Theory Seminar Uni Marburg

11 November, 2010

Bose-Einstein Condensation and correlations in magnon systems

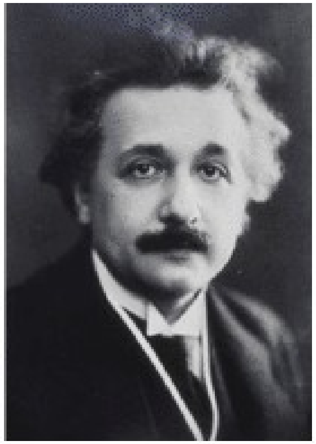
Peter Kopietz, Universität Frankfurt

1.) Bose-Einstein condensation
2.) Interacting magnons in yttrium-iron-garnet
3.) Magnon-phonon interactions in Cs_2CuCl_4

1. Bose-Einstein-Condensation: free bosons

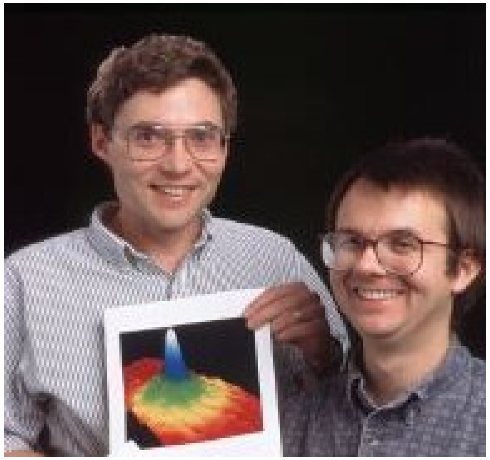
<u>predicted:</u>

S. Bose, 1924 A. Einstein, 1925



<u>first observed in atomic gases:</u>

C. Wieman, E. Cornell, 1995:



Nobel Prize 2001
(with W. Ketterle)

- Hamiltonian of free bosons: $H_0 = \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}^2}{2m} b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}} = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} n_{\boldsymbol{k}}$
- Bose-Einstein distribution: $\langle n_{\boldsymbol{k}} \rangle = \frac{1}{e^{(\epsilon_{\boldsymbol{k}} - \mu)/T} - 1}$
- relation between density and chemical potential: $\rho = \frac{\langle n_0 \rangle}{V} + \frac{1}{V} \sum_{\boldsymbol{k} \neq 0} \langle n_{\boldsymbol{k}} \rangle$

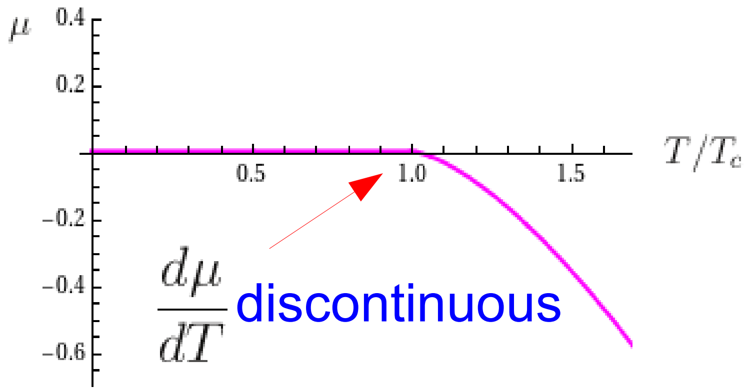
BEC for free bosons: first or second order?

- for high densities/low temperatures, density equation can only be satisfied if k=0 state is macroscopically occupied:

$$\rho\lambda^3 > \zeta(3/2) \qquad \text{De Broglie wavelength } \lambda = \sqrt{\frac{2\pi\hbar^2}{mT}}$$

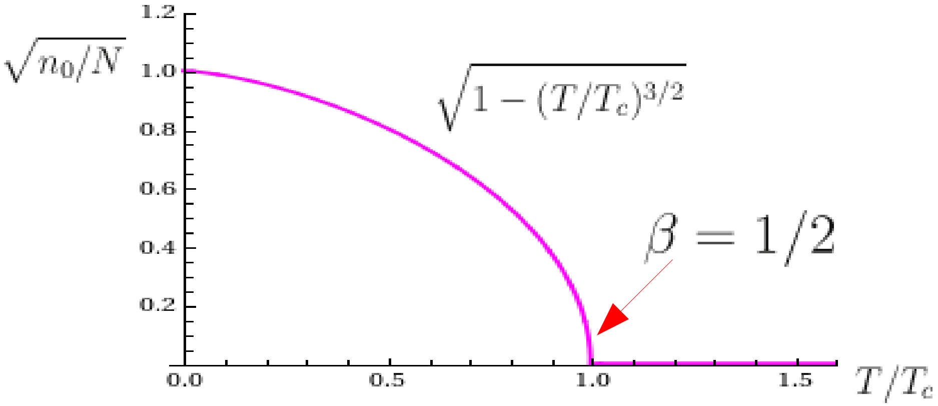
- critical temperature for BEC: $$T_c = \frac{2\pi\hbar^2}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3}$$

- chemical potential:



→ first order!

- order parameter:

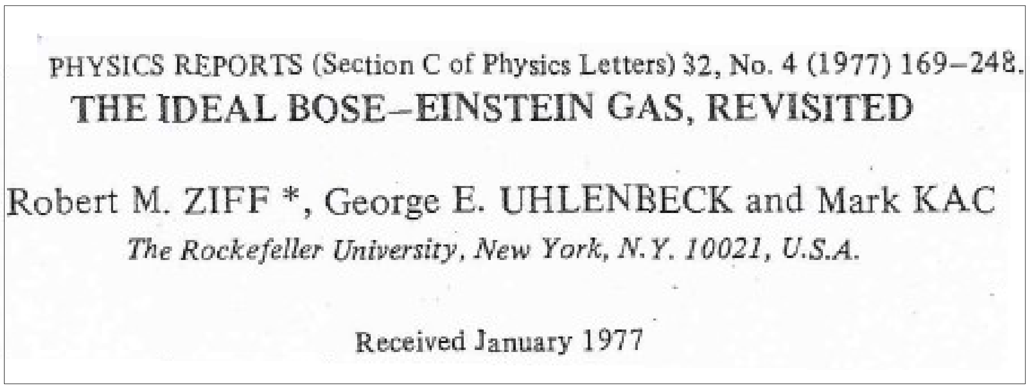


→ second order!

interacting bosons: $\beta_{XY} \approx 0.35$

BEC in grand canonical ensemble unphysical in condensed phase

- grand canonical ensemble unphysical in condensed phase of free bosons:


PHYSICS REPORTS (Section C of Physics Letters) 32, No. 4 (1977) 169–248.

THE IDEAL BOSE–EINSTEIN GAS, REVISITED

Robert M. ZIFF *, George E. UHLENBECK and Mark KAC

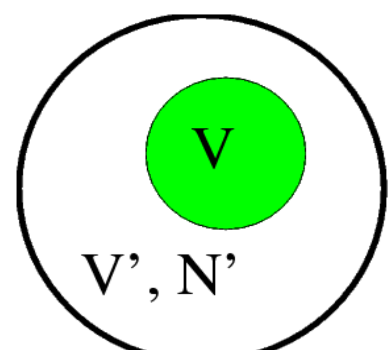
The Rockefeller University, New York, N.Y. 10021, U.S.A.

Received January 1977

(never mention this in Statistical Mechanics class...)

- grand canononical and canonical ensembles differ in their predictions for some bulk properties, e.g., $\langle (\Delta n_0)^2 \rangle$,reduced $n \geq 2$ particle density matrix

- properties of smaller subsystem are always identical to those of canonical ensemble



BEC in canonical ensemble

- canonical partition function: $Z_N = \mathrm{Tr}[\delta_{N,\hat{N}} e^{-\beta H_0}]$ $\hat{N} = \sum_{\boldsymbol{k}} b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}}$

- occupation number: $\langle n_{\boldsymbol{k}} \rangle = Z_N^{-1} \mathrm{Tr}[b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} \delta_{N,\hat{N}} e^{-\beta H_0}] \neq \text{Bose function}$

- probability for finding n particles in lowest single-particle state:

$$P_N(n) = Z_N^{-1} \mathrm{Tr}[\delta_{n,b_0^\dagger b_0} \delta_{N,\hat{N}} e^{-\beta H_0}] = e^{-N \mathcal{L}_N^{\mathrm{BEC}}(n/N)}$$

Landau function

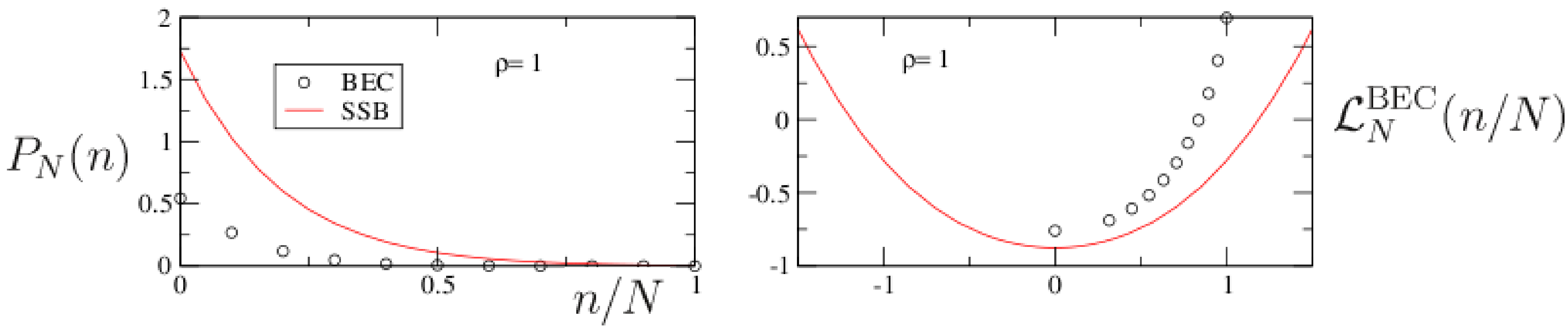
- efficient recursive algorithms: Landsberg 1961, Balazs+Bergeman 1998

$$Z_N(T) = \frac{1}{N} \sum_{k=1}^{N} Z_1(T/k) Z_{N-k}(T)$$

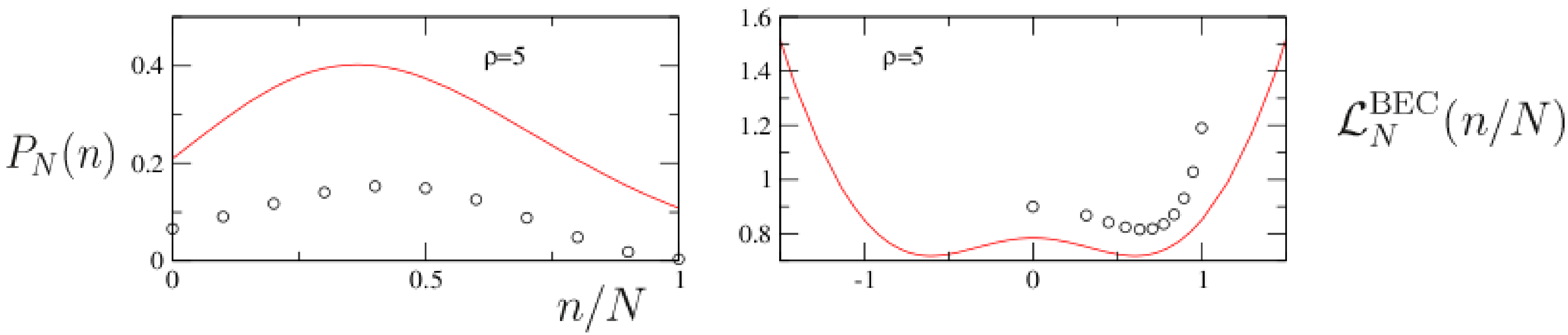
Statistics of BEC in finite systems

- 10 bosons in harmonic potential (Sinner, Schütz, PK, PRA 2006)

T > Tc:



T < Tc:



- BEC in finite and rather small systems is possible!

superfluidity and symmetry breaking

- BEC is not the same as superfluidity:

REVIEWS OF MODERN PHYSICS VOLUME 34 NUMBER 4 OCTOBER 1962

Concept of Off-Diagonal Long-Range Order and the Quantum Phases of Liquid He and of Superconductors

C. N. YANG

Institute for Advanced Study, Princeton, New Jersey

- one-body density matrix:

$$\rho^{(1)}(\boldsymbol{r},\boldsymbol{r}') = \langle \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}')\rangle = \psi^*(\boldsymbol{r})\psi(\boldsymbol{r}') + g(\boldsymbol{r},\boldsymbol{r}') \qquad \hat{\psi}(\boldsymbol{r}) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} b_{\boldsymbol{k}}$$

macroscopic wave-function: $\int d^3r|\psi(\boldsymbol{r})|^2 = \mathcal{O}(N)$ $\qquad g(\boldsymbol{r},\boldsymbol{r}') \to 0 \text{ for } |\boldsymbol{r}-\boldsymbol{r}'| \to \infty$

- superfluid: U(1) symmetry of Hamiltonian spontaneously broken
- non-interacting bosons: superfluid unstable and pathological!

Interacting bosons: Bogoliubov theory (1947)

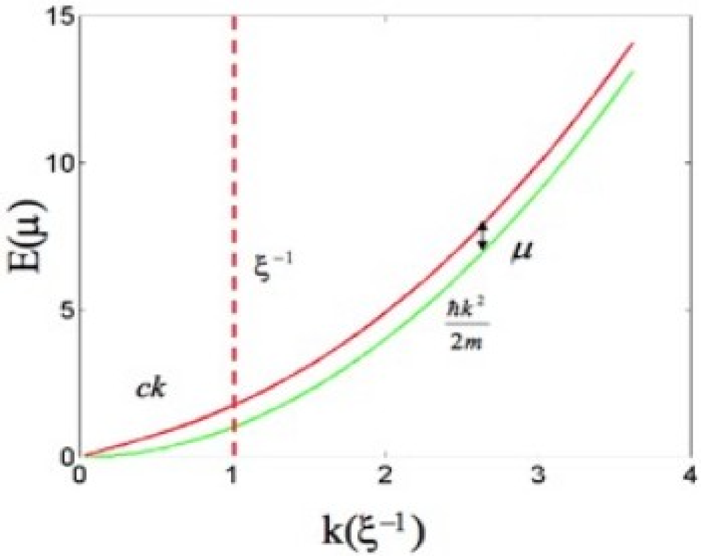
$$H = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + \frac{1}{2V} \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{q}} U_{\boldsymbol{q}} b^\dagger_{\boldsymbol{k}+\boldsymbol{q}} b^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} b_{\boldsymbol{k}'} b_{\boldsymbol{k}}$$

- Bogoliubov-shift: $b_{\boldsymbol{k}=0} \to \sqrt{N_0}$ $\qquad \rho_0 = \frac{N_0}{V}$
- Bogoliubov mean-field Hamiltonian:

$$H \approx \sum_{\boldsymbol{k}\neq 0} \left[[\epsilon_{\boldsymbol{k}} + \rho_0(U_0 + U_{\boldsymbol{k}})] b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + \frac{\rho_0 U_{\boldsymbol{k}}}{2} (b_{-\boldsymbol{k}} b_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b^\dagger_{-\boldsymbol{k}}) \right]$$

- condensate density: $\rho_0 = \frac{|b_0|^2}{V} = \frac{\mu}{U_0}$
- excitation energy: $E_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}}[\epsilon_{\boldsymbol{k}} + 2\rho_0 U_{\boldsymbol{k}}]}$
- long wavelength excitations: sound!

$$E_{\boldsymbol{k}} \sim c|\boldsymbol{k}| \qquad c = \sqrt{\rho_0 U_0 / m}$$



beyond mean-field: infrared divergencies

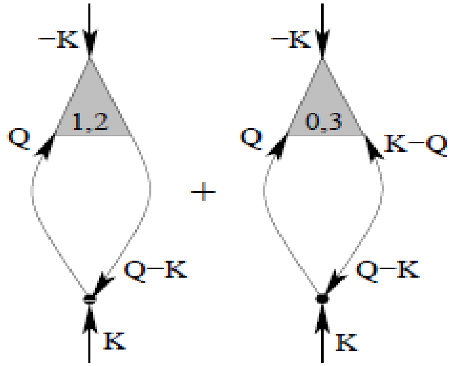
- Bogoliubov mean-field Hamiltonian:

$$H \approx \sum_{\boldsymbol{k}\neq 0} \left[[\epsilon_{\boldsymbol{k}} + \Sigma_N^{(1)}(\boldsymbol{k})] b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}} + \frac{\Sigma_A^{(1)}(\boldsymbol{k})}{2} (b_{-\boldsymbol{k}} b_{\boldsymbol{k}} + b_{\boldsymbol{k}}^\dagger b_{-\boldsymbol{k}}^\dagger) \right]$$

normal self-energy: $\Sigma_N^{(1)}(\boldsymbol{k}) \; = \; \rho_0 [U_0 + U_{\boldsymbol{k}}]$

anomalous self-energy: $\Sigma_A^{(1)}(\boldsymbol{k}) \; = \; \rho_0 U_{\boldsymbol{k}}$

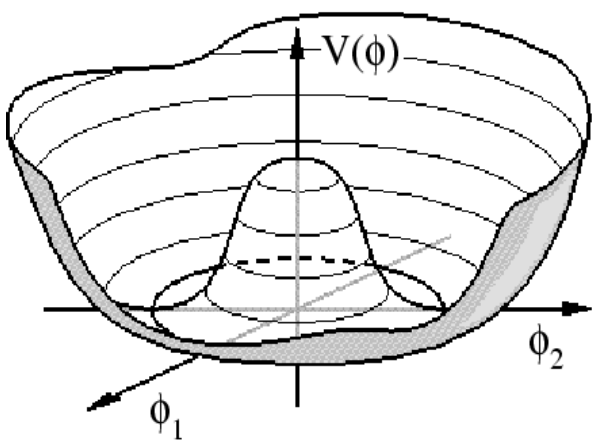
- mean-field fails due to infrared divergent fluctuation corrections:



$$\Sigma_A^{(2)}(\boldsymbol{k}) \propto \int_{|\boldsymbol{k}|}^{k_G} \frac{dq}{q^{4-D}} \propto \begin{cases} (k_G/|\boldsymbol{k}|)^{3-D} & \text{for } D < 3 \\ \ln(k_G/|\boldsymbol{k}|) & \text{for } D = 3 \end{cases}$$

$|\Sigma_A^{(2)}(\boldsymbol{k})| \gg |\Sigma_A^{(1)}(\boldsymbol{k})|$ for $|\boldsymbol{k}| \ll k_G$

- origin: coupling between transverse and longitudinal fluctuations in Mexican hat



exact result: Nepomnyashi-identity (1975)

- anomalous self-energy vanishes at zero momentum/frequency:

$$\boxed{\Sigma_A(0) = 0}$$

Письма в ЖЭТФ, том 21, вып. 1, стр. 3 – 6 *5 января 1975 г.*

К ТЕОРИИ СПЕКТРА БОЗЕ-СИСТЕМЫ
С КОНДЕНСАТОМ В ОБЛАСТИ МАЛЫХ ИМПУЛЬСОВ

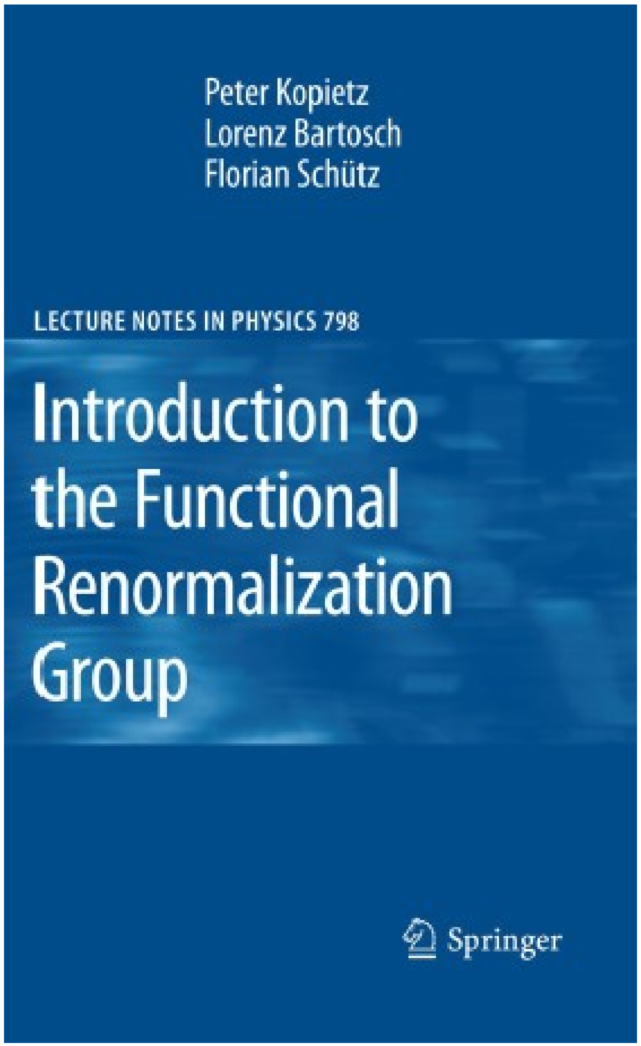
А.А.Непомнящий, Ю.А.Непомнящий

Получен результат $\Sigma_{02}(0) = 0$, устраняющий расходимости при выводе формул для функций Грина бозе-системы с конденсатом в области малых импульсов. Найдено простое сходящееся диаграммное выражение для $1/c^2$ (c – скорость звука). Обсуждаются условия применимости вычислений с использованием малого параметра.

- Bogoliubov approximation wrong: $\Sigma_A^{(1)}(0) = \rho_0 U_0$

- need non-perturbative methods!

Functional renormalization group

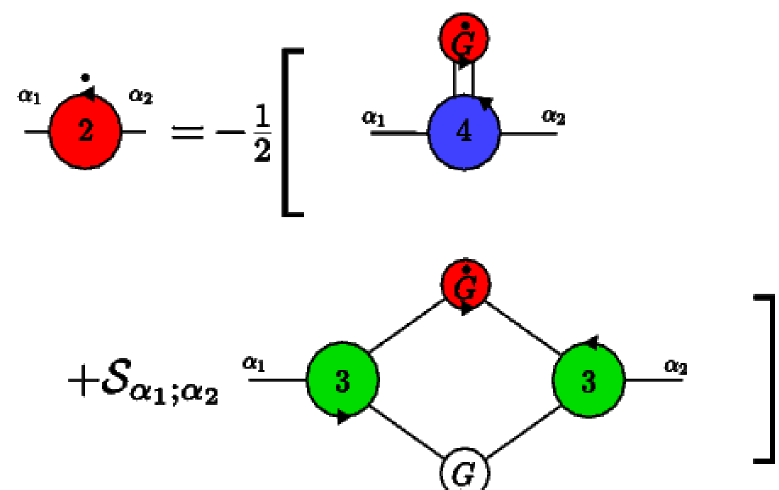


2010. XII, 380 p. (Lecture Notes in Physics, Vol. 798) Hardcover

- exact equation for change of generating functional of irreducible vertices as IR cutoff is reduced (Wetterich 1993)

$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2} \mathrm{Tr} \left[(\partial_\Lambda \boldsymbol{R}_\Lambda) \left(\frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \Gamma_\Lambda[\Phi] + \boldsymbol{R}_\Lambda \right)^{-1} \right]$$

- exact RG flow equations for all vertices
- flow of self-energy:



T_c-shift due to interactions

- critical temperature of free bosons: $T_c^0 = \frac{2\pi\hbar^2}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3}$

do weak interactions increase or decrease T_c?

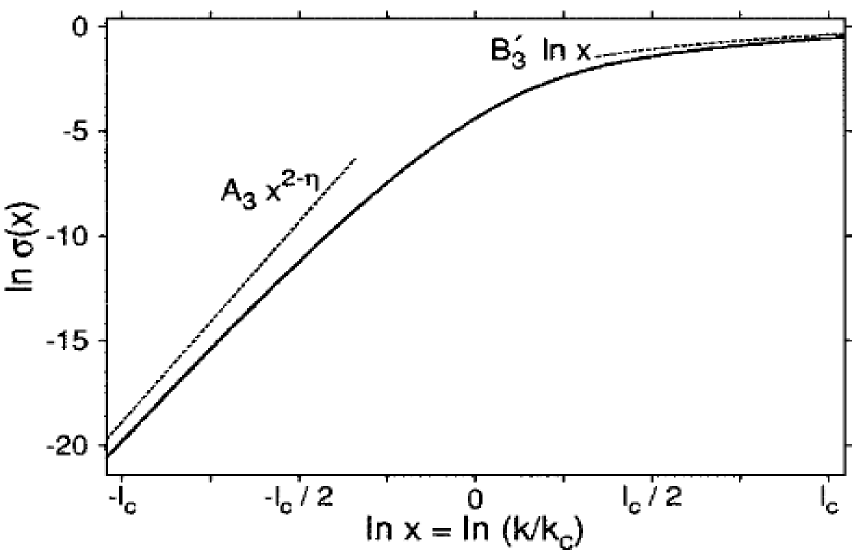
Tc-shift due to interactions

- critical temperature of free bosons: $T_c^0 = \frac{2\pi\hbar^2}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3}$

do weak interactions increase or decrease T_c?

- answer: (Baym et al 1999) $\frac{T_c - T_c^0}{T_c^0} = c_1 a_s \rho^{1/3} + \mathcal{O}(a_s^2 \ln a_s) \qquad c_1 \approx 2.9$
- Monte-Carlo simulations (Kashurnikov 2001): $c_1 \approx 1.29 \pm 0.05$
- FRG calculation $c_1 \approx 1.23$ (Ledowski, Hasselmann, PK 2004):

challenge: need momentum dependent self-energy for all wave-vectors



renormalized excitation spectrum

- experiments can measure renormalized excitation spectrum

PRL 101, 135301 (2008) PHYSICAL REVIEW LETTERS week ending 26 SEPTE

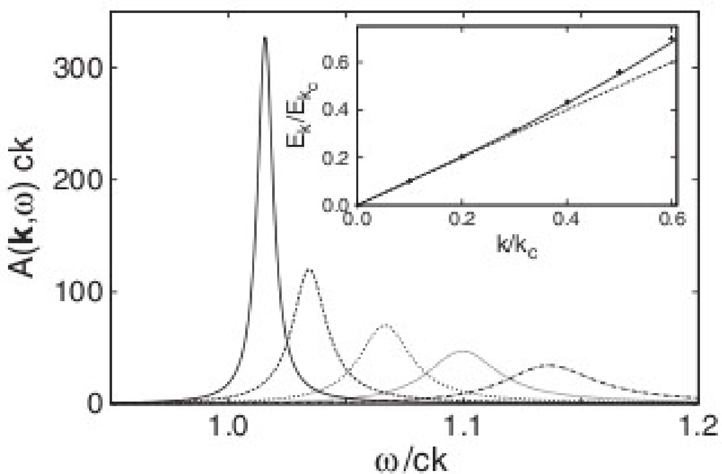
Bragg Spectroscopy of a Strongly Interacting ^{85}Rb Bose-Einstein Condensate

S. B. Papp,[1] J. M. Pino,[1] R. J. Wild,[1] S. Ronen,[1] C. E. Wieman,[2,1] D. S. Jin,[1] and E. A. Cornell[1,*]

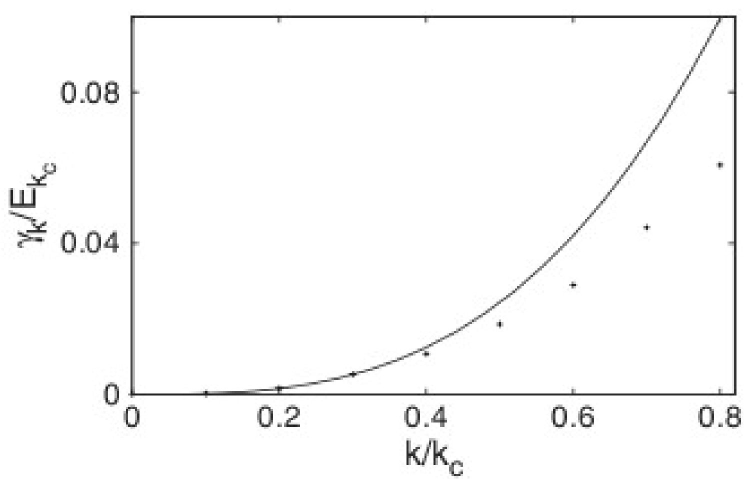
We report on measurements of the excitation spectrum of a strongly interacting Bose-Einstein condensate. A magnetic-field Feshbach resonance is used to tune atom-atom interactions in the conden-

- FRG calculation of spectral function in 2D

(Sinner, Hasselmann, PK, PRL 102, 2009, and arXiv: 1008.4521)



quasi-particle damping:



2. Interacting magnons in yttrium-iron-garnet (YIG)

•<u>Motivation</u>:

collaboration with experimental group of B. Hillebrands (Kaiserslautern)

non-equilibrium dynamics of interacting magnons in YIG

nature Vol 443|28 September 2006|doi:10.1038/nature0511

LETTERS

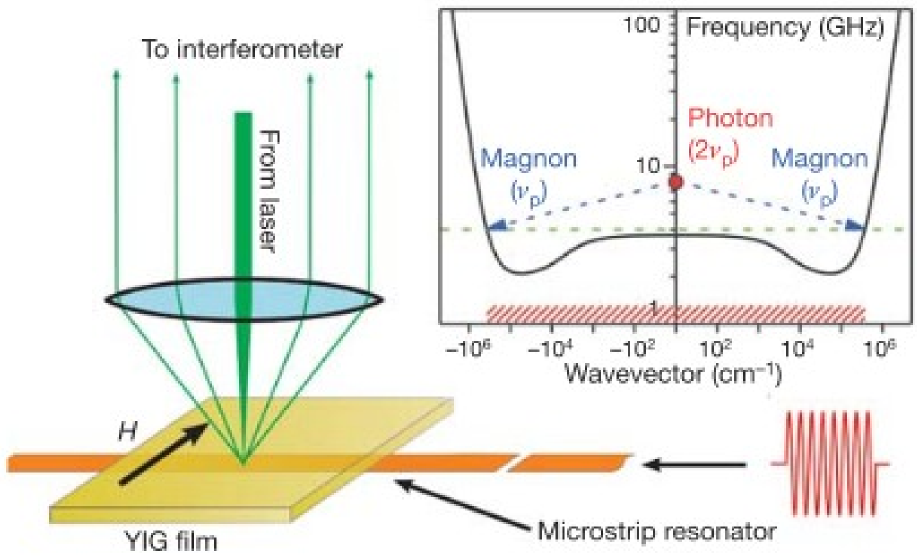
Bose–Einstein condensation of quasi-equilibrium magnons at room temperature under pumping

S. O. Demokritov[1], V. E. Demidov[1], O. Dzyapko[1], G. A. Melkov[2], A. A. Serga[3], B. Hillebrands[3] & A. N. Slavin[4]

•<u>Experiment:</u>

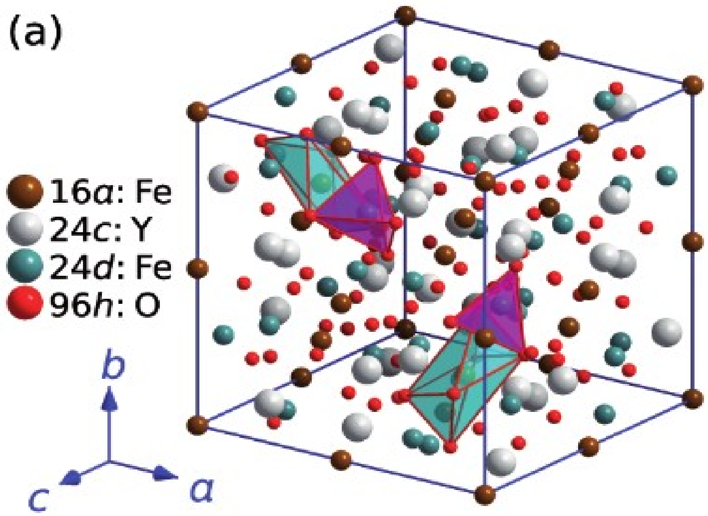
microwave-pumping of magnons in YIG

measurement of magnon distriubution via Brillouin light scattering



effective quantum spin model for YIG

- what is YIG?
  ferromagnetic insulator



◀ (a) Elementary cell of YIG with 160 atoms. The spins of the 16 Fe in positions a are coupled anti-ferromagnetically to the spins of the 24 in positions d and cause the ferrimagnetic ordering.

europhysicsnews
HIGHLIGHTS FROM EUROPEANS JOURNALS

A. Kreisel, F. Sauli, L. Bartosch, PK, 2009

- at the first sight:
  too complicated!

- effective quantum spin model for relevant magnon band:

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \mu \boldsymbol{H}_e \cdot \sum_i \boldsymbol{S}_i - \frac{1}{2}\sum_{ij,i\neq j}\frac{\mu^2}{|\boldsymbol{R}_{ij}|^3}\left[3(\boldsymbol{S}_i\cdot\hat{\boldsymbol{R}}_{ij})(\boldsymbol{S}_j\cdot\hat{\boldsymbol{R}}_{ij}) - \boldsymbol{S}_i\cdot\boldsymbol{S}_j\right]$$

exchange interaction: $J = 1.29\ \mathrm{K}$.

lattice spacing: $a = 12.376\ \text{Å}$

saturation magnetization: $4\pi M_S = 1750\ \mathrm{G}$

effective spin: $S = M_s a^3/\mu \approx 14.2$

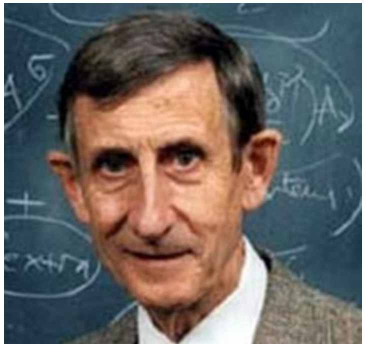
bosonization of spin operators
Holstein-Primakoff transformation

- **problem:** spin-algebra is very complicated: $[S_i^\alpha, S_j^\beta] = i\delta_{ij}\epsilon^{\alpha\beta\gamma}S_i^\gamma \quad \boldsymbol{S}_i^2 = S(S+1)$
- **solution:** for ordered magnets: bosonization of spins (Holstein,Primakoff 1940)

$$S_i^+ = S_i^x + iS_i^y = \sqrt{2S}\sqrt{1-\frac{b_i^\dagger b_i}{2S}}\,b_i = \sqrt{2S}\left[b_i - \frac{b_i^\dagger b_i b_i}{4S} + \dots\right]$$

$$S_i^z = S - b_i^\dagger b_i$$

- Bachelor project: spin algebra indeed satisfied if $[b_i, b_j^\dagger] = \delta_{ij}$
- proof that different dimension of Hilbert spaces does not matter by Dyson 1956:



PHYSICAL REVIEW VOLUME 102, NUMBER 5 JUNE 1, 1956

General Theory of Spin-Wave Interactions*

FREEMAN J. DYSON
Department of Physics, University of California, Berkeley, California, and Institute for Advanced Study, Princeton, New Jersey

(Received February 2, 1956)

magnon Hamiltonian for YIG

A. Kreisel, F. Sauli, L. Bartosch, PK, EPJB 2009

- 1/S expansion: $\hat{H} = H_0 + \sum_{n=2}^{\infty} \hat{H}_n \qquad \hat{H}_n / S^2 = \mathcal{O}(1/S^{n/2})$

- magnon dispersion is determined by quadratic part in bosons

$$\hat{H}_2 = \sum_{ij} \left[A_{ij} b_i^\dagger b_j + \frac{B_{ij}}{2} \left(b_i b_j + b_i^\dagger b_j^\dagger \right) \right]$$

$$A_{ij} = \delta_{ij} h + S(\delta_{ij} \sum_n J_{in} - J_{ij}) + S \left[\delta_{ij} \sum_n D_{in}^{zz} - \frac{D_{ij}^{xx} + D_{ij}^{yy}}{2} \right]$$

$$B_{ij} = -\frac{S}{2} \left[D_{ij}^{xx} - 2i D_{ij}^{xy} - D_{ij}^{yy} \right]$$

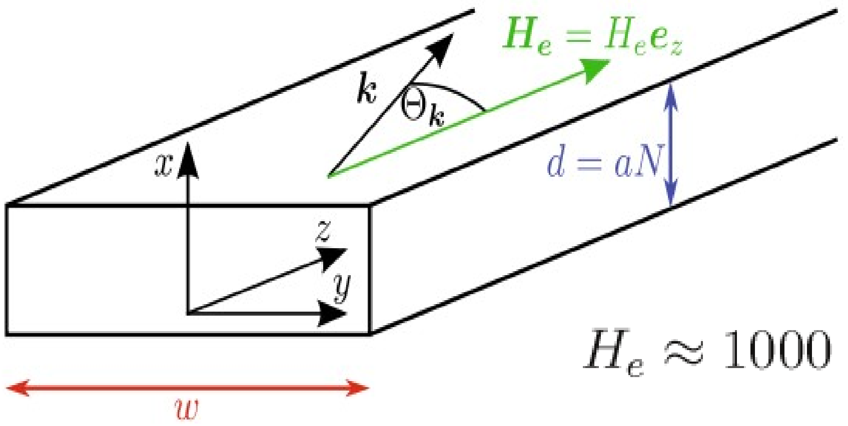
- dipolar tensor: $D_{ij}^{\alpha\beta} = (1 - \delta_{ij}) \frac{\mu^2}{|\boldsymbol{R}_{ij}|^3} \left[3 \hat{R}_{ij}^{\alpha} \hat{R}_{ij}^{\beta} - \delta^{\alpha\beta} \right]$

magnon spectrum of finite YIG films

- problem: experimentally relevant films have finite width and thickness

$$d \approx 5\mu m \approx 4000a$$

$$w \gg d$$



$H_e \approx 1000$ Oe

- phenomenological approach (Kalinikos, Slavin et al, 1986-today)

Landau-Lifshitz equation: $$\frac{\partial \boldsymbol{M}(\boldsymbol{r},t)}{\partial t} = \gamma \boldsymbol{M}(\boldsymbol{r},t) \times \boldsymbol{H}_{\text{eff}}(\boldsymbol{r},t)$$

- microscopic approach (Kreisel, Sauli, Bartosch, PK 2009)

partial Fourier transformation of Hamiltonian in yz-pane:

$$\hat{H}_2 = \sum_{\boldsymbol{k}} \sum_{x_i, x_j} \left[A_{\boldsymbol{k}}(x_{ij}) b^\dagger_{\boldsymbol{k}}(x_i) b_{\boldsymbol{k}}(x_j) + \frac{B_{\boldsymbol{k}}(x_{ij})}{2} b_{\boldsymbol{k}}(x_i) b_{-\boldsymbol{k}}(x_j) + \frac{B^*_{\boldsymbol{k}}(x_{ij})}{2} b^\dagger_{\boldsymbol{k}}(x_i) b^\dagger_{-\boldsymbol{k}}(x_j) \right]$$

$$b_i = \frac{1}{\sqrt{N_y N_z}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i} b_{\boldsymbol{k}}(x_i) \qquad A_{\boldsymbol{k}}(x_{ij}) = \sum_{\boldsymbol{r}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} A(x_i - x_j, \boldsymbol{r})$$

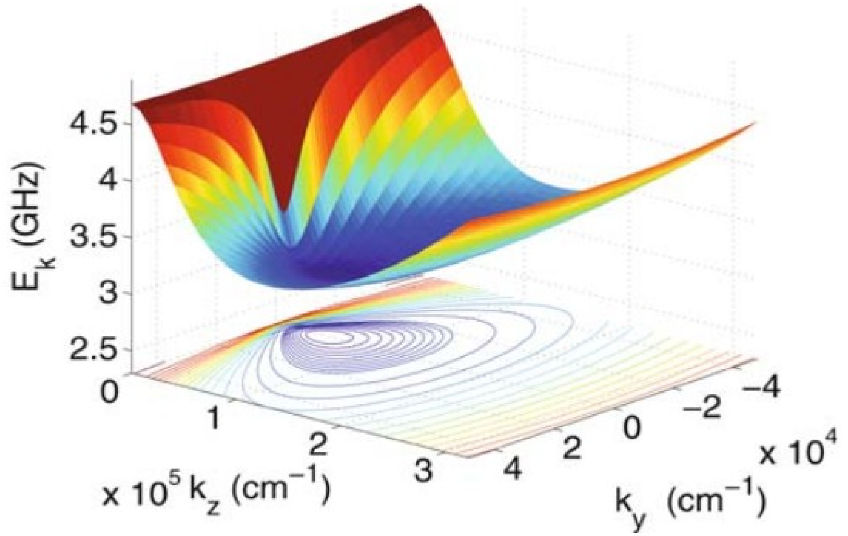
magnon spectrum of finite YIG films: results

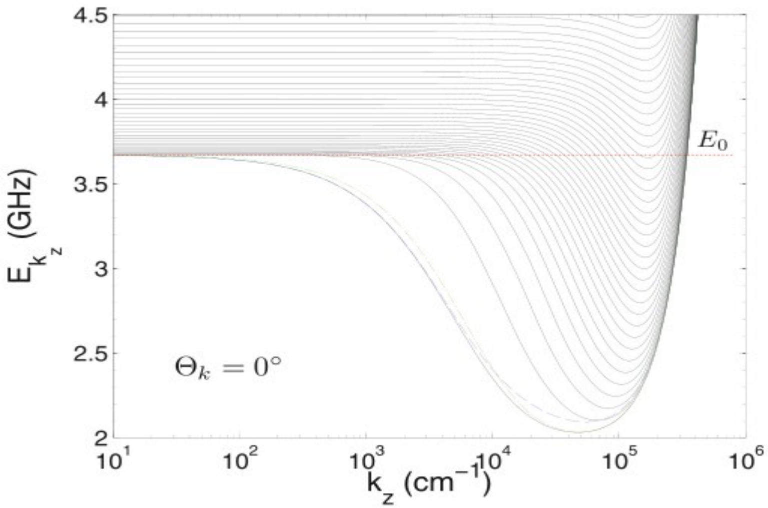
- calculate roots of secular determinant numerically:

$$\det \begin{pmatrix} E_{\boldsymbol{k}} - \mathbf{A}_{\boldsymbol{k}} & -\mathbf{B}_{\boldsymbol{k}} \\ -\mathbf{B}^*_{\boldsymbol{k}} & -E_{\boldsymbol{k}} - \mathbf{A}_{\boldsymbol{k}} \end{pmatrix} = 0 \qquad \begin{aligned} [\mathbf{A}_{\boldsymbol{k}}]_{ij} &= A_{\boldsymbol{k}}(x_{ij}) \\ [\mathbf{B}_{\boldsymbol{k}}]_{ij} &= B_{\boldsymbol{k}}(x_{ij}) \end{aligned}$$

- dispersion of lowest magnon mode has minimum at finite k

due to interplay between: 1. exchange interaction
2. dipole-dipole interaction
3. finite width of films





effective 2D model

- lowest magnon band is accurately described by uniform mode approximation:

$$b_{\boldsymbol{k}}(x_i) \approx \frac{1}{N}\sum_j b_{\boldsymbol{k}}(x_j) \equiv \frac{1}{\sqrt{N}} b_{\boldsymbol{k}}$$

→ effective translationally invariant 2D magnon Hamiltonian:

$$\hat{H}_2 = \frac{1}{2}\sum_{\boldsymbol{k}} \left[2A_{\boldsymbol{k}} b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + B^*_{\boldsymbol{k}} b_{-\boldsymbol{k}} b_{\boldsymbol{k}} + B_{\boldsymbol{k}} b^\dagger_{\boldsymbol{k}} b^\dagger_{-\boldsymbol{k}} \right]$$

- complete diagonalization via Bogoliubov transformation:

$$\begin{pmatrix} b_{\boldsymbol{k}} \\ b^\dagger_{-\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} u_{\boldsymbol{k}} & -v_{\boldsymbol{k}} \\ -v^*_{\boldsymbol{k}} & u_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{k}} \\ a^\dagger_{-\boldsymbol{k}} \end{pmatrix} \qquad u_{\boldsymbol{k}} = \sqrt{\frac{A_{\boldsymbol{k}} + \epsilon_{\boldsymbol{k}}}{2\epsilon_{\boldsymbol{k}}}} \qquad v_{\boldsymbol{k}} = \frac{B_{\boldsymbol{k}}}{|B_{\boldsymbol{k}}|}\sqrt{\frac{A_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}}}{2\epsilon_{\boldsymbol{k}}}}$$

→

$$\hat{H}_2 = \sum_{\boldsymbol{k}} \left[\epsilon_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{\epsilon_{\boldsymbol{k}} - A_{\boldsymbol{k}}}{2} \right] \qquad \epsilon_{\boldsymbol{k}} = \sqrt{A^2_{\boldsymbol{k}} - |B_{\boldsymbol{k}}|^2}$$

magnon-magnon interactions in Bogoliubov basis

- 3-magnon terms:

$$\hat{H}_3 = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,0} \Big[\frac{1}{2} \Gamma^{\bar{a}aa}_{1;23} a^\dagger_{-1} a_2 a_3 + \frac{1}{2} \Gamma^{\bar{a}\bar{a}a}_{12;3} a^\dagger_{-1} a^\dagger_{-2} a_3 + \frac{1}{3!} \Gamma^{aaa}_{123} a_1 a_2 a_3 + \frac{1}{3!} \Gamma^{\bar{a}\bar{a}\bar{a}}_{123} a^\dagger_{-1} a^\dagger_{-2} a^\dagger_{-3} \Big]$$

- 4-magnon terms:

$$\hat{H}_4 = \frac{1}{N} \sum_{\boldsymbol{k}_1 \ldots \boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\ldots+\boldsymbol{k}_4,0} \Big[\frac{1}{(2!)^2} \Gamma^{\bar{a}\bar{a}aa}_{12;34} a^\dagger_{-1} a^\dagger_{-2} a_3 a_4 + \frac{1}{3!} \Gamma^{\bar{a}aaa}_{1;234} a^\dagger_{-1} a_2 a_3 a_4 + \frac{1}{3!} \Gamma^{\bar{a}\bar{a}\bar{a}a}_{123;4} a^\dagger_{-1} a^\dagger_{-2} a^\dagger_{-3} a_4 + \frac{1}{4!} \Gamma^{aaaa}_{1234} a_1 a_2 a_3 a_4 + \frac{1}{4!} \Gamma^{\bar{a}\bar{a}\bar{a}\bar{a}}_{1234} a^\dagger_{-1} a^\dagger_{-2} a^\dagger_{-3} a^\dagger_{-4} \Big]$$

- all combinations allowed due to absence of U(1) symmetry

- explicit form of vertices: F. Sauli, PhD-thesis, 2010

BEC at finite momentum: analogy with liquid-solid transition

J. Hick, F. Sauli, A. Kreisel, PK, EPJB 2010 (in press)

- Landau function:

$$\mathcal{L}[\bar{\psi},\psi] = \frac{1}{2}\sum_{\boldsymbol{k}} \left[r_{\boldsymbol{k}} |\psi_{\boldsymbol{k}}| + \gamma_{\boldsymbol{k}} (\bar{\psi}_{\boldsymbol{k}} \bar{\psi}_{-\boldsymbol{k}} + \psi_{-\boldsymbol{k}} \psi_{\boldsymbol{k}}) \right] + \sum_{\boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,0} \frac{1}{2} \Gamma^{\bar{a}aa}_{1;23} \bar{\psi}_{-\boldsymbol{k}_1} \psi_{\boldsymbol{k}_2} \psi_{\boldsymbol{k}_3} + \ldots$$

$r_{\boldsymbol{k}} = 2(\epsilon_{\boldsymbol{k}} - \mu)$ has minimum at finite q

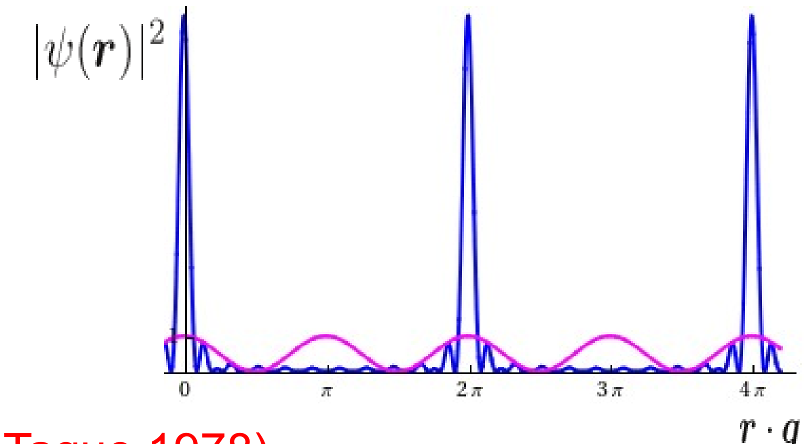
explicit breaking of U(1) symmetry

order parameter in real space:

- Gross-Pitaevskii- equation: $\dfrac{\delta \mathcal{L}[\bar{\psi},\psi]}{\delta \psi_{\boldsymbol{k}}} = 0$

solution has finite Fourier components for all integer multiples of q:

$$\psi_{\boldsymbol{k}} = \sum_{m=-\infty}^{\infty} \delta_{\boldsymbol{k},m\boldsymbol{q}} \psi_m$$



- analogy: liquid-solid transition: (Alexander, McTague 1978)

$$\mathcal{L}[\rho] = \frac{1}{2}\sum_{\boldsymbol{K}} r_{\boldsymbol{K}} |\rho_{\boldsymbol{K}}| - \Gamma_3 \sum_{\boldsymbol{K}_1 \boldsymbol{K}_2 \boldsymbol{K}_3} \delta_{\boldsymbol{K}_1+\boldsymbol{K}_2+\boldsymbol{K}_3,0} \rho_{\boldsymbol{K}_1} \rho_{\boldsymbol{K}_2} \rho_{\boldsymbol{K}_3} + \ldots$$

$r_{\boldsymbol{K}} = r_0 + c(\boldsymbol{K}^2 - k_0^2)$ has minimum on sphere in momentum space

Parallel pumping of magnons in YIG by microwaves

- what is parallel pumping of magnons?

Parallel pumping of magnons in YIG by microwaves

- what is parallel pumping of magnons?

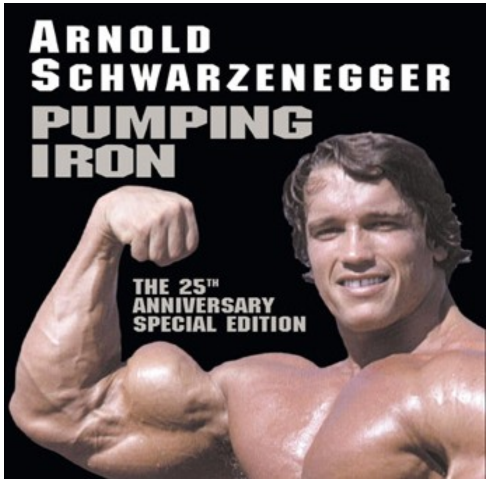
  two pumps in parallel:

Parallel pumping of magnons in YIG by microwaves

- what is parallel pumping of magnons?

two pumps in parallel:

one can also pump other things:



Parallel pumping of magnons in YIG by microwaves

• what is parallel pumping of magnons?

two pumps in parallel:

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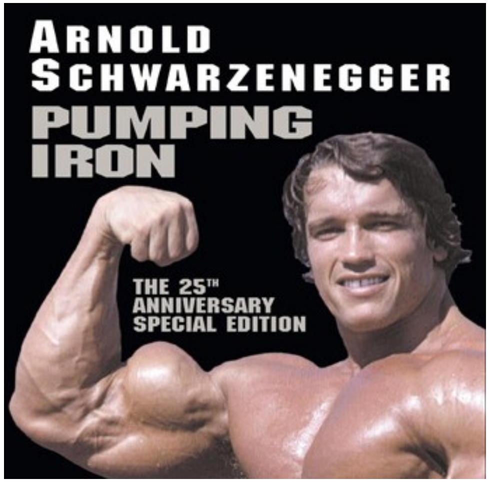


• in context of YIG: oscillating magnetic field is parallel to magnetization:

$$\hat{H}_{\mathrm{YIG}}(t) = -\frac{1}{2}\sum_{ij}\sum_{\alpha\beta}\left[J_{ij}\delta^{\alpha\beta} + D_{ij}^{\alpha\beta}\right] S_i^\alpha S_j^\beta - [h_0 + h_1\cos(\omega_0 t)]\sum_i S_i^z$$

➡ Hamiltonian in Bogoliubov basis: time-dependent off-diagonal terms:

$$\hat{H}_2(t) \approx \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k}}\left[\gamma_{\boldsymbol{k}} e^{-i\omega_0 t} a_{\boldsymbol{k}}^\dagger a_{-\boldsymbol{k}}^\dagger + \gamma_{\boldsymbol{k}}^* e^{i\omega_0 t} a_{-\boldsymbol{k}} a_{\boldsymbol{k}}\right]$$

Parametric resonance

- what is parametric resonance?
  - classical harmonic oscillator with harmonic frequency modulation:

$$\frac{d^2x(t)}{dt^2} + \Omega^2(t)x(t) = 0 \qquad \Omega(t) = \Omega_0 + \Omega_1 \cos(\omega_0 t)$$

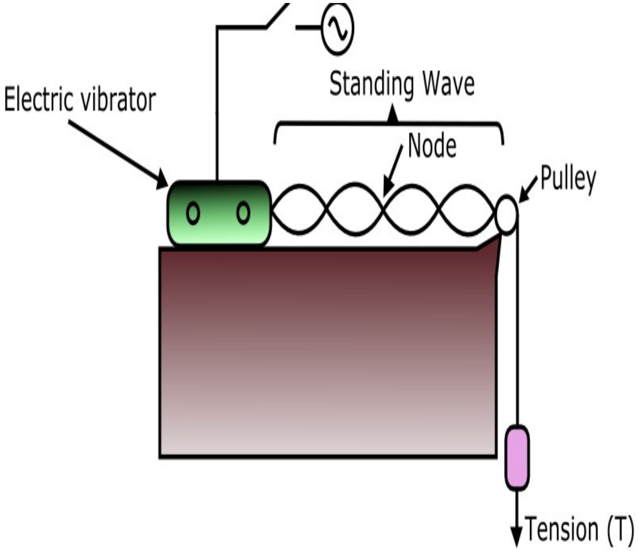
- resonance condition: $\omega_0 \approx 2\Omega_0$

→ oscillator absorbs energy at a rate proportional to the energy it already has!

- history:
  - discovered: Melde experiment, 1859

    excite oscillations of string by periodically varying its tension at twice its resonance frequency

  - theoretically explained: Rayleigh 1883



Parametric resonance of magnons in YIG

H. Suhl, 1957, E. Schlömann et al, 1960s,
V. E. Zakharov, V. S. L'vov, S. S. Starobionets, 1970s

- minimal model:

$$\hat{H}_{\mathrm{res}}(t) = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} a^{\dagger}_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k}} \left[\gamma_{\boldsymbol{k}} e^{-i\omega_0 t} a^{\dagger}_{\boldsymbol{k}} a^{\dagger}_{-\boldsymbol{k}} + \gamma^{*}_{\boldsymbol{k}} e^{i\omega_0 t} a_{-\boldsymbol{k}} a_{\boldsymbol{k}} \right] + \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} u(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}) a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}} a^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q}} a_{\boldsymbol{k}'} a_{\boldsymbol{k}}$$

- "S-theory": time-dependent self-consistent Hartree-Fock approximation for magnon distributions functions

$$n_{\boldsymbol{k}}(t) = \langle a^{\dagger}_{\boldsymbol{k}}(t) a_{\boldsymbol{k}}(t) \rangle$$

$$p_{\boldsymbol{k}}(t) = \langle a_{-\boldsymbol{k}}(t) a_{\boldsymbol{k}}(t) \rangle$$

weak points:
- no microscopic description of dissipation and damping
- possibility of BEC not included!

- goals:
  - consistent quantum kinetic theory for magnons in YIG beyond Hartree-Fock
  - include time-evolution of Bose-condendsate
  - develop functional renormalization group for non-equilibrium

toy model for parametric resonance

T. Kloss, A. Kreisel, PK, PRB 2010

- anharmonic oscillator with off-diagonal pumping:

$$\hat{H}(t) \;=\; \epsilon_0 a^\dagger a + \frac{\gamma_0}{2} e^{-i\omega_0 t} a^\dagger a^\dagger + \frac{\gamma_0^*}{2} e^{i\omega_0 t} aa + \frac{u}{2} a^\dagger a^\dagger aa$$

- rotating reference frame: $\tilde{a} \;=\; e^{\frac{i}{2}\omega_0 t} a \qquad \tilde{a}^\dagger \;=\; e^{-\frac{i}{2}\omega_0 t} a^\dagger$

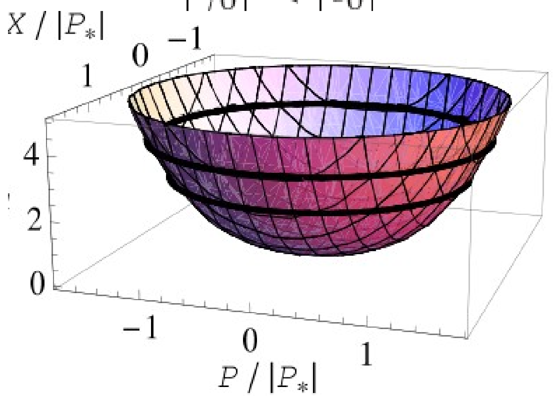
$$\tilde{H} \;=\; \tilde{\epsilon}_0 \tilde{a}^\dagger \tilde{a} + \frac{\gamma_0}{2} \tilde{a}^\dagger \tilde{a}^\dagger + \frac{\gamma_0^*}{2} \tilde{a}\tilde{a} + \frac{u}{2} \tilde{a}^\dagger \tilde{a}^\dagger \tilde{a}\tilde{a} \qquad \tilde{\epsilon}_0 = \epsilon_0 - \frac{\omega_0}{2}$$

- instability of non-interacting system for large pumping:

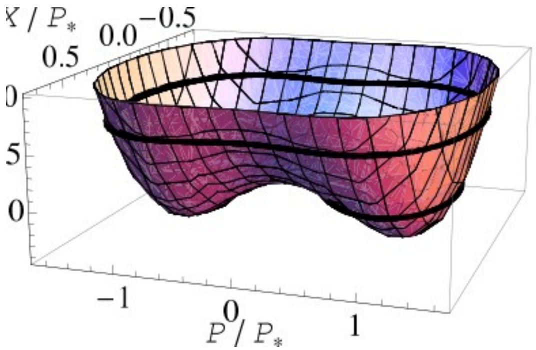
$$\tilde{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}} \qquad \tilde{a}^\dagger = \frac{\hat{X} - i\hat{P}}{\sqrt{2}} \qquad \tilde{\epsilon}_0 \tilde{a}^\dagger \tilde{a} + \frac{\gamma_0}{2} [\tilde{a}^\dagger \tilde{a}^\dagger + \tilde{a}\tilde{a}] = \frac{\tilde{\epsilon}_0 - \gamma_0}{2} \hat{P}^2 + \frac{\tilde{\epsilon}_0 + \gamma_0}{2} \hat{X}^2.$$

- order parameter:
Hamiltonian dynamics
in effective potential
(Hartree-Fock)

$|\gamma_0| < |\tilde{\epsilon}_0|$



$|\gamma_0| > |\tilde{\epsilon}_0|$



Towards a non-equilibrium many-body theory for YIG

T. Kloss, PK, cond-mat soon to appear...

- 6 types of non-equilibrium Green functions:
  - retarded: $g^R(t,t') = -i\Theta(t-t')\langle[a(t),a^\dagger(t')]\rangle$ $\quad p^R(t,t') = -i\Theta(t-t')\langle[a(t),a(t')]\rangle$
  - advanced: $g^A(t,t') = i\Theta(t'-t)\langle[a(t),a^\dagger(t')]\rangle$ $\quad p^A(t,t') = i\Theta(t'-t)\langle[a(t),a(t')]\rangle$
  - Keldysh: $g^K(t,t') = -i\langle\{a(t),a^\dagger(t')\}\rangle$ $\quad p^K(t,t') = -i\langle\{a(t),a(t')\}\rangle$

- Keldysh component at equal times gives distribution function

$$G^K(t,t) = \begin{pmatrix} p^K(t,t) & g^K(t,t) \\ g^K(t,t) & p^K(t,t)^* \end{pmatrix} = -2i \begin{pmatrix} p(t) & n(t)+\frac{1}{2} \\ n(t)+\frac{1}{2} & p^*(t) \end{pmatrix}$$

- functional integral formulation (Kamenev 2004)

$$iG^{\lambda\lambda'}_{\sigma\sigma'}(t,t') = \int \mathcal{D}[\Phi] e^{iS_0[\Phi]+iS_1[\Phi]} \Phi^\lambda_\sigma(t)\Phi^{\lambda'}_{\sigma'}(t')$$

$$S_0[\Phi] = \frac{1}{2}\int dt dt' \sum_{\sigma\sigma'}\sum_{\lambda\lambda'} \Phi^\lambda_\sigma(t)[\mathbf{G}_0^{-1}]^{\lambda\lambda'}_{\sigma\sigma'}(t,t')\Phi^{\lambda'}_{\sigma'}(t')$$

$$S_1[\Phi] = -\frac{u_0}{2}\int dt \sum_{\sigma=a,\bar{a}} \Phi^C_{\bar\sigma}(t)\Phi^Q_{\bar\sigma}(t)\Big[[\Phi^C_\sigma(t)]^2 + [\Phi^Q_\sigma(t)]^2\Big]$$

$\lambda,\lambda' \in \{C,Q\}$

$\sigma,\sigma' \in \{a,a^\dagger\}$

$G^R = G^{CQ}$

$G^A = G^{QC}$

$G^K = G^{CC}$

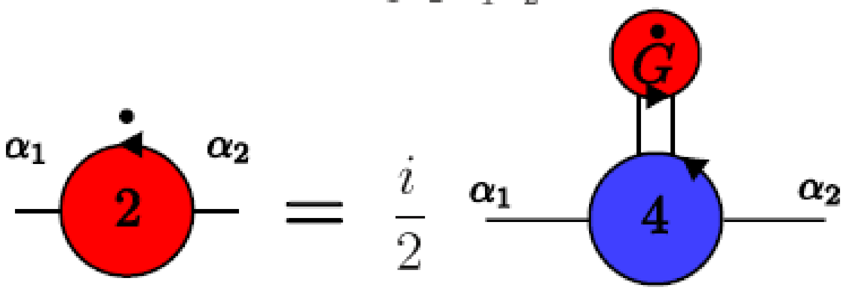
FRG for bosons out of equilibrium

- introduce RG flow parameter Λ which cuts off long-time behavior

$$\mathbf{G} \to \mathbf{G}_\Lambda \qquad \mathbf{\Sigma} \to \mathbf{\Sigma}_\Lambda \qquad \text{eventually} \qquad \Lambda \to 0$$

- exact RG flow equation for non-equilibrium self-energy:

$$\partial_\Lambda \Sigma^{\lambda\lambda'}_{\Lambda,\sigma\sigma'}(t,t') = \frac{i}{2} \sum_{\sigma_1\sigma_2} \sum_{\lambda_1\lambda_2} \int dt_1 dt_2 [\dot{\mathbf{G}}_\Lambda]^{\lambda_1\lambda_2}_{\sigma_1\sigma_2}(t_1,t_2) \Gamma^{(4),\lambda_2\lambda_1\lambda\lambda'}_{\Lambda,\sigma_2\sigma_1\sigma\sigma'}(t_2,t_1,t,t')$$



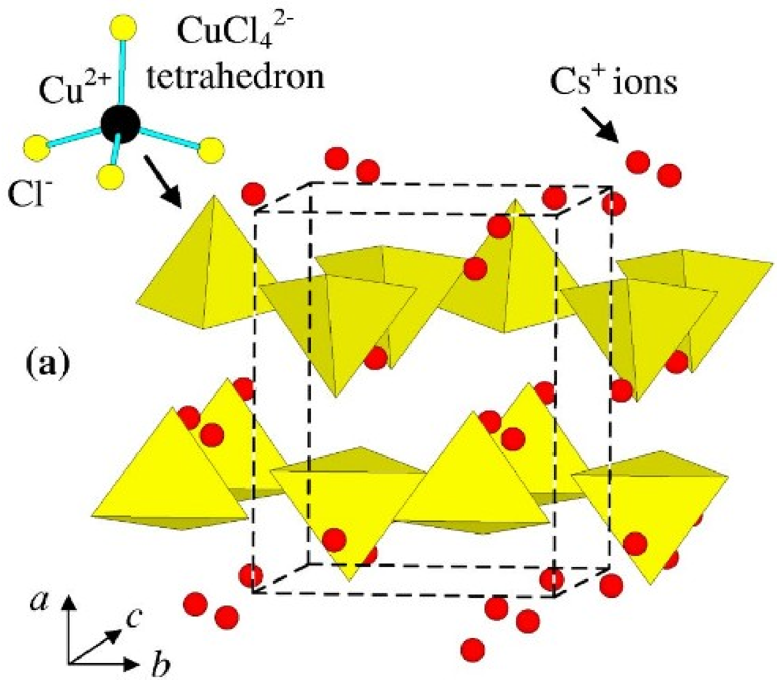
see also Gasenzer, Pawlowski 2008

- self-energy defines collision integral in quantum kinetic equation:

$$\begin{aligned} i\partial_t G^K_\Lambda(t,t) - M G^K_\Lambda(t,t) - G^K_\Lambda(t,t) M^T &= \int_{t_0}^{t} dt_1 [Z \Sigma^K_\Lambda(t,t_1) G^A_\Lambda(t_1,t) - G^R_\Lambda(t,t_1) \Sigma^K_\Lambda(t_1,t) Z] \\ &- \int_{t_0}^{t} dt_1 [G^K_\Lambda(t,t_1) \Sigma^A_\Lambda(t_1,t) Z - Z \Sigma^R_\Lambda(t,t_1) G^K_\Lambda(t_1,t)] \end{aligned}$$

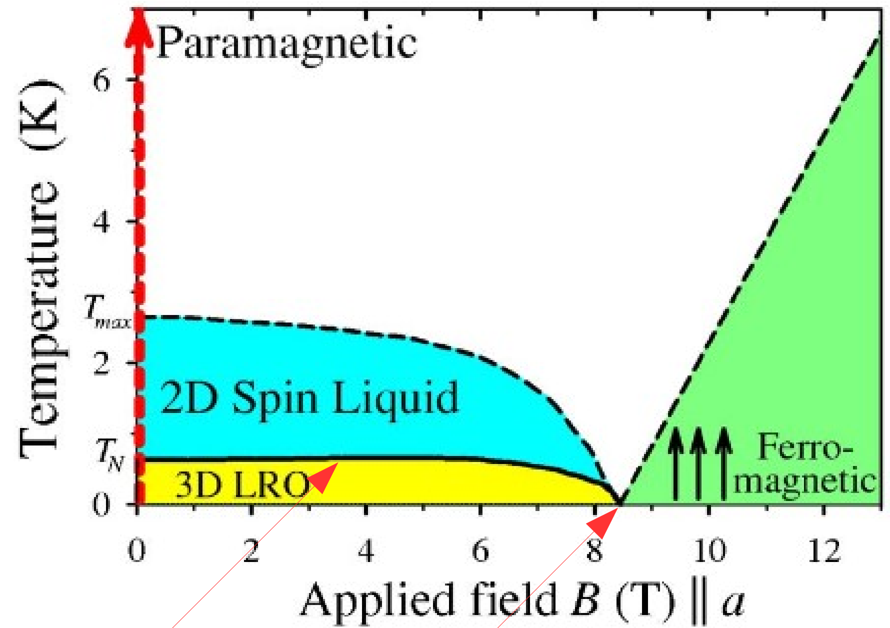
3. Magnon-phonon interactions in Cs_2CuCl_4

•quasi 2D QAFM on anisotropic triangular lattice



(figures from Coldea, Tennant, Tylczynski PRB 2003)

•phase diagram for magnetic field along a-direction:



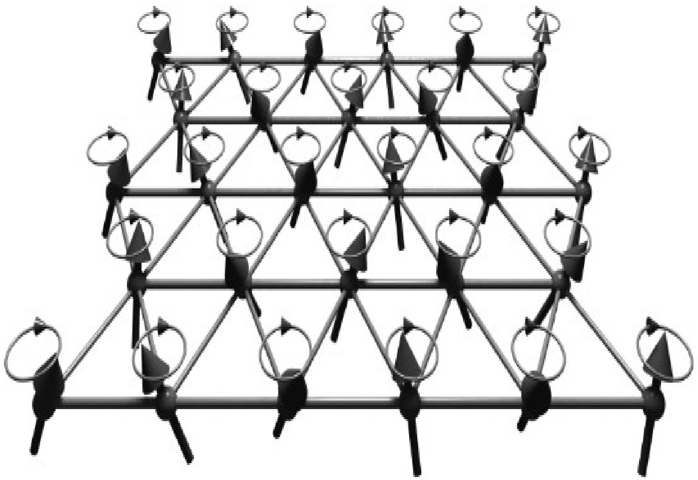
cone-state

quantum critical point:

change in magnetic order = BEC of magnons

challenges for theory

- spin excitations in cone-state
  - neutron scattering: extended scattering continua
  - spin-wave expansion: infrared divergencies!



- magnetic field dependence of
  elastic constants and ultrasound attenuation in cone state

- Spin excitations in spin-liquid phase

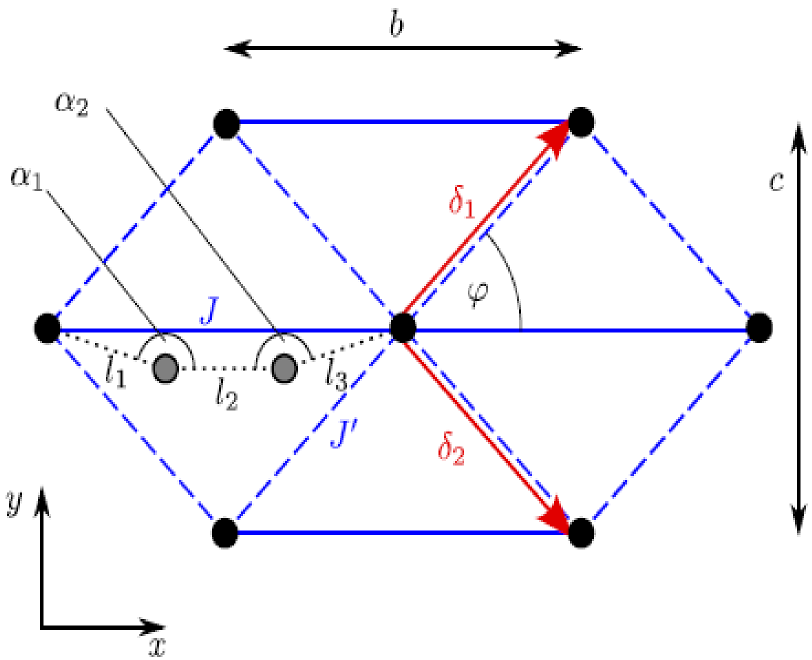
- critical behavior close to quantum critical point:
  can one see experimental signatures for breakdown of mean-field theory?
  (A. Kreisel, N. Hasselmann, PK, PRL 2007)

Magnon-phonon Hamiltonian for Cs_2CuCl_4

$$H = \frac{1}{2}\sum_{ij}\left[J_{ij}\boldsymbol{S}_i\cdot\boldsymbol{S}_j + \boldsymbol{D}_{ij}\cdot(\boldsymbol{S}_i\times\boldsymbol{S}_j)\right] - \sum_i \boldsymbol{h}\cdot\boldsymbol{S}_i$$

$$+ \sum_{\boldsymbol{k}\lambda}\omega_{\boldsymbol{k}\lambda}\left(a^\dagger_{\boldsymbol{k}\lambda}a_{\boldsymbol{k}\lambda} + \frac{1}{2}\right)$$

$$\boldsymbol{r}_i = \boldsymbol{R}_i + \boldsymbol{X}_i$$



$$\boldsymbol{X}_i = \frac{1}{\sqrt{N}}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{R}_i}\frac{\boldsymbol{e}_{\boldsymbol{k}\lambda}}{\sqrt{2M\omega_{\boldsymbol{k}\lambda}}}(a_{\boldsymbol{k}\lambda} + a^\dagger_{-\boldsymbol{k}\lambda}) \qquad \boldsymbol{X}_{ij} = \boldsymbol{X}_i - \boldsymbol{X}_j$$

$$J_{ij} = J(\boldsymbol{R}_{ij}) + (\boldsymbol{X}_{ij}\cdot\nabla_{\boldsymbol{r}})\, J(\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{R}_{ij}} + \frac{1}{2}(\boldsymbol{X}_{ij}\cdot\nabla_{\boldsymbol{r}})^2\, J(\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{R}_{ij}} + \ldots$$

$$H = H_{\rm spin} + H^{\rm pho} + H^{\rm 1pho}_{\rm spin} + H^{\rm 2pho}_{\rm spin} + \ldots$$

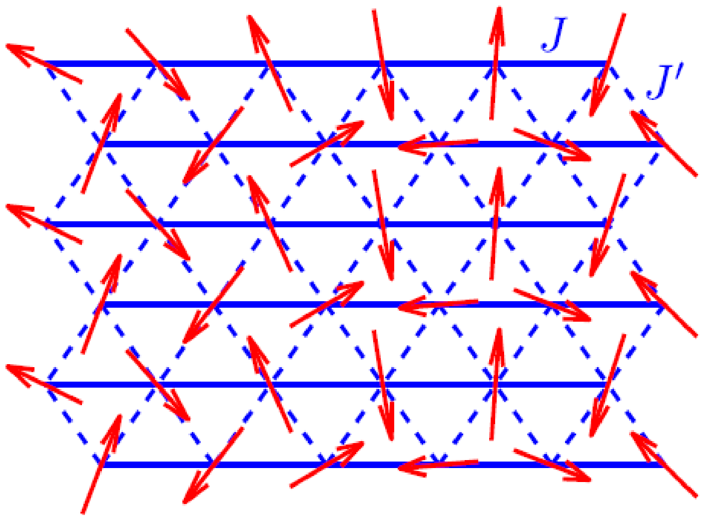
magnon dispersion in cone state

•classical ground state:

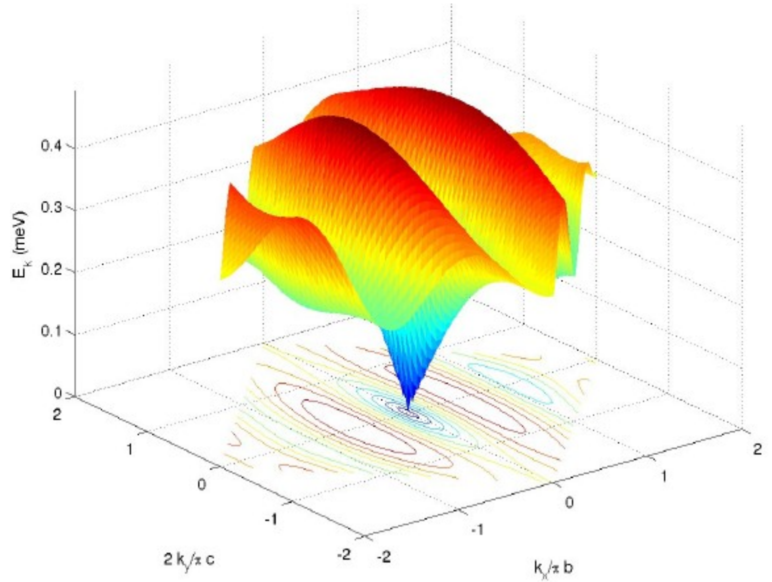
$$\hat{\boldsymbol{m}}_i = \cos\theta[\cos(\boldsymbol{Q}\cdot\boldsymbol{R}_i)\boldsymbol{e}_x + \sin(\boldsymbol{Q}\cdot\boldsymbol{R}_i)\boldsymbol{e}_y] + \sin\theta\,\boldsymbol{e}_z$$

$$J^D_{\boldsymbol{k}} = J_{\boldsymbol{k}} - iD_{\boldsymbol{k}} \qquad \nabla_{\boldsymbol{Q}} J^D_{\boldsymbol{Q}} = 0$$

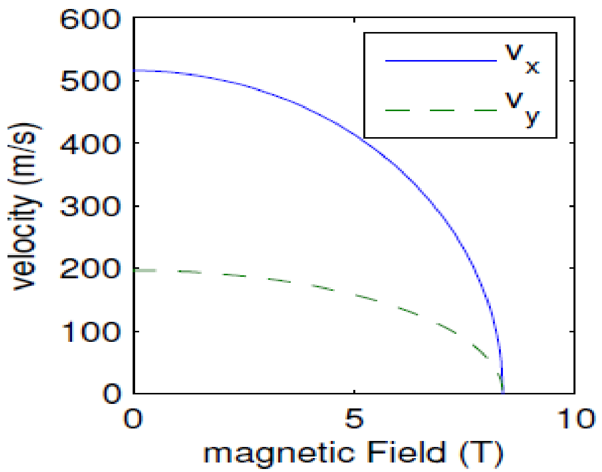
$$\sin\theta = h/h_c \qquad h_c = S(J^D_0 - J^D_{\boldsymbol{Q}})$$



•magnon dispersion:



•magnon velocities:



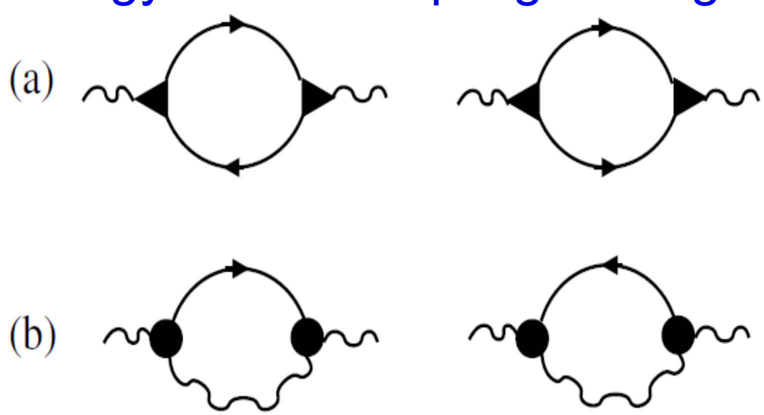
Calculation of elestic constants and ultrasound attenuation

A. Kreisel, PK, B. Wolf, M. Lang et al, in preparation

- magnetic field dependence of elastic constants:
  - calculate: renormalization of phonon velocities due to coupling to magnons $\frac{\Delta c_\lambda}{c_\lambda}$
  - dominant contribution from magnon-phonon hybridisation:

$$H_{1\mathrm{mag}}^{1\mathrm{pho}} = \sum_{\boldsymbol{k}} \boldsymbol{\Gamma}_{\boldsymbol{k}}^{Xb} \cdot \left(\boldsymbol{X}_{-\boldsymbol{k}} b_{\boldsymbol{k}} + \boldsymbol{X}_{\boldsymbol{k}} b_{\boldsymbol{k}}^{\dagger} \right)$$

- ultrasound attentuation rate:
  - calculate: imaginary part of phonon self-energy due to coupling to magnons
  - dominant contributions from 1 phonon+2magnon and 2 phonon+1magnon scattering (many-body version of Fermi's golden rule)

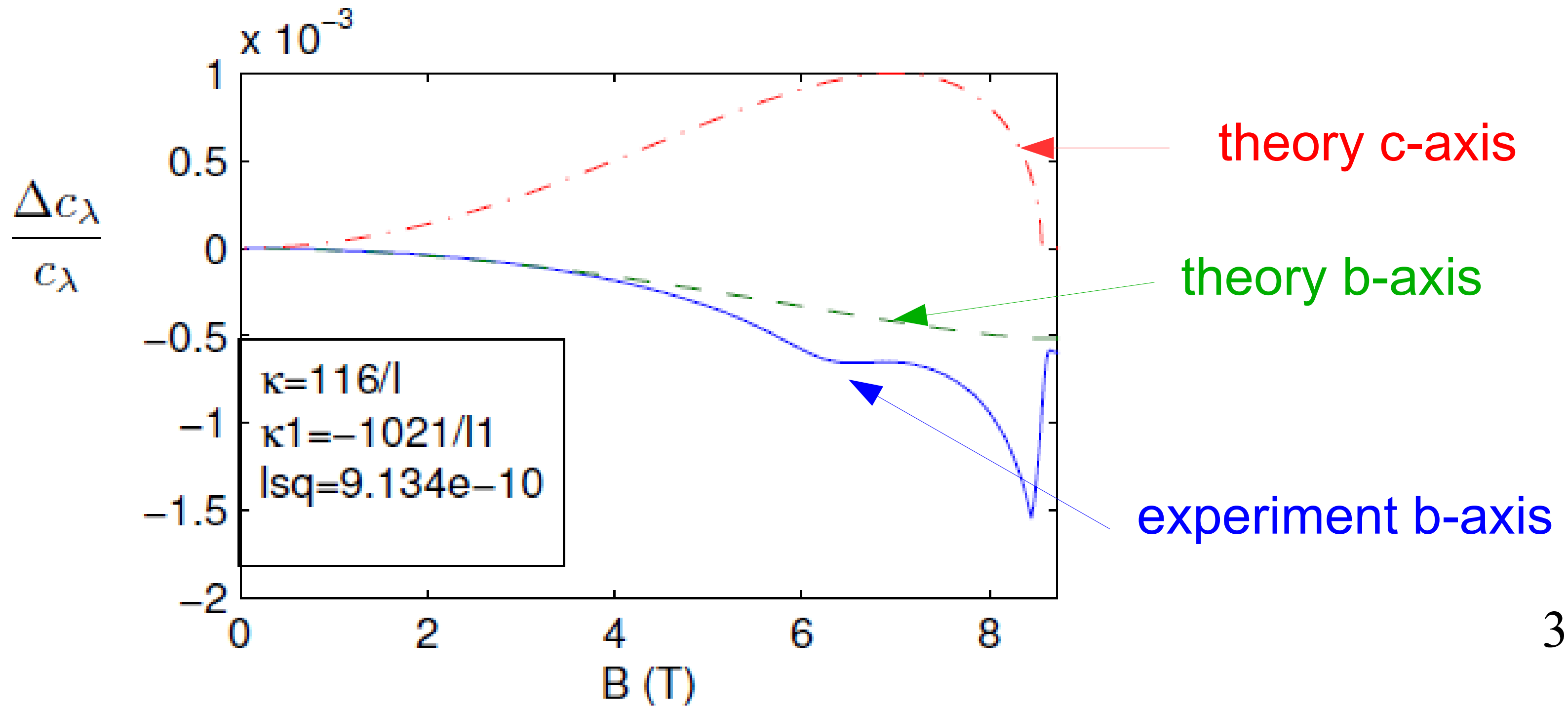


Result: magnetic field dependence of elastic constants

$$\frac{\Delta c_\lambda}{c_\lambda} = \frac{S^3}{4}\left(\frac{v(\hat{\boldsymbol{k}})}{c_\lambda}\right)\left(\frac{h_c}{M c_\lambda^2}\right)\left|\boldsymbol{F}^{X\beta}(\hat{\boldsymbol{k}})\cdot \boldsymbol{e}_{\boldsymbol{k}\lambda}\right|^2$$

$$\boldsymbol{F}^{X\beta}(\hat{\boldsymbol{k}}) = \frac{s_\theta}{h_c}(\hat{\boldsymbol{k}}\cdot\nabla_{\boldsymbol{Q}})\left[\boldsymbol{J}^{(1)}_{\boldsymbol{Q}}\Big|_{\boldsymbol{Q}=0} - \boldsymbol{J}^{(1)}_{\boldsymbol{Q}}\right] - \frac{c_\theta^2}{2v(\hat{\boldsymbol{k}})}(\hat{\boldsymbol{k}}\cdot\nabla_{\boldsymbol{Q}})^2\boldsymbol{J}^{(1)}_{\boldsymbol{Q}}$$

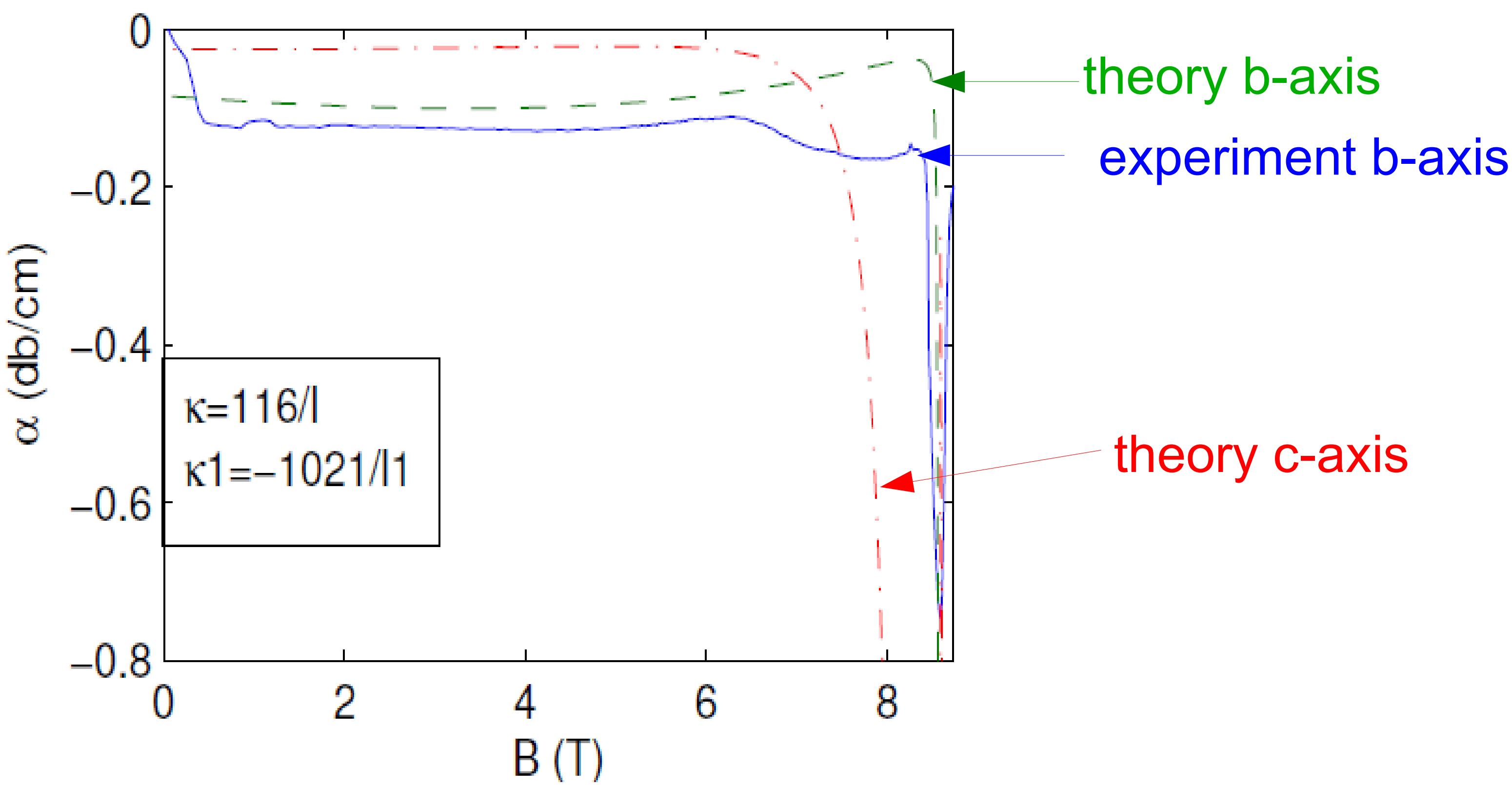
need: fitting parameter: $\boldsymbol{J}^{(1)}(\boldsymbol{R}_{ij}) = \nabla_{\boldsymbol{r}}\, J(\boldsymbol{r})|_{\boldsymbol{r}=\boldsymbol{R}_{ij}}$ from Valenti group



Results: sound attenuation rate

$$\gamma_{\boldsymbol{k}\lambda} = \frac{\pi^2}{64}\left(\frac{\boldsymbol{k}^2}{2M}\right)\left(\frac{S^2 c_\lambda^2 \boldsymbol{k}^2}{V_{\mathrm{BZ}} v_x v_y}\right)\frac{I_\lambda(\hat{\boldsymbol{k}})}{\sqrt{1-r_{\boldsymbol{k}\lambda}^2}}$$

$$\begin{aligned} I_\lambda(\hat{k}) = {} & \left(\frac{2s_\theta^2}{c_\theta^2}+1\right)^2\left(1-r_{k\lambda}^2+\frac{3}{8}r_{k\lambda}^4\right)\left[\frac{(\hat{\boldsymbol{k}}\cdot\nabla_Q)\left(\boldsymbol{J}_Q^{(1)}\big|_{Q=0}-\boldsymbol{J}_Q^{(1)}\right)\cdot\boldsymbol{e}_{k\lambda}}{h_c}\right]^2 \\ & + 2s_\theta\left(\frac{2s_\theta^2}{c_\theta^2}+1\right)\left(1-\frac{3}{4}r_{k\lambda}^2\right)\left[\frac{(\hat{\boldsymbol{k}}\cdot\nabla_Q)\left(\boldsymbol{J}_Q^{(1)}\big|_{Q=0}-\boldsymbol{J}_Q^{(1)}\right)\cdot\boldsymbol{e}_{k\lambda}}{h_c}\right]\left[\frac{(\hat{\boldsymbol{k}}\cdot\nabla_Q)^2\boldsymbol{J}_Q^{(1)}\cdot\boldsymbol{e}_{k\lambda}}{c_\lambda}\right] \\ & + \frac{s_\theta^2}{2}\left\{3\left[\frac{(\hat{\boldsymbol{k}}\cdot\nabla_Q)^2\boldsymbol{J}_Q^{(1)}\cdot\boldsymbol{e}_{k\lambda}}{c_\lambda}\right]^2+(1-r_{k\lambda}^2)\left[\frac{(\hat{\boldsymbol{k}}_\perp\cdot\nabla_Q)(\hat{\boldsymbol{k}}\cdot\nabla_Q)\boldsymbol{J}_Q^{(1)}\cdot\boldsymbol{e}_{k\lambda}}{c_\lambda}\right]^2\right\}. \end{aligned}$$



Summary+Outlook

- Theoretical description of BEC of interacting bosons requires non-perturbative methods
- Magnon gas in yttrium-iron garnet: parametric resonance of interacting bosons
- Frustrated antiferromagnet $\mathrm{Cs_2CuCl_4}$ elastic constants, ultrasound attenuation
- Outlook:
  - YIG: develop FRG to study bosons out of equilibrium
  - $\mathrm{Cs_2CuCl_4}$:spin excitations, quantum critical point

Acknowledgments

1.) Thanks all members of my group:

special thanks to A. Kreisel!

2.) Collaborations on FRG with
L. Bartosch, A. Isidori,
N. Hasselmann, A. Sinner,
S. Ledowski, A. Ferraz

3.) Financial support from
SFB/TRR49

FOR723
Functional Renormalization Group for Correlated Fermion Systems

DAAD

4.) Collaboration on YIG: with group of B. Hillebrands (Kaiserslautern)

A. Serga

5.) Collaboration on Cs_2CuCl_4 with group of M. Lang (Frankfurt)

Wolf

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