Bose-Einstein condensation of magnons and spin-wave interactions in quantum antiferromagnets

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1. Some not-so-well-known aspects of BEC
2. New formulation of spin-wave expansion for QAF
3. QAF in a uniform magnetic field: QEC of magnons
1. Some not-so-well known aspects of Bose-Einstein condensation

Hamiltonian of the interacting Bose gas:

\[ \hat{H} = \sum_k \frac{k^2}{2m} b_k^\dagger b_k + \frac{1}{2V} \sum_{q,k,k'} V_q b_{k+q}^\dagger b_{k'-q}^\dagger b_{k'} b_k \]

Bose-Einstein condensation (BEC):
expectation value of \( \hat{\phi} = b_0^\dagger b_0 / N \) is of order unity

Spontaneous symmetry breaking (SSB):
expectation value of \( \hat{\phi} = b_0 / \sqrt{N} \) is of order unity

Thermodynamic limit \( \langle \hat{\phi} \rangle = |\langle \hat{\phi} \rangle|^2 \) for \( N \to \infty \)

Finite systems: no SSB \( \langle \hat{\phi} \rangle = 0 \) but BEC possible!
a) non-interacting bosons: Landau functions


probability distributions of eigenvalues \( q \) and \( \phi \) of operators \( \hat{q} \) and \( \hat{\phi} \)

\[
P_{N}^{SSB}(\phi) = Z_{N}^{-1} e^{-N\mathcal{L}_{N}^{SSB}(\phi)}
\]

\[
P_{N}^{BEC}(q) = Z_{N}^{-1} e^{-N\mathcal{L}_{N}^{BEC}(q)}
\]

canonical partition function:

\[
Z_{N} = \text{Tr}_{N} \exp[-\beta \hat{H}]
= \int d^{2}\phi e^{-N\mathcal{L}_{N}^{SSB}(\phi)}
\]

FIG. 2: (Color online) Order parameter probability distributions (left column) and Landau functions (right column) of \( N = 10 \) bosons in a three-dimensional harmonic potential for different dimensionless densities \( \rho = (\lambda_{th}/L)^{3}N \). The plots in the second row correspond to the critical density \( \rho_{c} \approx 2.85 \) for \( N = 10 \). Discrete points denote \( P_{N}^{BEC} \) (plotted versus \( q = n/N \)) and \( \mathcal{L}_{N}^{BEC} \) (versus \( \sqrt{q} \)), while the continuous curves represent \( P_{N}^{SSB} \) (versus \( |\phi|^{2} \)) and \( \mathcal{L}_{N}^{SSB} \) (versus \( \text{Re}\phi \)).

Landau functions develop minima if relevant dimensionless density

\[
\rho = (\lambda_{th}/L)^{D}N \quad \text{is sufficiently large.}
\]
b) effect of interactions on the critical temperature for BEC


Critical temperature for BEC in the free Bose gas (D=3):

\[ T_c^0 = \frac{2\pi \hbar^2 n^{2/3}}{m [\zeta(3/2)]^{2/3}} \]

Question: Do interactions increase or decrease the critical temperature?

This is a non-perturbative problem! Need sophisticated field-theoretical methods, such as functional renormalization group flow equations:

\[ \Delta \frac{T_c}{T_0} = c_1 an^{1/3} + O[(an^{1/3})^2] \]

\[ a = \frac{m V_0}{4\pi} \quad \text{s-wave scattering length} \quad c_1 \approx 1.23 \]
c) interacting bosons at T=0: Bogoliubov theory

(N. Bogoliubov, 1947)

Euclidian action for interacting Bose gas (constant chemical potential):

\[
S[\bar{\psi}, \psi] = \int_K \left( -i \omega + \epsilon_k - \mu \right) \bar{\psi}_K \psi_K + \frac{1}{(2!)^2} \int_{K_1'} \int_{K_2'} \int_{K_2} \int_{K_1} \delta_{K_1' + K_2', K_2 + K_1} \times \Gamma_{\Lambda_0}^{(4)}(K_1', K_2'; K_2, K_1) \bar{\psi}_{K_1'} \bar{\psi}_{K_2'} \psi_{K_2} \psi_{K_1}
\]

symmetrized interaction:

\[
\Gamma_{\Lambda_0}^{(4)}(K_1', K_2'; K_2, K_1) = V(k_1' - k_1) + V(k_2' - k_1)
\]

Bogoliubov-shift:

\[
\psi_K = \psi_K^0 + \Delta \psi_K \\
\psi_K^0 = \langle \psi_K \rangle = \delta_{K,0} \sqrt{\rho_0}
\]

\[
S[\Delta \bar{\psi}, \Delta \psi; \psi^0] = \beta V \rho_0 \left[ \frac{\rho_0}{4} \Gamma_{\Lambda_0}^{(4)}(0, 0; 0, 0) - \mu \right] + (\Delta \psi_0 + \Delta \bar{\psi}_0) \sqrt{\rho_0} \left[ \frac{\rho_0}{2} \Gamma_{\Lambda_0}^{(4)}(0, 0; 0, 0) - \mu \right] \\
+ \int_K \left\{ [-i \omega + \epsilon_k - \mu + \rho_0 \Gamma_{\Lambda_0}^{(4)}(0, K; K, 0)] \Delta \bar{\psi}_K \Delta \psi_K + \frac{\rho_0}{4} \left[ \Gamma_{\Lambda_0}^{(4)}(0, 0; K, -K) \Delta \psi_K \Delta \psi_{-K} + \Gamma_{\Lambda_0}^{(4)}(K, -K; 0, 0) \Delta \bar{\psi}_K \Delta \bar{\psi}_{-K} \right] \right\}
\]

\[
+ \frac{\sqrt{\rho_0}}{2} \int_{K_1} \int_{K_2} \int_{K_3} \delta_{K_1 + K_2, K_3} \left[ \Gamma_{\Lambda_0}^{(4)}(K_1, K_2; K_3, 0) \Delta \bar{\psi}_{K_1} \Delta \bar{\psi}_{K_2} \Delta \psi_{K_3} + \Gamma_{\Lambda_0}^{(4)}(0, K_3; K_2, K_1) \Delta \bar{\psi}_{K_3} \Delta \psi_{K_2} \Delta \psi_{K_1} \right] \\
+ \frac{1}{(2!)^2} \int_{K_1'} \int_{K_2'} \int_{K_2} \int_{K_1} \delta_{K_1' + K_2', K_2 + K_1} \Gamma_{\Lambda_0}^{(4)}(K_1', K_2'; K_2, K_1) \Delta \bar{\psi}_{K_1'} \Delta \bar{\psi}_{K_2'} \Delta \psi_{K_2} \Delta \psi_{K_1}
\]
...Bogoliubov continued...

dispersion of single-particle excitations: \[ E_k = \sqrt{\epsilon_k^2 + 2\rho_0 U(k)\epsilon_k} \sim c_0 |k| \]

condensate density at zero temperature: \[ \rho_0 = \frac{\mu}{V(0)} \]

velocity of Goldstone mode: \[ c_0 = \sqrt{\frac{\rho_0 V(0)}{m}} \]

Correlation functions:

\[ \Delta \psi_K = \psi^l_K + i\psi^t_K \]

longitudinal transverse

\[ \langle \psi^t_K \psi^t_{-K} \rangle \propto \frac{c_0^2}{\omega^2 + c_0^2 k^2} \]

\[ \langle \psi^l_K \psi^l_{-K} \rangle \propto \frac{\omega}{\omega^2 + c_0^2 k^2} \]

\[ \langle \psi^l_K \psi^t_{-K} \rangle \propto \frac{k^2}{\omega^2 + c_0^2 k^2} \]

\[ \langle \psi^t_K \psi^l_{-K} \rangle \propto \frac{k^2}{\omega^2 + c_0^2 k^2} \]
(d) beyond Bogoliubov

- in dimensions $d \leq 3$ Bogoliubov fixed point is unstable towards a different fixed point characterized by the divergence of the longitudinal correlation function

$$\langle \psi^l_K \psi^l_{-K} \rangle \propto -Z_\parallel Z_\perp \frac{\omega^2}{\omega^2 + c^2 k^2} + K_{D+1} \frac{(mc)^3}{Z_\rho^3 \rho_0} \left\{ \ln \left[ \frac{(mc)^2}{\omega^2/c^2 + k^2} \right] \frac{2}{3-D} \left[ \frac{\omega^2}{c^2} + k^2 \right] \frac{D-3}{2} \right\}, \quad D = 3, \quad D < 3$$

- lots of divergencies in perturbation theory
- lots of cancellations controlled by Ward identities
- critical continuum in longitudinal structure factor
- observable? Yes: in BEC of magnons!
2. New formulation of the spin-wave expansion for antiferromagnets


(a) Warmup: Heisenberg ferromagnet

Hamiltonian

\[ H = \frac{1}{2} \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - h \sum_i S_i^z \]

\[ J_{ij} = J < 0 \]

for nearest neighbors

Map to Boson-problem via Holstein-Primakoff transformation:

\[ S_i^- = \sqrt{2S} b_i^\dagger [1 - n_i/(2S)]^{1/2} = (S_i^+)\dagger \]

\[ S_i^z = S - n_i \]

\[ n_i = b_i^\dagger b_i \]

\[ H \approx E_0 + H_2 + H_4 \]

\[ E_0 = N[\tilde{J}_0 S^2/2 - hS] \]

\[ H_2 = -\frac{S}{2} \sum_{ij} J_{ij} [n_i + n_j - b_i^\dagger b_j - b_j^\dagger b_i] + h \sum_i n_i \]

\[ H_4 = \frac{1}{4} \sum_{ij} J_{ij} [2n_i n_j - b_i^\dagger n_i b_j - b_j^\dagger n_j b_i] \]

\[ \tilde{J}_0 = \tilde{J}_{k=0} \]

\[ \tilde{J}_k = \sum_j e^{ik \cdot r_j} J_{ij} \]
Why not BEC of Holstein-Primakoff bosons in ferromagnets?

Fourier transform of two-body boson interaction:

\[ \hat{H}_4 = \frac{1}{2V} \sum_{k_1', k_2', k_2, k_1} \delta_{k_1' + k_2', k_2 + k_1} V(k_1', k_2', k_2, k_1) b_{k_1'}^\dagger b_{k_2'}^\dagger b_{k_2} b_{k_1} \]

Effective interaction vanishes at long wavelengths:

\[ V(k_1', k_2', k_2, k_1) \propto k_1' \cdot k_2' + k_1 \cdot k_2 \]

Ferromagnetic spin-waves (Goldstone bosons) interact too weak at long wavelengths!
(b) how about antiferromagnetic magnons?

Look at conventional spin-wave expansion for QAF:
(Anderson 1952, Kubo 1952, Oguchi 1960)

\[ \hat{H} = \frac{1}{2} \sum_{ij} J_{ij} S_i \cdot S_j \quad J_{ij} = J > 0 \]

3 Transformations:

1.) Holstein-Primakoff: mapping onto boson-problem

\[ \hat{H}_2 = S \sum_{ij} J_{ij} [b_i^\dagger b_i + b_j^\dagger b_j + b_i b_j + b_i^\dagger b_j^\dagger] \]

2.) Fourier transformation in sublattice basis:

on sublattice A:

\[ b_i = (2/N)^{1/2} \sum_k e^{i \mathbf{k} \cdot \mathbf{r}_i} A_k \]

on sublattice B:

\[ b_i = (2/N)^{1/2} \sum_k e^{i \mathbf{k} \cdot \mathbf{r}_i} B_k \]

3.) Bogoliubov transformation:

\[ \begin{pmatrix} A_k \\ B_{-\mathbf{k}}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}}^\dagger \\ \beta_{-\mathbf{k}}^\dagger \end{pmatrix} \]

\[ u_k = 2^{-1/2}[\epsilon_k^{-1} + 1]^{1/2} \]

\[ v_k = 2^{-1/2}[\epsilon_k^{-1} - 1]^{1/2} \]

\[ \epsilon_k = [1 - \gamma_k^2]^{1/2} \]

\[ \gamma_k = D^{-1} \sum_\mu \cos(k_\mu a) \]
...spin-wave theory for QAF...

quadratic part of boson hamiltonian is diagonal

\[
\hat{H}_2 = -NDJS + \hat{H}'_2,
\]

\[
\hat{H}'_2 = \sum_k E_k [\alpha^+_k \alpha_k + \beta^+_k \beta_k + 1]
\]

spin-wave interactions: very complicated!
(Harris, Kumar, Halperin, Hohenberg, PRB 1971, PK, PRB 1990)

\[
\hat{H}^{DM}_4 = \frac{\chi_0^{-1}}{4V} \sum_{k_1, \ldots, k_4} \left\{ V^{(1)}_{1234} (\beta_1^+ \beta_2^+ \beta_3 \beta_4 + \alpha_3^+ \alpha_4^+ \alpha_1 \alpha_2) - 2V^{(2)}_{1234} (\alpha_3^+ \beta_4 \alpha_1 \alpha_2 + \alpha_4^+ \beta_1^+ \beta_2^+ \beta_3) \\
+ 2V^{(3)}_{1234} (\alpha_3^+ \alpha_4^+ \alpha_2 \beta_1^+ + \alpha_1 \beta_2^+ \beta_3 \beta_4) - 2V^{(4)}_{1234} (\alpha_3^+ \alpha_1 \beta_2^+ \beta_4 + \alpha_4^+ \alpha_2 \beta_1^+ \beta_3) \\
+ V^{(5)}_{1234} (\alpha_3^+ \alpha_4^+ \beta_1^+ \beta_2^+ + \alpha_1 \alpha_2 \beta_3 \beta_4) \right\} \delta_{k_1 + k_2, k_3 + k_4}
\]

interaction vertices are infrared-singular in some limits:

\[
V^{(j)}_{1234} \sim \frac{1}{2} \sqrt{\frac{|k_1||k_2|}{|k_3||k_4|}} \left( 1 + \xi_j \frac{k_1 \cdot k_2}{|k_1||k_2|} \right)
\]

\[\xi_1 = \xi_2 = \xi_5 = 1, \quad \xi_3 = \xi_4 = -1\]

miracle: all divergencies cancel in physical quantities!
effective action for antiferromagnetic spin fluctuations

alternative description of antiferromagnetic spin-fluctuations: non-linear sigma model
(Chakravarty, Halperin, Nelson, PRL and PRB 1988)

\[ S_{\text{NLSM}}[\Omega] = \frac{\rho_0}{2} \int_0^\beta d\tau \int d^D r \left[ (\partial_\mu \Omega)^2 + c_0^{-2} (\partial_\tau \Omega)^2 \right] \]

\[ \Omega(r, \tau) \] represents slowly fluctuating part of staggered magnetization

- interactions between AF spin fluctuations involve derivatives and therefore vanish at long wavelengths
- what is relation between NLsM and conventional spin-wave theory?
c) hermitian operator approach
(Anderson 1952, N. Hasselmann and PK, 2006)

• introduce symmetric and antisymmetric combinations of Bogoliubov operators:

\[ \hat{\Psi}_{k \pm} = \frac{1}{\sqrt{2}} (\alpha_k \pm \beta_k) \]

• express each of them in terms of two canonically conjugate hermitian operators:

\[
\hat{\Psi}_{k+} = -i \sqrt{\frac{\chi_0}{2VE_k}} [E_k \hat{\Pi}_{k+} + i\chi_0^{-1} \hat{\Phi}_{k+}] \]
\[ \chi_0 = \frac{\rho_0}{c_0^2} \]

\[
\hat{\Psi}_{k-} = \sqrt{\frac{\chi_0}{2VE_k}} [E_k \hat{\Pi}_{k-} + i\chi_0^{-1} \hat{\Phi}_{k-}] \]

\[
[\hat{\Pi}_{k\sigma}, \hat{\Phi}_{k'\sigma'}] = iV \delta_{k,-k'} \delta_{\sigma,\sigma'} \]

• quadratic part of spin-wave hamiltonian:

\[
\hat{H}_2 = \frac{1}{V} \sum_k \left[ \frac{\hat{\Phi}_{-k} \cdot \hat{\Phi}_k}{2\chi_0} + \frac{\chi_0 E_k^2}{2} \hat{\Pi}_{-k} \cdot \hat{\Pi}_k \right] \]
\[ \hat{\Phi} = (\Phi_+, \Phi_-) \]
\[ \hat{\Pi} = (\Pi_+, \Pi_-) \]
advantages of hermitian operator approach

• simple relation to staggered and uniform components of spin

\[
S_k = \sqrt{\frac{1}{N}} \sum_i e^{-i k \cdot r_i} S_i
\]

\[
S_{st,k} = \sqrt{\frac{1}{N}} \sum_i e^{-i [k + Q_{af}] \cdot r_i} S_i
\]

\[
\begin{align*}
S^x_{st,k} & \approx \frac{1}{\sqrt{N}} (S/a^D) \\
S^y_{st,k} & \approx -\frac{1}{\sqrt{N}} \\
S^z_{st,k} & \approx \hat{\Phi}_k / \sqrt{N}
\end{align*}
\]

• precise relation between non-linear sigma-model and SWT established:

\[
\Omega_z = \sqrt{1 - \Pi^2} \approx 1 - \frac{1}{2} \Pi^2 + \ldots
\]

• allows to derive new effective action of staggered fluctuations alone

• very convenient to understand spin-waves in finite systems

• suppression of effective interaction between AF magnons at long wavelengths becomes manifest

• for AF in uniform field: relation to divergencies in the interacting Bose gas!

(A. Kreisel, N. Hasselmann, PK, PRL 2007)
zero modes in finite quantum antiferromagnets (Anderson, 1952)

- Contribution of \( k = 0 \) mode to quadratic spin-wave Hamiltonian:

  \[
  \hat{H}_2^0 = \frac{P^2}{2m}
  \]

  where

  \[
  P = (SN)^{-1/2} \Phi_0 \\
  X = (S/N)^{1/2} a^{-D} \Pi_0 \\
  m = 4DJS
  \]

- Groundstate has \( \langle P^2 \rangle = 0 \).

- Uncertainly principle: \( \langle X^2 \rangle = \infty \)

- Staggered magnetization diverges

  \[
  M_{st} = N(S + 1/2) - \langle P^2 \rangle - \langle X^2 \rangle - \frac{1}{2V} \sum_{k \neq 0} \langle f_k \Phi_{-k} \cdot \Phi_k + f_k^{-1} \Pi_{-k} \cdot \Pi_k \rangle
  \]

  spin-wave approach appears inconsistent!?

- Solution: for Gaussian wavepackt with \( \langle P^2 \rangle = \langle X^2 \rangle = 1 \) spin wave approach remains consistent for macroscopic times:

  \[
  \langle X^2 \rangle_t = \langle X^2 \rangle_0 + \frac{t^2}{m^2} \langle P^2 \rangle_0
  \]

  \[
  t \approx \sqrt{N m} = \sqrt{N/(4DJS)}
  \]
spin-wave interactions in quantum antiferromagnets

\[
\hat{H}_4 = -\frac{1}{8} \sum_{i,j} J_{ij} \left[ 4n_i n_j + n_ib_i b_j + b_i n_j b_j + b_i^\dagger b_j^\dagger n_j + b_i^\dagger n_i b_j^\dagger \right]
\]

\[
\hat{H}_4 = E_4 + \frac{\hat{H}_2'}{2S} + \frac{\rho_0}{2V^3} \frac{D}{a^2} \sum_{k_1, \ldots, k_4} \delta_{k_1+k_2+k_3+k_4,0} \left[ \Gamma_{\Pi}^{\Pi} (k_1, k_2, k_3, k_4) \prod_{k_1} \cdot \prod_{k_2} \prod_{k_3} \cdot \prod_{k_4} \right.
\]

\[
+ f_0^2 \Gamma_{\parallel}^{\Phi \Pi} (k_1, k_2, k_3, k_4) \sum_{\sigma}^{1/2} \left\{ \Phi_{k_1 \sigma}^\dagger \Phi_{k_2 \sigma}, \prod_{k_3 \sigma} \prod_{k_4 \sigma} \right\}
\]

\[
+ f_0^2 \Gamma_{\perp}^{\Phi \Pi} (k_1, k_2, k_3, k_4) \sum_{\sigma} \left\{ \Phi_{k_1 \sigma}^\dagger \Phi_{k_2 \sigma} \Pi_{k_3, -\sigma} \Pi_{k_4, -\sigma} \right\}
\]

\[
+ f_0^2 \Gamma_{u}^{\Phi \Phi} (k_1, k_2, k_3, k_4) \left\{ \Phi_{k_1 +}^\dagger, \prod_{k_2 +} \right\} \left\{ \prod_{k_3 -}, \Phi_{k_4 -} \right\}
\]

\[
f_0 = a^D / S
\]

important: effective interaction between staggered spin fluctuations vanishes at long wavelengths (see ferromagnet!):

\[
\Gamma_{\Pi}^{\Pi} (k_1, k_2, k_3, k_4) = (a^2/16D)[k_1^2 + k_2^2 + k_3^2 + k_4^2 + 4k_1 \cdot k_2 + 4k_3 \cdot k_4] + \mathcal{O}(k_i^4)
\]

\[
\Gamma_{\parallel}^{\Phi \Pi} (0) = \Gamma_{\perp}^{\Phi \Pi} (0) = \Gamma_{\Phi \Phi} (0) = -1 \quad \Gamma_{\perp}^{\Phi \Pi} (0) = 1
\]
3. BEC of magnons: realized in QAFs in a uniform magnetic field

(Matsubara+Matsuda 1956, Batyev+Braginsky 1984,...)  

\[ H = \frac{1}{2} \sum_{ij} J_{ij} S_i \cdot S_j - \hbar \sum_i S_i^z \]

classical ground state:

experiment: Cs$_2$CuCl$_4$  
(Radu et al, PRL 2005)

FIG. 1: (Color online) Spin configuration in the ground state of a QAF in a uniform magnetic field for \( h < h_c \). The local moments \( \langle S_i \rangle \) are represented by dashed arrows. We choose the coordinate system such that the magnetic field \( h \) points in the z-direction and the staggered magnetization \( M_s \) points in the \( y \)-direction. The sublattices are labelled A and B.
(a) mapping onto interacting boson problem

for large external magnetic field: ground state is saturated ferromagnet:

\[
\begin{align*}
\mathbf{h} & \quad \text{spin-wave expansion usual for quantum ferromagnets:} \\
\end{align*}
\]

- Holstein-Primakoff:
  \[
  H \approx E_0 + H_2 + H_4 \quad \text{with } \quad E_0 = N\frac{\tilde{J}_0 S^2}{2} - hS \\
  H_2 = -\frac{S}{2} \sum_{ij} J_{ij} [n_i + n_j - b_i^\dagger b_j - b_j^\dagger b_i] + h \sum_i n_i \\
  H_4 = \frac{1}{4} \sum_{ij} J_{ij} [2n_i n_j - b_i^\dagger n_i b_j - b_j^\dagger n_i b_i]
  \]

- Fourier trafo in sublattice basis
  \[
  b_{k\sigma} = N^{-1/2} \left[ \sum_{i \in A} e^{-ik \cdot r_i} b_i + \sigma \sum_{i \in B} e^{-ik \cdot r_i} b_i \right]
  \]

  \[
  H_2 = \sum_{k\sigma} (\epsilon_{k\sigma} - \mu) b_{k\sigma}^\dagger b_{k\sigma} \
  \quad \text{with } \quad \mu = h_c - h \quad \text{and } \quad h_c = 2\tilde{J}_0 S
  \]

- Two magnon branches in reduced BZ:
  \[
  \epsilon_{k\sigma} = (\tilde{J}_0 + \sigma\tilde{J}_k) S \quad \sigma = \pm
  \]

  gapless: \( \epsilon_{k,-} \approx \frac{k^2}{2m} \) \quad \text{gapped: } \epsilon_{k=0,+} = h_c
(b) BEC of magnons and hermitian field parametrization

- retain only gapless mode and perform continuum limit: 
  \[ \hat{\psi}_k = V^{1/2} b_k, \]

spin-Hamiltonian is mapped on interacting Bose gas:

\[
H = E_0 + \int_k \left( \frac{k^2}{2m} - \mu \right) \hat{\psi}_k^\dagger \hat{\psi}_k + \frac{1}{2} \int_q \int_k \int_{k'} U_q \hat{\psi}_{k+q}^\dagger \hat{\psi}_{k'}^\dagger \hat{\psi}_k \hat{\psi}_{k'}
\]

two body interaction has finite long-wavelength limit:

\[
U_q = \Theta (\Lambda_0 - |q|) \chi_0^{-1}, \quad \chi_0 = (2J_0 a^D)^{-1}
\]

- BEC of magnons: for \( \mu > 0 \) (i.e. \( h < h_c \)) field acquires finite expectation value:
  \[
  \hat{\psi}_k = (2\pi)^D \delta(k) \psi_0 + \Delta \hat{\psi}_k
  \]

Bogoliubov approximation:

\[
|\psi_0|^2 U_0 = \mu, \quad \rho_0 = |\psi_0|^2
\]

\[
\epsilon_k = c_0 |k|, \quad c_0 = \sqrt{\mu/m} = 2\sqrt{DJSa^D}
\]

- Physical meaning: condensation density corresponds to staggered magnetization:
  \[
  M_s = V^{-1} \sum_i \zeta_i \langle S_{i}^y \rangle
  \]
  \[
  \theta^2 \approx M_s^2/s^2 \approx 2\rho_0/s = 2(1 - h/h_c)
  \]
  \[ s = \tilde{S}/a^D \]
hermitian field parameterization and correlation functions

What are spin-correlation functions in condensed phase?
Idea: Use non-perturbative results by Castellani and co-workers!

- hermitian operator parametrization

\[ \hat{\psi}_k = \rho_0^{1/2} \hat{\Pi}_k + i(4\rho_0)^{-1/2} \hat{\Phi}_k \]
\[ [\hat{\Pi}_k, \hat{\Phi}_{k'}] = i(2\pi)^D \delta(k + k') \]

- physical meaning in underlying spin-problem:
  components of staggered magnetization

\[ \hat{\Pi}_k \approx \frac{1}{M_s} \sum_i \zeta_i e^{-ik \cdot r_i} S_i^x \]
\[ \hat{\Phi}_k \approx \frac{M_s}{s} \sum_i \zeta_i e^{-ik \cdot r_i} S_i^y \]

- correlation functions in linear spin-wave theory:

\[ \langle \Pi_K \Pi_{K'} \rangle_0 = \delta_{K,-K'} \frac{\chi_0^{-1}}{\omega^2 + c_0^2 k^2} \]
\[ \langle \Pi_K \Phi_{K'} \rangle_0 = \delta_{K,-K'} \frac{\omega}{\omega^2 + c_0^2 k^2} \]
\[ \langle \Phi_K \Phi_{K'} \rangle_0 = \delta_{K,-K'} \frac{\chi_0 c_0^2 k^2}{\omega^2 + c_0^2 k^2} \]

qualitatively wrong!
c) critical continuum in longitudinal structure factor

True asymptotics of correlation functions:

\[ \langle \Pi_K \Pi_{K'} \rangle = \delta_{K,-K'} \frac{\chi^{-1}}{\omega^2 + c^2 k^2} \]

\[ \langle \Pi_K \Phi_{K'} \rangle = \delta_{K,-K'} \frac{Z_\| \omega}{\omega^2 + c^2 k^2} \]

\[ \langle \Phi_K \Phi_{K'} \rangle = \delta_{K,-K'} \chi \left[ -Z_\| \frac{\omega^2}{\omega^2 + c^2 k^2} + K_{D+1} \frac{(mc)^3}{Z^3 \rho_0} \left\{ \ln \left[ \frac{(mc)^2}{\omega^2/c^2 + k^2} \right] \frac{2}{3-D} \frac{(mc)^2}{\omega^2/c^2 + k^2} \right\}^{\frac{D-3}{2}}, \quad D = 3 \right] \]

\[ \quad \text{critical continuum in longitudinal structure factor:} \]

\[ S_\| (k, \omega) = \frac{\chi s^2}{M_s^2} \left[ \frac{Z_\|^2}{2} c |k| \delta(\omega - c|k|) + C_D \frac{(mc)^3}{Z^3 \rho_0} \frac{\Theta(\omega - c|k|)}{(\omega^2/c^2 - k^2)^{\frac{D-3}{2}}} \right] \]

\[ \quad 1 < D \leq 3 \]

\[ 0 < \omega/c \leq k_G \]

Ginzburg scale:

\[ k_G \approx mc \left[ (mc)^D / \rho_0 \right]^{\frac{1}{3-D}} \quad D < 3 \]

\[ k_G \approx mc \exp \left[ -\rho_0 / (mc)^3 \right] \quad D = 3 \]

\[ \frac{I_c}{I_\delta} \propto \left( k_G / |k| \right) \ln(k_G / |k|) \]

\[ \frac{I_c}{I_\delta} \propto \left( k_G / |k| \right)(mc)^3 / \rho_0 \]
Summary: Interacting bosons and magnons

- Some not-so-well known facts about the Bose gas
  - BEC is not the same as spontaneous symmetry breaking
  - Landau functions for free Bose gas
  - Shift in the critical temperature due to interactions
  - Corrections to Bogoliubov approximation large in D <= 3

- Spin-wave interactions in quantum antiferromagnets
  - Usual spin-wave expansion leads to rather nasty interaction vertices
  - Hermitian-field parameterization has many advantages:
    - Weak interactions between Goldstone modes manifest
    - Relation between SWT and non-linear sigma model precise
    - Zero modes and spin-waves infinite systems

- QAF in a uniform magnetic field: BEC of magnons