf{\Sigma}]^{QC} & [\mathbf{\Sigma}]^{QQ} \end{pmatrix} = \begin{pmatrix} 0 & \hat{\Sigma}^A \\ \hat{\Sigma}^R & \hat{\Sigma}^K \end{pmatrix}$$

$$\left(\mathbf{G}_0^{-1} - \mathbf{\Sigma}\right)\mathbf{G} = \mathbf{I} \qquad iG^K(\boldsymbol{k},t,t) = 1 + 2n_{\boldsymbol{k}}(t)$$

$$\begin{aligned} i\partial_t G^K(\boldsymbol{k},t,t) &= \int_{t_0}^{t} dt_1 [\Sigma^K(\boldsymbol{k},t,t_1)G^A(\boldsymbol{k},t_1,t) - G^R(\boldsymbol{k},t,t_1)\Sigma^K(\boldsymbol{k},t_1,t)] \\ &+ \int_{t_0}^{t} dt_1 [\Sigma^R(\boldsymbol{k},t,t_1)G^K(\boldsymbol{k},t_1,t) - G^K(\boldsymbol{k},t,t_1)\Sigma^A(\boldsymbol{k},t_1,t)] \end{aligned}$$

- functional integral representation with Keldysh action:

$$\begin{aligned} i[\mathbf{G}]^{\lambda\lambda'}_{\sigma\boldsymbol{k}t,\sigma'\boldsymbol{k}'t'} &\equiv iG^{\lambda\lambda'}_{\sigma\sigma'}(\boldsymbol{k}t,\boldsymbol{k}'t') = \langle \Phi^{\lambda}_{\sigma}(\boldsymbol{k}t)\Phi^{\lambda'}_{\sigma'}(\boldsymbol{k}'t')\rangle \\ &= \int \mathcal{D}[\Phi] e^{iS[\Phi]} \Phi^{\lambda}_{\sigma}(\boldsymbol{k}t)\Phi^{\lambda'}_{\sigma'}(\boldsymbol{k}'t') \end{aligned}$$

non-equilibrium FRG vertex expansion

(Gezzi et al, 2007; Gasenzer+Pawlowski, 2008)

- trivial generalization of equilibrium vertex expansion:

$$\Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n} \to i\Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n}; \qquad \mathbf{G}_\Lambda \to -i\mathbf{G}_\Lambda \;,\; \dot{\mathbf{G}}_\Lambda \to -i\dot{\mathbf{G}}_\Lambda;$$

$$\partial_\Lambda \Gamma^{(2)}_{\Lambda,\alpha_1\alpha_2} = \frac{i}{2}\int_{\beta_1}\int_{\beta_2} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2}\Gamma^{(4)}_{\Lambda,\beta_2\beta_1\alpha_1\alpha_2}$$

$$\partial_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{i}{2}\int_{\beta_1}\int_{\beta_2} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2}\Gamma^{(6)}_{\Lambda,\beta_2\beta_1\alpha_1\alpha_2\alpha_3\alpha_4}$$

$$+\frac{i}{2}\int_{\beta_1}\int_{\beta_2}\int_{\beta_3}\int_{\beta_4} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2}[\mathbf{G}_\Lambda]_{\beta_3\beta_4}$$

$$\times\Big[\Gamma^{(4)}_{\Lambda,\beta_2\beta_3\alpha_3\alpha_4}\Gamma^{(4)}_{\Lambda,\beta_4\beta_1\alpha_1\alpha_2} + \Gamma^{(4)}_{\Lambda,\beta_2\beta_3\alpha_1\alpha_2}\Gamma^{(4)}_{\Lambda,\beta_4\beta_3\alpha_3\alpha_4} + (\alpha_1 \leftrightarrow \alpha_2) + (\alpha_1 \leftrightarrow \alpha_4)\Big]$$

gives scale-dependent self-energies in kinetic equations

what is the best cutoff scheme?

- many possibilities, some violate causality
- long-time cutoff (Gasenzer, Pawlowski, 2008)
- out-scattering rate cutoff (Kloss, P.K., 2010)

$$\mathbf{G}_0^{-1} = \begin{pmatrix} 0 & (\hat{G}_0^A)^{-1} \\ (\hat{G}_0^R)^{-1} & -(\hat{G}_0^R)^{-1}\hat{G}_0^K(\hat{G}_0^A)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \hat{D}_0 - i\eta \\ \hat{D}_0 + i\eta & 2i\eta\hat{F}_0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \hat{D}_0 - i\Lambda \\ \hat{D}_0 + i\Lambda & 2i\eta\hat{F}_{0,\Lambda} \end{pmatrix}$$

- hybridization cutoff (Jakobs, Pletyukhov, Schoeller, 2010)

$$\mathbf{G}_{0,\Lambda}^{-1} = \begin{pmatrix} 0 & \hat{D}_0 - i\Lambda \\ \hat{D}_0 + i\Lambda & 2i\Lambda\hat{F}_0 \end{pmatrix}$$

bosonic toy model with out-scattering rate cutoff

(T. Kloss, P.K., arXiv: 1011.4943v2 [cond-mat.stat-mech])

$$\mathcal{H}(t) = \epsilon a^\dagger a + \frac{1}{2}\left[\gamma e^{-i\omega_0 t} a^\dagger a^\dagger + \gamma^* e^{i\omega_0 t} aa\right] + \frac{u}{2} a^\dagger a^\dagger aa.$$

- exact numerical time evolution from Schrödinger equation
- no intrinsic dissipation
- out-scattering cutoff FRG gives best results (first order truncation)