Bose-Einstein condensation of magnons and spin-wave interactions in quantum antiferromagnets

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S. Ledowski, N. Hasselmann, PK, Phys Rev A 69, 061601(R) (2004)
A. Sinner, F. Schütz, PK, Phys Rev A 74, 023608 (2006)
N. Hasselmann and PK, Europhys. Lett. 74, 1067 (2006)
A. Kreisel, N. Hasselmann, and PK, Phys. Rev. Lett. 98, 067203 (2007)

1. Some not-so-well-known aspects of BEC
2. New formulation of spin-wave expansion for QAF
3. QAF in a uniform magnetic field: QEC of magnons

1.Some not-so-well known aspects of Bose-Einstein condensation

Hamiltonian of the interacting Bose gas:

$$\hat{H} = \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}^2}{2m} b_{\boldsymbol{k}}^\dagger b_{\boldsymbol{k}} + \frac{1}{2\mathcal{V}} \sum_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{q}} b_{\boldsymbol{k}+\boldsymbol{q}}^\dagger b_{\boldsymbol{k}'-\boldsymbol{q}}^\dagger b_{\boldsymbol{k}'} b_{\boldsymbol{k}}$$

Bose-Einstein condensation (BEC):

expectation value of $\hat{q} = b_0^\dagger b_0 / N$ is of order unity

Spontaneous symmetry breaking (SSB):

expectation value of $\hat{\phi} = b_0 / \sqrt{N}$ is of order unity

Thermodynamic limit $\langle \hat{q} \rangle = |\langle \hat{\phi} \rangle|^2$ for $N \to \infty$

Finite systems: no SSB $\langle \hat{\phi} \rangle = 0$ but BEC possible!

a) non-interacting bosons: Landau functions

A. Sinner, F. Schütz, PK, Phys Rev A 74, 023608 (2006)

probability distributions of eigenvalues q and ϕ of operators $\hat{q}$ and $\hat{\phi}$

$$P_N^{\mathrm{SSB}}(\phi) = Z_N^{-1} e^{-N\mathcal{L}_N^{\mathrm{SSB}}(\phi)}$$

$$P_N^{\mathrm{BEC}}(q) = Z_N^{-1} e^{-N\mathcal{L}_N^{\mathrm{BEC}}(q)}$$

canonical partition function:

$$Z_N = \mathrm{Tr}_N \exp[-\beta \hat{H}] = \int d^2\phi e^{-N\mathcal{L}_N^{\mathrm{SSB}}(\phi)}$$

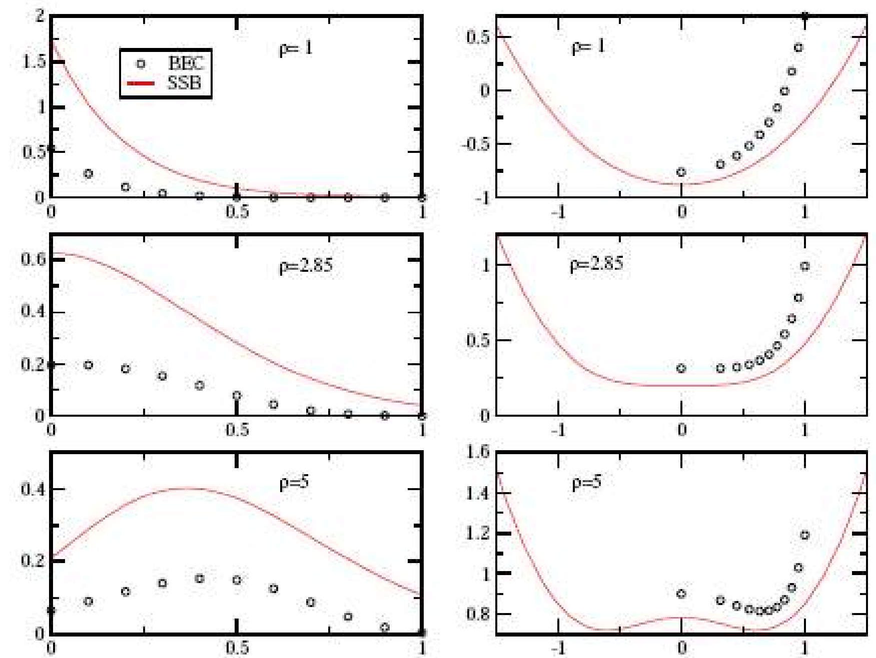


FIG. 2: (Color online) Order parameter probability distributions (left column) and Landau functions (right column) of $N = 10$ bosons in a three-dimensional harmonic potential for different dimensionless densities $\rho = (\lambda_{\mathrm{th}}/L)^3 N$. The plots in the second row correspond to the critical density $\rho_c \approx 2.85$ for $N = 10$. Discrete points denote P_N^{BEC} (plotted versus $q = n/N$) and $\mathcal{L}_N^{\mathrm{BEC}}$ (versus $\sqrt{q}$), while the continuous curves represent P_N^{SSB} (versus $|\phi|^2$) and $\mathcal{L}_N^{\mathrm{SSB}}$ (versus $\mathrm{Re}\phi$).

Landau functions develop minima if relevant dimensioless density $\rho = (\lambda_{\mathrm{th}}/L)^D N$ is sufficiently large.

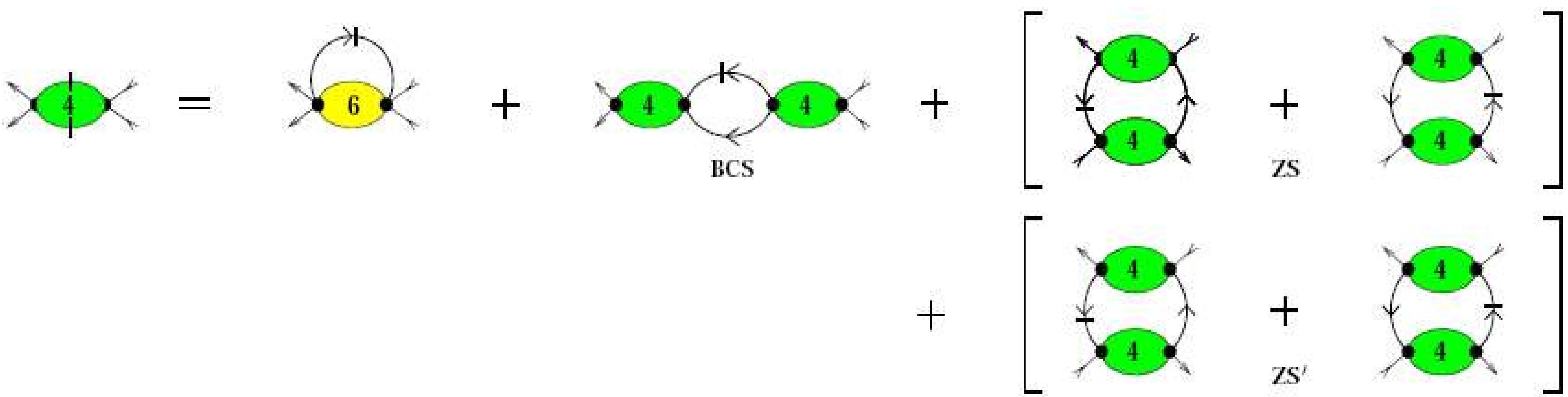
b) effect of interactions on the critical temperature for BEC

G. Baym et al, PRL 1999,
functional RG: S. Ledowski, N. Hasselmann, PK, PRA 2004

Critical temperature for BEC in the free Bose gas (D=3): $T_c^0 = \frac{2\pi\hbar^2 n^{2/3}}{m[\zeta(3/2)]^{2/3}}$

Question: Do interactions increase or decrease the critical temperature?

This is a non-perturbative problem! Need sophisticated field-theoretical methods, such as functional renormalization group flow equations:



Answer: $\frac{\Delta T_c}{T^0} = c_1 \; a n^{1/3} + O[(a n^{1/3})^2]$

$a = \frac{mV_0}{4\pi}$ s-wave scattering length $c_1 \approx 1.23$

c) interacting bosons at T=0: Bogoliubov theory

(N. Bogoliubov, 1947)

Euclidian action for interacting Bose gas (constant chemical potential):

$$S[\bar{\psi},\psi] = \int_K (-i\omega + \epsilon_{\boldsymbol{k}} - \mu)\bar{\psi}_K \psi_K + \frac{1}{(2!)^2}\int_{K_1'}\int_{K_2'}\int_{K_2}\int_{K_1} \delta_{K_1'+K_2',K_2+K_1} \times \Gamma^{(4)}_{\Lambda_0}(K_1',K_2';K_2,K_1)\bar{\psi}_{K_1'}\bar{\psi}_{K_2'}\psi_{K_2}\psi_{K_1}$$

symmetrized interaction: $\Gamma^{(4)}_{\Lambda_0}(K_1',K_2';K_2,K_1) = V(\boldsymbol{k}_1' - \boldsymbol{k}_1) + V(\boldsymbol{k}_2' - \boldsymbol{k}_1)$

Bogoliubov-shift: $\psi_K = \psi_K^0 + \Delta\psi_K$ $\qquad \psi_K^0 = \langle \psi_K \rangle = \delta_{K,0}\sqrt{\rho_0}$

$$S[\Delta\bar{\psi},\Delta\psi;\psi^0] = \beta V \rho_0 \left[\frac{\rho_0}{4}\Gamma_{\Lambda_0}(0,0;0,0) - \mu\right] + (\Delta\psi_0 + \Delta\bar{\psi}_0)\sqrt{\rho_0}\left[\frac{\rho_0}{2}\Gamma_{\Lambda_0}(0,0;0,0) - \mu\right]$$

$$+\int_K \left\{ \left[-i\omega + \epsilon_{\boldsymbol{k}} - \mu + \rho_0 \Gamma^{(4)}_{\Lambda_0}(0,K;K,0)\right]\Delta\bar{\psi}_K \Delta\psi_K + \frac{\rho_0}{4}\left[\Gamma^{(4)}_{\Lambda_0}(0,0;K,-K)\Delta\psi_K\Delta\psi_{-K} + \Gamma^{(4)}_{\Lambda_0}(K,-K;0,0)\Delta\bar{\psi}_K\Delta\bar{\psi}_{-K}\right]\right\}$$

$$+\frac{\sqrt{\rho_0}}{2}\int_{K_1}\int_{K_2}\int_{K_3}\delta_{K_1+K_2,K_3}\left[\Gamma^{(4)}_{\Lambda_0}(K_1,K_2;K_3,0)\Delta\bar{\psi}_{K_1}\Delta\bar{\psi}_{K_2}\Delta\psi_{K_3} + \Gamma^{(4)}_{\Lambda_0}(0,K_3;K_2,K_1)\Delta\bar{\psi}_{K_3}\Delta\psi_{K_2}\Delta\psi_{K_1}\right]$$

$$+\frac{1}{(2!)^2}\int_{K_1'}\int_{K_2'}\int_{K_2}\int_{K_1}\delta_{K_1'+K_2',K_2+K_1}\Gamma^{(4)}_{\Lambda_0}(K_1',K_2';K_2,K_1)\Delta\bar{\psi}_{K_1'}\Delta\bar{\psi}_{K_2'}\Delta\psi_{K_2}\Delta\psi_{K_1}$$

...Bogoliubov continued...

dispersion of single-particle excitations: $E_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}}^2 + 2\rho_0 U(\boldsymbol{k})\epsilon_{\boldsymbol{k}}} \sim c_0|\boldsymbol{k}|$

condensate density at zero temperature: $\rho_0 = \dfrac{\mu}{V(0)}$

velocity of Goldstone mode: $c_0 = \sqrt{\dfrac{\rho_0 V(0)}{m}}$

Correlation functions: $\Delta\psi_K = \psi_K^l + i\psi_K^t$

longitudinal (→ ψ_K^l) transverse (→ ψ_K^t)

$$\langle \psi_K^t \psi_{-K}^t \rangle \propto \frac{c_0^2}{\omega^2 + c_0^2 \boldsymbol{k}^2}$$

$$\langle \psi_K^t \psi_{-K}^l \rangle \propto \frac{\omega}{\omega^2 + c_0^2 \boldsymbol{k}^2}$$

$$\langle \psi_K^l \psi_{-K}^l \rangle \propto \frac{\boldsymbol{k}^2}{\omega^2 + c_0^2 \boldsymbol{k}^2}$$

(d) beyond Bogoliubov

C. Castellani, C. Di Castro, F. Pistolesi, G. C. Strinati, PRL 1997, PRB 2004

- in dimensions d <= 3 Bogoliubov fixed point is unstable towards a different fixed point characterized by the divergence of the longitudinal correlation function

$$\langle \psi^l_K \psi^l_{-K} \rangle \propto \quad -Z_{\parallel}^2 \frac{\omega^2}{\omega^2 + c^2 \boldsymbol{k}^2} \; + K_{D+1} \frac{(mc)^3}{Z_\rho^3 \rho_0} \begin{cases} \ln[\frac{(mc)^2}{\omega^2/c^2 + \boldsymbol{k}^2}] & , D = 3 \\ \frac{2}{3-D} [\frac{\omega^2}{c^2} + \boldsymbol{k}^2]^{\frac{D-3}{2}} & , D < 3 \end{cases}$$

- lots of divergencies in perturbation theory
- lots of cancellations controlled by Ward identities
- critical contiuum in longitudinal structure factor
- observable? Yes: in BEC of magnons!

2.New formulation of the spin-wave expansion for antiferromagnets

N. Hasselmann, PK, Europhys. Lett. 74, 1067 (2006)

(a) Warmup: Heisenberg ferromagnet

Hamiltonian $H = \frac{1}{2}\sum_{ij} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - h \sum_i S_i^z$

$J_{ij} = J < 0$ for nearest neighbors

Map to Boson-problem via Holstein-Primakoff transformation:

$$S_i^- = \sqrt{2S} b_i^\dagger [1 - n_i/(2S)]^{1/2} = (S_i^+)^\dagger \qquad S_i^z = S - n_i \qquad n_i = b_i^\dagger b_i$$

$$H \approx E_0 + H_2 + H_4$$

$$E_0 = N[\tilde{J}_0 S^2/2 - hS]$$

$$H_2 = -\frac{S}{2}\sum_{ij} J_{ij}[n_i + n_j - b_i^\dagger b_j - b_j^\dagger b_i] + h \sum_i n_i$$

$$H_4 = \frac{1}{4}\sum_{ij} J_{ij}[2n_i n_j - b_i^\dagger n_i b_j - b_j^\dagger n_i b_i]$$

$$\tilde{J}_0 = \tilde{J}_{\boldsymbol{k}=0}$$

$$\tilde{J}_{\boldsymbol{k}} = \sum_j e^{i\boldsymbol{k}\cdot\boldsymbol{r}_j} J_{ij}$$

Why not BEC of Holstein-Primakoff bosons in ferromagnets?

Fourier transform of two-body boson interaction:

$$\hat{H}_4 = \frac{1}{2\mathcal{V}} \sum_{\boldsymbol{k}_1', \boldsymbol{k}_2', \boldsymbol{k}_2, \boldsymbol{k}_1} \delta_{\boldsymbol{k}_1' + \boldsymbol{k}_2', \boldsymbol{k}_2 + \boldsymbol{k}_1} V(\boldsymbol{k}_1', \boldsymbol{k}_2', \boldsymbol{k}_2, \boldsymbol{k}_1) b^\dagger_{\boldsymbol{k}_1'} b^\dagger_{\boldsymbol{k}_2'} b_{\boldsymbol{k}_2} b_{\boldsymbol{k}_1}$$

effective interaction vanishes at long wavelenths:

$$V(\boldsymbol{k}_1', \boldsymbol{k}_2', \boldsymbol{k}_2, \boldsymbol{k}_1) \propto \boldsymbol{k}_1' \cdot \boldsymbol{k}_2' + \boldsymbol{k}_1 \cdot \boldsymbol{k}_2$$

ferromagnetic spin-waves (Goldstone bosons) interact
too weak at long wavelengths!

(b) how about antiferromagnetic magnons?

Look at conventional spin-wave expansion for QAF:

(Anderson 1952, Kubo 1952, Oguchi 1960)

$$\hat{H} = \frac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j \qquad J_{ij} = J > 0$$

3 Transformations:

1.) Holstein-Primakoff: mapping onto boson-problem $\hat{H} = -DNJS^2 + \hat{H}_2 + \hat{H}_{\text{int}}$

$$\hat{H}_2 = S\sum_{ij} J_{ij}[b_i^\dagger b_i + b_j^\dagger b_j + b_i b_j + b_i^\dagger b_j^\dagger]$$

2.) Fourier transformation in sublattice basis:

on sublattice A:

$$b_i = (2/N)^{1/2}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i} A_{\boldsymbol{k}}$$

on sublattice B:

$$b_i = (2/N)^{1/2}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i} B_{\boldsymbol{k}}$$

3.)Bogoliubov transformation:

$$\begin{pmatrix} A_{\boldsymbol{k}} \\ B^\dagger_{-\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} u_{\boldsymbol{k}} & -v_{\boldsymbol{k}} \\ -v_{\boldsymbol{k}} & u_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} \alpha_{\boldsymbol{k}} \\ \beta^\dagger_{-\boldsymbol{k}} \end{pmatrix}$$

$$\begin{aligned} u_{\boldsymbol{k}} &= 2^{-1/2}[\epsilon_{\boldsymbol{k}}^{-1} + 1]^{1/2} \\ v_{\boldsymbol{k}} &= 2^{-1/2}[\epsilon_{\boldsymbol{k}}^{-1} - 1]^{1/2} \\ \epsilon_{\boldsymbol{k}} &= [1 - \gamma_{\boldsymbol{k}}^2]^{1/2} \\ \gamma_{\boldsymbol{k}} &= D^{-1}\sum_\mu \cos(k_\mu a) \end{aligned}$$

...spin-wave theory for QAF...

quadratic part of boson hamiltonian is diagonal

$$\hat{H}_2 = -NDJS + \hat{H}_2', \qquad \hat{H}_2' = \sum_{\boldsymbol{k}} E_{\boldsymbol{k}} [\alpha_{\boldsymbol{k}}^\dagger \alpha_{\boldsymbol{k}} + \beta_{\boldsymbol{k}}^\dagger \beta_{\boldsymbol{k}} + 1]$$

spin-wave interactions: very complicated!
(Harris,Kumar,Halperin, Hohenberg, PRB 1971, PK, PRB 1990)

$$\begin{aligned}\hat{H}_4^{\mathrm{DM}} = \frac{\chi_0^{-1}}{4V} \sum_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4} \Big\{ & V_{1234}^{(1)} (\beta_1^\dagger \beta_2^\dagger \beta_3 \beta_4 + \alpha_3^\dagger \alpha_4^\dagger \alpha_1 \alpha_2) - 2V_{1234}^{(2)} (\alpha_3^\dagger \beta_4 \alpha_1 \alpha_2 + \alpha_4^\dagger \beta_1^\dagger \beta_2^\dagger \beta_3) \\ & + 2V_{1234}^{(3)} (\alpha_3^\dagger \alpha_4^\dagger \alpha_2 \beta_1^\dagger + \alpha_1 \beta_2^\dagger \beta_3 \beta_4) - 2V_{1234}^{(4)} (\alpha_3^\dagger \alpha_1 \beta_2^\dagger \beta_4 + \alpha_4^\dagger \alpha_2 \beta_1^\dagger \beta_3) \\ & + V_{1234}^{(5)} (\alpha_3^\dagger \alpha_4^\dagger \beta_1^\dagger \beta_2^\dagger + \alpha_1 \alpha_2 \beta_3 \beta_4 \Big\} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2, \boldsymbol{k}_3+\boldsymbol{k}_4}\end{aligned}$$

interaction vertices are infrared-singular in some limits:

$$V_{1234}^{(j)} \sim \frac{1}{2} \sqrt{\frac{|\boldsymbol{k}_1||\boldsymbol{k}_2|}{|\boldsymbol{k}_3||\boldsymbol{k}_4|}} \left(1 + \xi_j \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{|\boldsymbol{k}_1||\boldsymbol{k}_2|} \right)$$

$$\xi_1 = \xi_2 = \xi_5 = 1 \qquad \xi_3 = \xi_4 = -1$$

miracle: all divergencies cancel
in physical quantities!

effective action for antiferromagnetic spin fluctuations

alternative description of antiferromagnetic spin-fluctuations:
non-linear sigma model
(Chakravarty, Halperin, Nelson, PRL and PRB 1988)

$$S_{\mathrm{NLSM}}[\mathbf{\Omega}] = \frac{\rho_0}{2}\int_0^\beta d\tau \int d^D r \left[(\partial_\mu \mathbf{\Omega})^2 + c_0^{-2}(\partial_\tau \mathbf{\Omega})^2\right]$$

$\mathbf{\Omega}(r,\tau)$ represents slowly fluctuating part of staggered magnetization

- interactions between AF spin fluctations involve derivatives and therefore vanish at long wavelengths
- what is relation between NLsM and conventional spin-wave theory?

c) hermitian operator approach

(Anderson 1952, N. Hasselmann and PK, 2006)

- introduce symmetric and antisymmetric combinations of Bogoliubov operators:

$$\hat{\Psi}_{\boldsymbol{k}\pm} = \frac{1}{\sqrt{2}}(\alpha_{\boldsymbol{k}} \pm \beta_{\boldsymbol{k}})$$

- express each of them in terms of two canonically conjugate hermitian operators:

$$\hat{\Psi}_{\boldsymbol{k}+} = -i\sqrt{\frac{\chi_0}{2VE_{\boldsymbol{k}}}}\left[E_{\boldsymbol{k}}\hat{\Pi}_{\boldsymbol{k}+} + i\chi_0^{-1}\hat{\Phi}_{\boldsymbol{k}+}\right]$$

$$\hat{\Psi}_{\boldsymbol{k}-} = \sqrt{\frac{\chi_0}{2VE_{\boldsymbol{k}}}}\left[E_{\boldsymbol{k}}\hat{\Pi}_{\boldsymbol{k}-} + i\chi_0^{-1}\hat{\Phi}_{\boldsymbol{k}-}\right]$$

$$\chi_0 = \rho_0/c_0^2$$

$$\left[\hat{\Pi}_{\boldsymbol{k}\sigma}, \hat{\Phi}_{\boldsymbol{k}'\sigma'}\right] = iV\delta_{\boldsymbol{k},-\boldsymbol{k}'}\delta_{\sigma,\sigma'}$$

- quadratic part of spin-wave hamiltonian:

$$\hat{H}_2 = \frac{1}{V}\sum_{\boldsymbol{k}}\left[\frac{\hat{\boldsymbol{\Phi}}_{-\boldsymbol{k}} \cdot \hat{\boldsymbol{\Phi}}_{\boldsymbol{k}}}{2\chi_0} + \frac{\chi_0 E_{\boldsymbol{k}}^2}{2}\hat{\boldsymbol{\Pi}}_{-\boldsymbol{k}} \cdot \hat{\boldsymbol{\Pi}}_{\boldsymbol{k}}\right]$$

$$\hat{\boldsymbol{\Phi}} = (\Phi_+, \Phi_-)$$

$$\hat{\boldsymbol{\Pi}} = (\Pi_+, \Pi_-)$$

advantages of hermitian operator approach

- simple relation to staggered and uniform components of spin

$$\begin{aligned} \boldsymbol{S}_{\boldsymbol{k}} &= \sqrt{1/N}\sum_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}\,\boldsymbol{S}_i \\ \boldsymbol{S}_{\mathrm{st},\boldsymbol{k}} &= \sqrt{1/N}\sum_i e^{-i[\boldsymbol{k}+\boldsymbol{Q}_{\mathrm{af}}]\cdot\boldsymbol{r}_i}\,\boldsymbol{S}_i \end{aligned}$$

$$\begin{aligned} S^x_{\mathrm{st},\boldsymbol{k}} &\approx \hat{\Pi}_{\boldsymbol{k}+}(S/a^D)/\sqrt{N} \\ S^y_{\mathrm{st},\boldsymbol{k}} &\approx \hat{\Pi}_{\boldsymbol{k}-}(S/a^D)/\sqrt{N} \\ S^x_{\boldsymbol{k}} &\approx -\hat{\Phi}_{\boldsymbol{k}-}/\sqrt{N} \\ S^y_{\boldsymbol{k}} &\approx \hat{\Phi}_{\boldsymbol{k}+}/\sqrt{N} \end{aligned}$$

- precise relation between non-linear sigma-model and SWT established:

$$\Omega_z = \sqrt{1-\boldsymbol{\Pi}^2} \approx 1 - \frac{1}{2}\boldsymbol{\Pi}^2 + \dots$$

- allows to derive new effective action of staggered fluctuations alone
- very convenient to understand spin-waves in finite systems
- suppression of effective interaction between AF magnons at long wavelengths becomes manifest
- for AF in uniform field: relation to divergencies in the interacting Bose gas! (A. Kreisel, N. Hasselmann, PK, PRL 2007)

zero modes in finite quantum antiferromagnets (Anderson, 1952)

- Contribution of $k = 0$ mode to quadratic spin-wave Hamiltonian:

$$\hat{H}_2^0 = \frac{\boldsymbol{P}^2}{2m} \quad \text{where} \quad \boldsymbol{P} = (SN)^{-1/2}\hat{\boldsymbol{\Phi}}_0 \qquad [X_\sigma, P_{\sigma'}] = i\delta_{\sigma,\sigma'}$$

$$\boldsymbol{X} = (S/N)^{1/2} a^{-D}\hat{\boldsymbol{\Pi}}_0 \qquad m = 4DJS$$

- Groundstate has $\langle \boldsymbol{P}^2 \rangle = 0$
- Uncertainly principle: $\langle \boldsymbol{X}^2 \rangle = \infty$
- Staggered magnetization diverges

$$M_{\rm st} = N(S + 1/2) - \langle \boldsymbol{P}^2 \rangle - \langle \boldsymbol{X}^2 \rangle - \frac{1}{2V} \sum_{\boldsymbol{k} \neq 0} \langle f_{\boldsymbol{k}} \hat{\boldsymbol{\Phi}}_{-\boldsymbol{k}} \cdot \hat{\boldsymbol{\Phi}}_{\boldsymbol{k}} + f_{\boldsymbol{k}}^{-1} \hat{\boldsymbol{\Pi}}_{-\boldsymbol{k}} \cdot \hat{\boldsymbol{\Pi}}_{\boldsymbol{k}} \rangle$$

spin-wave approach appears inconsistent!?

- Solution: for Gaussian wavepackt with $\langle \boldsymbol{P}^2 \rangle = \langle \boldsymbol{X}^2 \rangle = 1$ spin wave approach remains consistent for macroscopic times: $t \approx \sqrt{N} m = \sqrt{N}/(4DJS)$

$$\langle \boldsymbol{X}^2 \rangle_t = \langle \boldsymbol{X}^2 \rangle_0 + \frac{t^2}{m^2} \langle \boldsymbol{P}^2 \rangle_0$$

spin-wave interactions in quantum antiferromagnets

$$\hat{H}_4 = -\frac{1}{8}\sum_{ij} J_{ij}\left[4n_i n_j + n_i b_i b_j + b_i n_j b_j + b_i^\dagger b_j^\dagger n_j + b_i^\dagger n_i b_j^\dagger\right]$$

$$\hat{H}_4 = E_4 + \frac{\hat{H}_2'}{2S} + \frac{\rho_0}{2V^3}\frac{D}{a^2}\sum_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,0}\Big[\Gamma^{\Pi\Pi}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\hat{\boldsymbol{\Pi}}_{\boldsymbol{k}_1}\cdot\hat{\boldsymbol{\Pi}}_{\boldsymbol{k}_2}\hat{\boldsymbol{\Pi}}_{\boldsymbol{k}_3}\cdot\hat{\boldsymbol{\Pi}}_{\boldsymbol{k}_4}$$

$$+f_0^2\Gamma_{\parallel}^{\Phi\Pi}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\sum_\sigma \frac{1}{2}\{\hat{\Phi}_{\boldsymbol{k}_1\sigma}\hat{\Phi}_{\boldsymbol{k}_2\sigma},\hat{\Pi}_{\boldsymbol{k}_3\sigma}\hat{\Pi}_{\boldsymbol{k}_4\sigma}\}$$

$$+f_0^2\Gamma_{\perp}^{\Phi\Pi}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\sum_\sigma \hat{\Phi}_{\boldsymbol{k}_1\sigma}\hat{\Phi}_{\boldsymbol{k}_2\sigma}\hat{\Pi}_{\boldsymbol{k}_3,-\sigma}\hat{\Pi}_{\boldsymbol{k}_4,-\sigma}$$

$$f_0 = a^D/S \qquad +f_0^2\Gamma_{u}^{\Phi\Pi}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\{\hat{\Phi}_{\boldsymbol{k}_1+},\hat{\Pi}_{\boldsymbol{k}_2+}\}\{\hat{\Pi}_{\boldsymbol{k}_3-},\hat{\Phi}_{\boldsymbol{k}_4-}\}$$

$$+f_0^4\Gamma^{\Phi\Phi}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\hat{\boldsymbol{\Phi}}_{\boldsymbol{k}_1}\cdot\hat{\boldsymbol{\Phi}}_{\boldsymbol{k}_2}\hat{\boldsymbol{\Phi}}_{\boldsymbol{k}_3}\cdot\hat{\boldsymbol{\Phi}}_{\boldsymbol{k}_4}\Big]$$

important: effective interaction between staggered spin fluctuations vanishes at long wavelenths (see ferromagnet!):

$$\Gamma^{\Pi\Pi}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = (a^2/16D)[\boldsymbol{k}_1^2+\boldsymbol{k}_2^2+\boldsymbol{k}_3^2+\boldsymbol{k}_4^2+4\boldsymbol{k}_1\cdot\boldsymbol{k}_2+4\boldsymbol{k}_3\cdot\boldsymbol{k}_4] + \mathcal{O}(k_i^4)$$

$$\Gamma_{\parallel}^{\Phi\Pi}(0) = \Gamma_{u}^{\Phi\Pi}(0) = \Gamma^{\Phi\Phi}(0) = -1 \qquad \Gamma_{\perp}^{\Phi\Pi}(0) = 1$$

3.BEC of magnons: realized in QAFs in a uniform magnetic field

(Matsubara+Matsuda 1956, Batyev+Braginsky 1984,...)
A. Kreisel, N. Hasselmann, PK, PRL 2007: connection with anomalous longitudinal fluctuations in the interacting Bose gas.

$$H = \frac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j - h\sum_i S_i^z$$

experiment: Cs_2CuCl_4

(Radu et al, PRL 2005)

classical ground state:

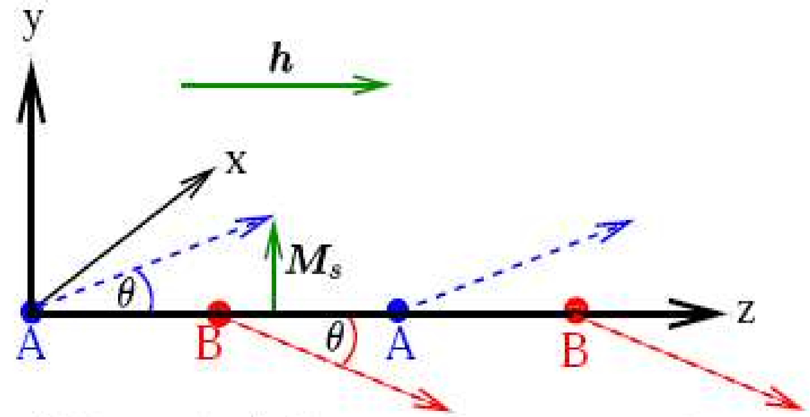
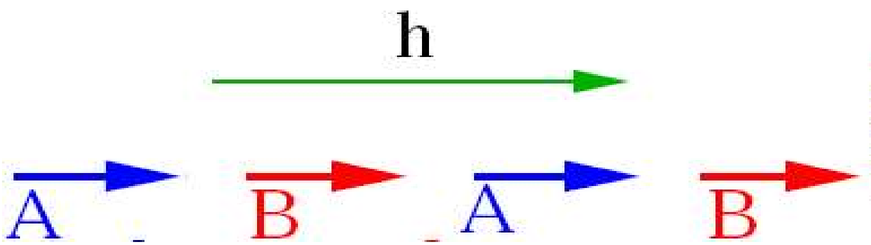


FIG. 1: (Color online) Spin configuration in the ground state of a QAF in a uniform magnetic field for $h < h_c$. The local moments $\langle \boldsymbol{S}_i \rangle$ are represented by dashed arrows. We choose the coordinate system such that the magnetic field $\boldsymbol{h}$ points in the z-direction and the staggered magnetization $\boldsymbol{M}_s$ points in the y-direction. The sublattices are labelled A and B.

(a) mapping onto interacting boson problem

for large external magnetic field: ground state is saturated ferromagnet:



spin-wave expansion usual for quantum ferromagnets:

- Holstein-Primakoff: $H \approx E_0 + H_2 + H_4$ $\qquad E_0 = N[\tilde{J}_0 S^2/2 - hS]$

$$H_2 = -\frac{S}{2}\sum_{ij} J_{ij}[n_i + n_j - b_i^\dagger b_j - b_j^\dagger b_i] + h\sum_i n_i$$

$$H_4 = \frac{1}{4}\sum_{ij} J_{ij}[2n_i n_j - b_i^\dagger n_i b_j - b_j^\dagger n_i b_i]$$

- Fourier trafo in sublattice basis $b_{\boldsymbol{k}\sigma} = N^{-1/2}[\sum_{i\in A} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i} b_i + \sigma \sum_{i\in B} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i} b_i]$

$H_2 = \sum_{\boldsymbol{k}\sigma}(\epsilon_{\boldsymbol{k}\sigma} - \mu) b_{\boldsymbol{k}\sigma}^\dagger b_{\boldsymbol{k}\sigma}$ $\qquad \mu = h_c - h$ $\qquad h_c = 2\tilde{J}_0 S$

- Two magnon branches in reduced BZ: $\epsilon_{\boldsymbol{k}\sigma} = (\tilde{J}_0 + \sigma \tilde{J}_{\boldsymbol{k}})S$ $\qquad \sigma = \pm$

gapless: $\epsilon_{\boldsymbol{k},-} \approx \boldsymbol{k}^2/(2m)$ $\qquad$ gapped: $\epsilon_{\boldsymbol{k}=0,+} = h_c$

(b) BEC of magnons and hermitian field parametrization

- retain only gapless mode and perform continuum limit $\hat{\psi}_{\boldsymbol{k}} = V^{1/2} b_{\boldsymbol{k},-}$

  spin-Hamiltonian is mapped on interacting Bose gas:

$$H = E_0 + \int_{\boldsymbol{k}} \left(\frac{\boldsymbol{k}^2}{2m} - \mu\right) \hat{\psi}^\dagger_{\boldsymbol{k}} \hat{\psi}_{\boldsymbol{k}} + \frac{1}{2} \int_{\boldsymbol{q}} \int_{\boldsymbol{k}} \int_{\boldsymbol{k}'} U_{\boldsymbol{q}} \hat{\psi}^\dagger_{\boldsymbol{k}+\boldsymbol{q}} \hat{\psi}^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} \hat{\psi}_{\boldsymbol{k}'} \hat{\psi}_{\boldsymbol{k}}$$

  two body interaction has finite long-wavelength limit $U_{\boldsymbol{q}} = \Theta(\Lambda_0 - |\boldsymbol{q}|)\chi_0^{-1}$

$$\chi_0 = (2\tilde{J}_0 a^D)^{-1}$$

- BEC of magnons: for $\mu > 0$ (i.e. $h < h_c$) field acquires finite expectation value:

$$\hat{\psi}_{\boldsymbol{k}} = (2\pi)^D \delta(\boldsymbol{k}) \psi_0 + \Delta\hat{\psi}_{\boldsymbol{k}}$$

  Bogoliubov approximation: $|\psi_0|^2 U_0 = \mu$ $\qquad \rho_0 = |\psi_0|^2$

$$\epsilon_{\boldsymbol{k}} = c_0 |\boldsymbol{k}| \qquad c_0 = \sqrt{\mu/m} = 2\sqrt{D} J S a \theta$$

- Physical meaning: condendsate density corresponds to staggered magnetization

$$M_s = V^{-1} \sum_i \zeta_i \langle S_i^y \rangle$$

$$\theta^2 \approx M_s^2/s^2 \approx 2\rho_0/s = 2(1 - h/h_c) \qquad s = \tilde{S}/a^D$$

hermitian field parameterization and correlation functions

What are spin-correlation functions in condensed phase?
Idea: Use non-perturbative results by Castellani and co-workers!

- hermitian operator parametrization

$$\hat{\psi}_{\boldsymbol{k}} = \rho_0^{1/2}\hat{\Pi}_{\boldsymbol{k}} + i(4\rho_0)^{-1/2}\hat{\Phi}_{\boldsymbol{k}}$$

$$[\hat{\Pi}_{\boldsymbol{k}}, \hat{\Phi}_{\boldsymbol{k}'}] = i(2\pi)^D \delta(\boldsymbol{k} + \boldsymbol{k}')$$

- physical meaning in underlying spin-problem:

  components of staggered magnetization

$$\hat{\Pi}_{\boldsymbol{k}} \approx \frac{1}{M_s} \sum_i \zeta_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i} S_i^x$$

$$\hat{\Phi}_{\boldsymbol{k}} \approx \frac{M_s}{s} \sum_i \zeta_i e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i} S_i^y$$

- correlation functions in linear spin-wave theory:

$$\langle \Pi_K \Pi_{K'} \rangle_0 = \delta_{K,-K'} \frac{\chi_0^{-1}}{\omega^2 + c_0^2 \boldsymbol{k}^2}$$

$$\langle \Pi_K \Phi_{K'} \rangle_0 = \delta_{K,-K'} \frac{\omega}{\omega^2 + c_0^2 \boldsymbol{k}^2}$$

$$\langle \Phi_K \Phi_{K'} \rangle_0 = \delta_{K,-K'} \frac{\chi_0 c_0^2 \boldsymbol{k}^2}{\omega^2 + c_0^2 \boldsymbol{k}^2}$$

qualitatively wrong!

c) critical continuum in longitudinal structure factor

True asymptotics of correlation functions:

$$\langle \Pi_K \Pi_{K'} \rangle = \delta_{K,-K'} \frac{\chi^{-1}}{\omega^2 + c^2 \boldsymbol{k}^2}$$

$$\langle \Pi_K \Phi_{K'} \rangle = \delta_{K,-K'} \frac{Z_{\parallel}\omega}{\omega^2 + c^2 \boldsymbol{k}^2}$$

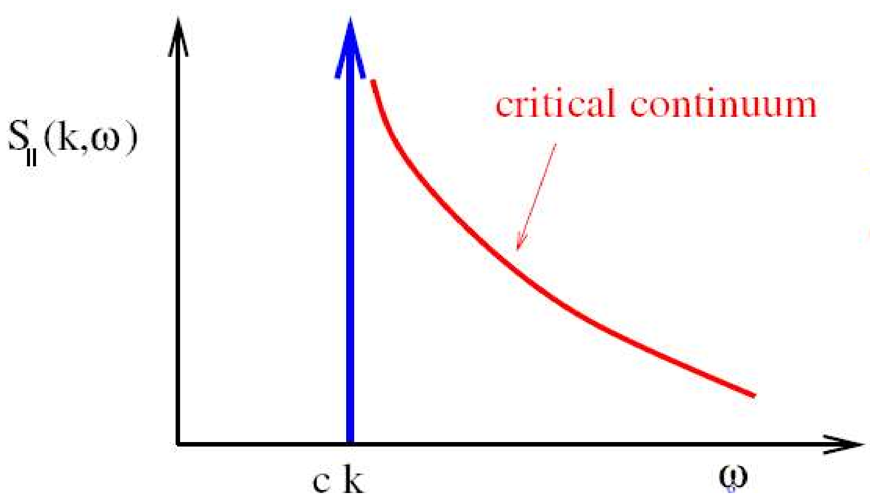
$$\langle \Phi_K \Phi_{K'} \rangle = \delta_{K,-K'} \chi \left[-Z_{\parallel}^2 \frac{\omega^2}{\omega^2 + c^2 \boldsymbol{k}^2} + K_{D+1} \frac{(mc)^3}{Z^3 \rho_0} \begin{cases} \ln[\frac{(mc)^2}{\omega^2/c^2 + \boldsymbol{k}^2}] & , D = 3 \\ \frac{2}{3-D} [\frac{\omega^2}{c^2} + \boldsymbol{k}^2]^{\frac{D-3}{2}} & , D < 3 \end{cases} \right]$$

critical continuum in longitudinal structure factor:

$$S_{\parallel}(\boldsymbol{k},\omega) = \frac{\chi s^2}{M_s^2} \left[\frac{Z_{\parallel}^2}{2} c|\boldsymbol{k}| \delta(\omega - c|\boldsymbol{k}|) + C_D \frac{(mc)^3}{Z_\rho^3 \rho_0} \frac{\Theta(\omega - c|\boldsymbol{k}|)}{(\omega^2/c^2 - \boldsymbol{k}^2)^{\frac{3-D}{2}}} \right]$$

$$1 < D \leq 3$$

$$0 < \omega/c \lesssim k_G$$



Ginzburg scale:

$$k_G \approx mc \left[(mc)^D / \rho_0 \right]^{\frac{1}{3-D}} \qquad D<3$$

$$k_G \approx mc \exp[-\rho_0/(mc)^3] \qquad D=3$$

$$I_c / I_\delta \propto (k_G/|\boldsymbol{k}|) \ln(k_G/|\boldsymbol{k}|)$$

$$I_c / I_\delta \propto (k_G/|\boldsymbol{k}|)(mc)^3/\rho_0$$

Summary: Interacting bosons and magnons

- Some not-so-well known facts about the Bose gas
    - BEC is not the same as spontaneous symmetry breaking
    - Landau functions for free Bose gas
    - shift in the critical temperature due to interactions
    - corrections to Bogoliubov approximation large in D <= 3
- Spin-wave interactions in quantum antiferromagnets
    - Usual spin-wave expansion leads to rather nasty interaction vertices
    - hermitian-field parameterization has many advantages:
        - weak interactions between Golstone modes manifest
        - relation between SWT and non-linear sigma model precise
        - zero modes and spin-waves infinite systems
- QAF in a uniform magnetic field: BEC of magnons