Exact solution of the Tomonaga-Luttinger model by means of the functional renormalization group

Florian Schütz[1], Lorenz Bartosch[1,2], and Peter Kopietz[2]
kopietz@itp.uni-frankfurt.de

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[1] Institut für Theoretische Physik, J.W.Goethe-Universität Frankfurt/Main

[2] Department of Physics, Yale University, New Haven

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Outline

- Introduction
- HS transformation, derivation of functional RG equations
- Rescaling, classification of vertices
- Interaction cutoff scheme and Ward identities
- Exact solution of TL model

Introduction

- functional renormalization group:
  - Wegner, Houghton, 1973: infinite hierarchy of coupled integro-differential equations for momentum-dependent vertex functions
  - (functional) RG for Grassmannian functional integral (Shankar ’94; Zanchi & Schulz ’96; Halboth & Metzner ’00; Honerkamp, Salmhofer, Furukawa, Rice ’01; Kopietz & Busche ’01; . . .).
  - functional integro-differential equations: a mathematical nightmare! severe truncations and hard numerical work necessary to make progress.
  - honest two-loop calculation in 2D still to be done!

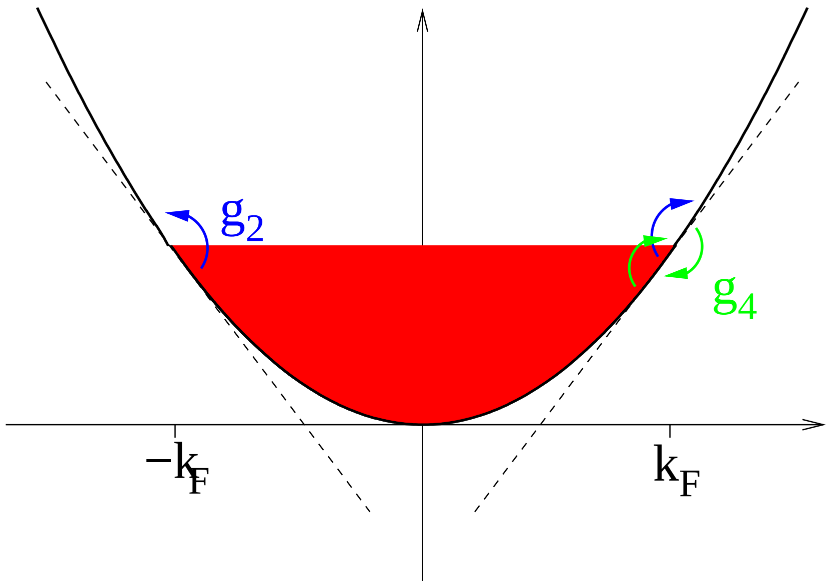
- are there non-trivial examples where infinite hierachy of coupled integro-differential equations can be solved exactly?

- YES: interacting electrons with dominant forward scattering!
  - solutions of infinite hierarchy of flow equations are given by infinite hierarchy of Ward identities
  - flow equation for two-point vertex can be closed and solved exactly
  - agreement with bosonization result (of course!)
  - possibility to study non-universal effects (band curvature, crossover scales etc...)

- general problem: how to calculate spectral functions of strongly correlated systems using RG methods?

Fermions in 1d

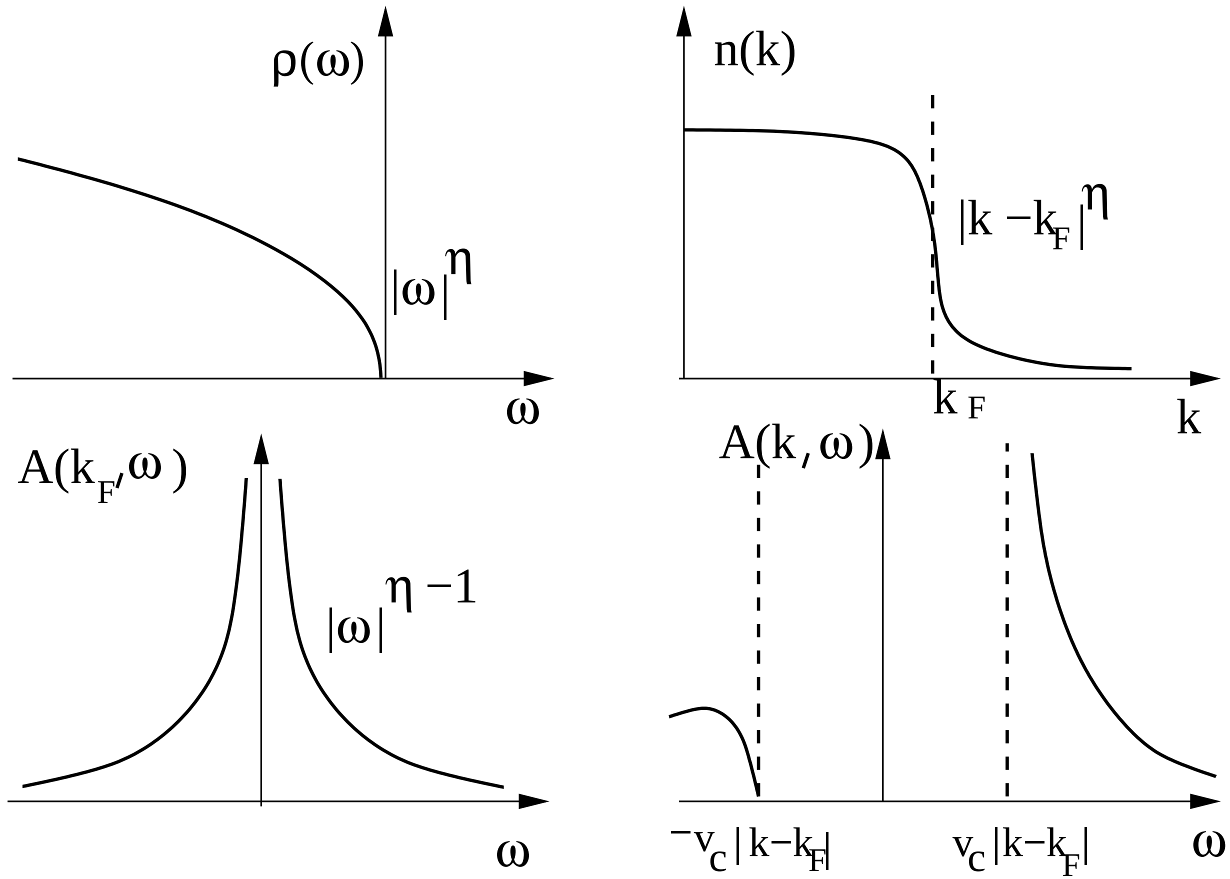
- interaction can be parameterized in terms of four marginal couplings
  - g_2: forward scattering, opposite Fermi points
  - g_4: forward scattering, same Fermi point
  - g_1: backward scattering
  - g_3: Umklapp scattering (commensurate fi llings)



- RG flow in coupling space has been discussed for many years (Solyom, 1979; field theory RG).
- no RG calculations of single-particle spectral function
- Tomonaga-Luttinger model: only forward scattering: g_2, g_4;, linear dispersion

exactly solvable via bosonization (Luther and Peschel, 1974)

Spectral function of the TLM



- vanishing density of states at Fermi energy
- no jump in momentum distribution at Fermi surface (no quasiparticles)
- anomalous scaling
- spin-charge separation

- Question: is it possible to obtain the spectral properties of the TLM entirely within the framework of the RG?

- strategy:
  - decouple electron-electron interaction in the zero-sound channel via Hubbard-Stratonovich transformation.
  - derive exact infinite hierarchy of RG flow equations for coupled Fermi-Bose field theory
  - try so solve hierarchy exactly, guided by Ward identities

- strategy can be generalized to include other scattering channels: both zero-sound channels, particle-particle on the same footing

HS-Transformation

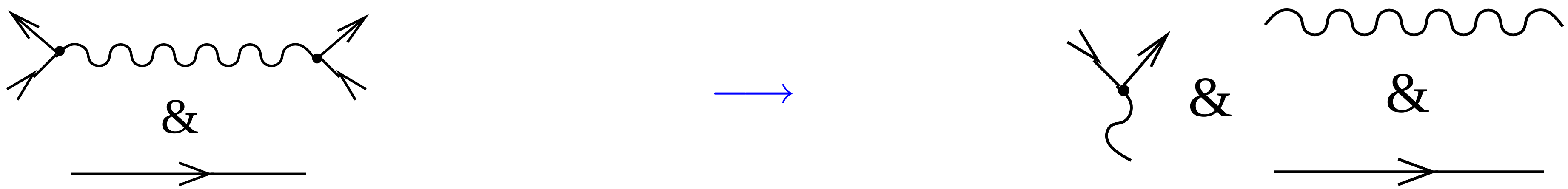
- aim: functional RG with collective fields (Correia, Polonyi, Richert ’01)
- start from action with only density-density interaction:
- density operators:

$$S[\psi] = \sum_\sigma \int_K \bar{\psi}_{K\sigma}[-i\omega + \xi_k]\psi_{K\sigma} + \frac{1}{2}\sum_{\sigma\sigma'}\int_{\bar{K}} f_{\bar{k}}^{\sigma\sigma'} \bar{\rho}_{\bar{K}\sigma}\rho_{\bar{K}\sigma'} \qquad \rho_{\bar{K}\sigma} = \int_K \bar{\psi}_{K\sigma}\psi_{K+\bar{K},\sigma}$$

$$\Downarrow$$

Hubbard-Stratonovich transformation

$$S[\psi,\varphi] = \sum_\sigma \int_K \bar{\psi}_{K\sigma}[-i\omega + \xi_k]\psi_{K\sigma} + \frac{1}{2}\sum_{\sigma\sigma'}\int_{\bar{K}} [f_{\bar{k}}^{-1}]^{\sigma\sigma'} \varphi^*_{\bar{K}\sigma}\varphi_{\bar{K}\sigma'} + i\sum_\sigma \int_{\bar{K}} \bar{\rho}_{\bar{K}\sigma}\varphi_{\bar{K}\sigma}\,.$$



Efficient notation

- collect fields in vector:

$$\boldsymbol{\Phi} = (\psi, \bar{\psi}, \varphi)$$

- symmetrized quadratic part of action:

$$S_0[\boldsymbol{\Phi}] = -\frac{1}{2}\left(\boldsymbol{\Phi}, [\mathbf{G}_0]^{-1}\boldsymbol{\Phi}\right)$$

- generalized free propagator ($\zeta = \pm 1$ for bosons/fermions):

$$\mathbf{G}_0^{-1} = \begin{pmatrix} 0 & \zeta[G_0^{-1}]^T & 0 \\ G_0^{-1} & 0 & 0 \\ 0 & 0 & -F_0^{-1} \end{pmatrix}, \quad \mathbf{G}_0^T = \mathbf{Z}\mathbf{G}_0\,, \quad \mathbf{Z} = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- with bare Green function and bare interaction:

$$\begin{aligned} [G_0]_{K\sigma,K'\sigma'} &= \delta_{K,K'}\delta_{\sigma\sigma'}\, G_{0,\sigma}(K) \quad G_{0,\sigma}(K) = [i\omega - \xi_{k\sigma}]^{-1}\,, \\ [F_0]_{\bar{K}\sigma,\bar{K}'\sigma'} &= \delta_{\bar{K},-\bar{K}'}\, F_{0,\sigma\sigma'}(\bar{K}) \quad F_{0,\sigma\sigma'}(\bar{K}) = f_k^{\sigma\sigma'} \end{aligned}$$

Generating functionals

- generating functional for Green functions:

$$\mathcal{G}[J] = e^{\mathcal{G}_c[J]} = \frac{1}{Z_0} \int D\Phi \, e^{-S_0 - S_{\text{int}} + (J,\Phi)} \, ,$$

- with:

$$(J,\Phi) = (\bar{j},\psi) + (\bar{\psi}, j) + (J^*,\varphi) \, .$$

- Legendre transformation:

$$\Phi := \frac{\delta \mathcal{G}_c}{\delta J} \, , \qquad \mathcal{L}[\Phi] := (J[\Phi],\Phi) - \mathcal{G}_c[J[\Phi]] \, , \quad \Rightarrow \quad J = \mathbf{Z} \frac{\delta \mathcal{L}}{\delta \Phi} \, .$$

- effective action generates one-line-irreducible (1LI) vertices:

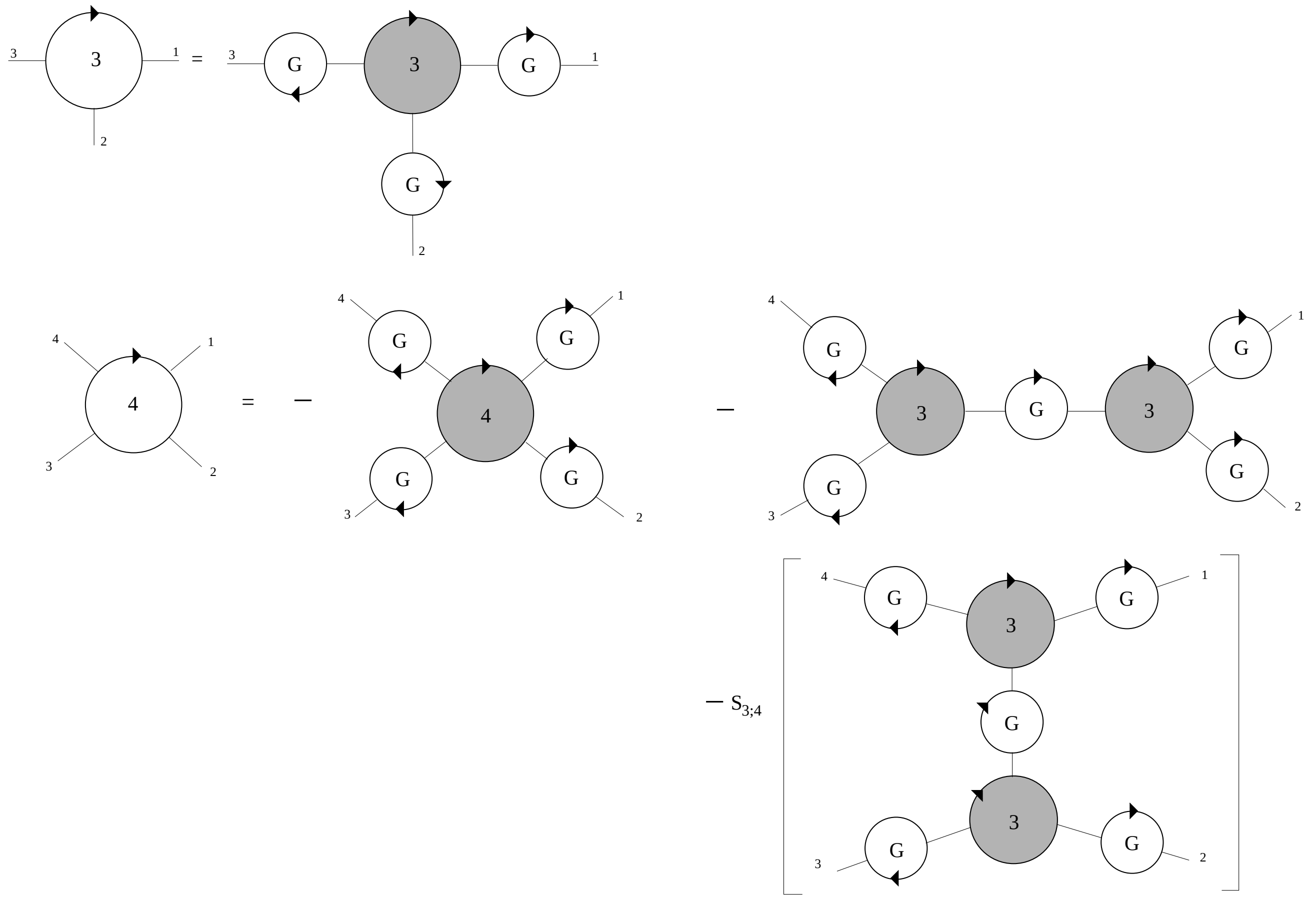
$$\Gamma[\Phi] := \mathcal{L}[\Phi] - S_0[\Phi] \, .$$

Tree expansion

- relation between connected and 1LI vertices:

$$\mathbf{1} = \frac{\delta\Phi}{\delta\Phi} = \frac{\delta^2 \mathcal{L}}{\delta\Phi\delta\Phi}\mathbf{Z}\frac{\delta^2 \mathcal{G}_c}{\delta J\delta J}\,.$$

- higher derivatives:



Cutoffs

- band cutoff:

$$
\begin{aligned}
G_0(K) &\longrightarrow \theta(\Lambda < D_K < \Lambda_0)\, G_0(K) \\
D_K &= |\varepsilon_k - \varepsilon_{k_F}|/v_0
\end{aligned}
$$

- interaction cutoff:

$$
\begin{aligned}
F_0(\bar{K}) &\longrightarrow \theta(\Lambda < \bar{D}_{\bar{K}} < \Lambda_0)\, F_0(\bar{K}) \\
\bar{D}_{\bar{K}} &= |\bar{k}|
\end{aligned}
$$

- or: only one of these

$\Rightarrow$ new RG schemes due to interaction cutoff $\Leftarrow$

Functional RG equations

- functional RG equation:

$$\begin{aligned} \partial_\Lambda \Gamma &= -\tfrac{1}{2}\mathrm{Tr}\left[\mathbf{Z}\dot{\mathbf{G}}^T\mathbf{U}^T\left\{\mathbf{1}-\mathbf{G}^T\mathbf{U}^T\right\}^{-1}\right] \\ &\quad -\tfrac{1}{2}\mathrm{Tr}\left[\mathbf{Z}\dot{\mathbf{G}}_0^T\mathbf{\Sigma}^T\left\{\mathbf{1}-\mathbf{G}^T\mathbf{\Sigma}^T\right\}^{-1}\right]. \end{aligned}$$

- with

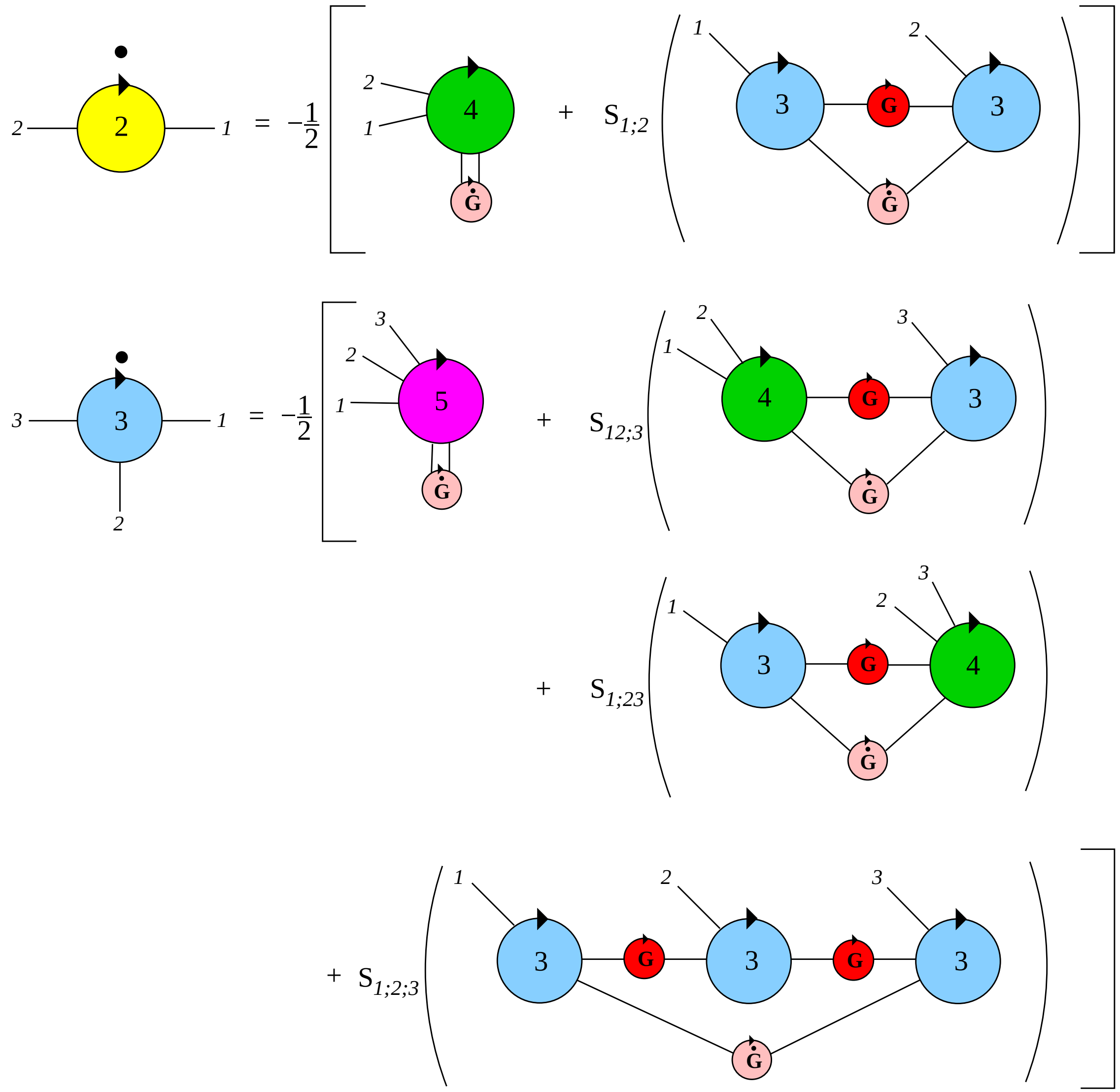
$$\mathbf{U}^T := \frac{\delta^2\Gamma}{\delta\Phi\delta\Phi} - \left.\frac{\delta^2\Gamma}{\delta\Phi\delta\Phi}\right|_{\Phi=0} = \frac{\delta^2\Gamma}{\delta\Phi\delta\Phi} - \mathbf{\Sigma}^T,$$

- and single scale propagator:

$$\dot{\mathbf{G}} = -\mathbf{G}\partial_\Lambda[\mathbf{G}_0^{-1}]\mathbf{G}.$$

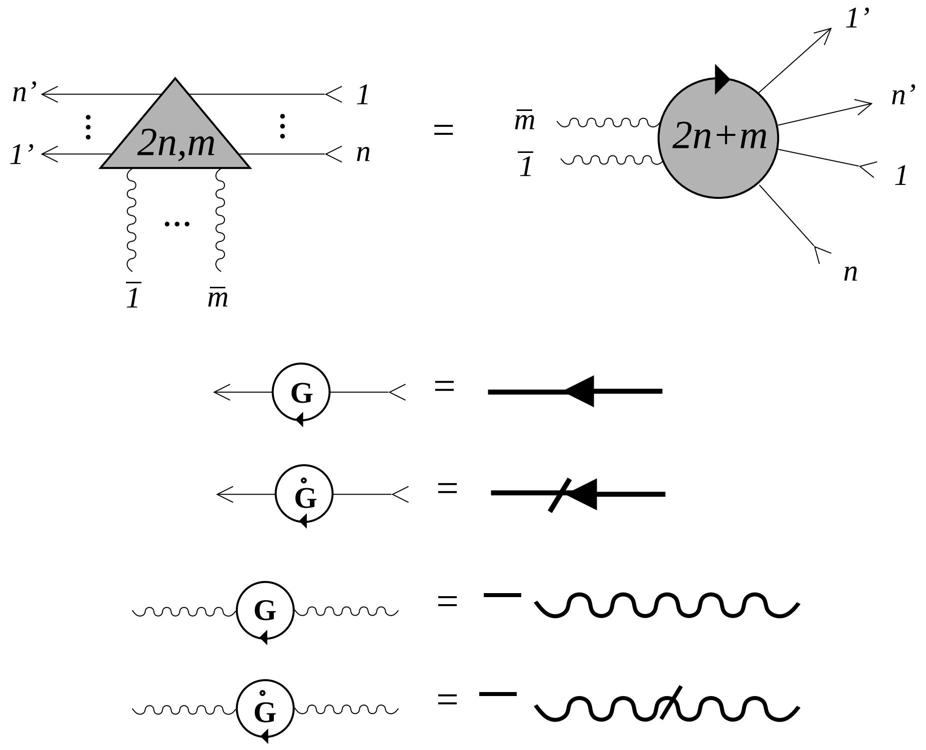
Flow of irreducible vertices

- expand Γ in powers of Φ
- diagrammatics:

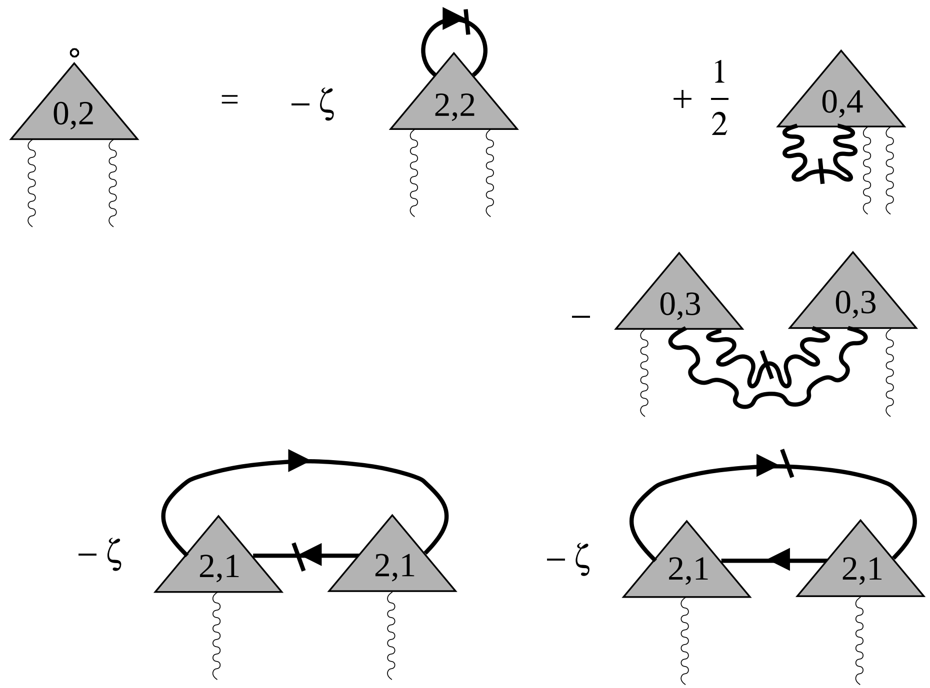


RG eqs. for physical correlation functions

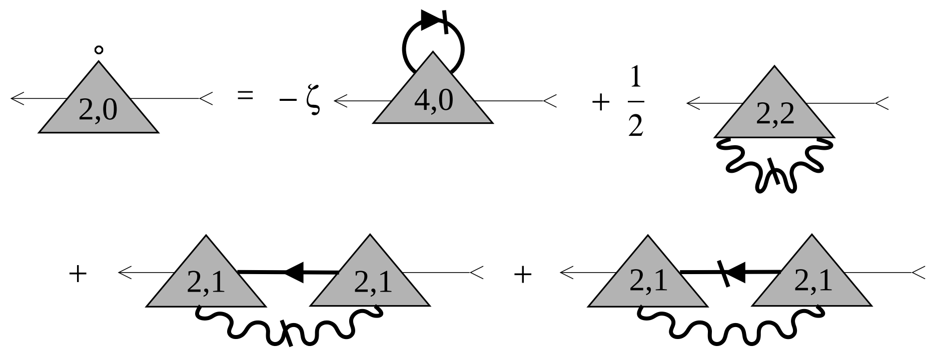
- use unsymmetrized vertices and usual propagators
- pictorial dictionary:



- flow equation for irreducible polarization:

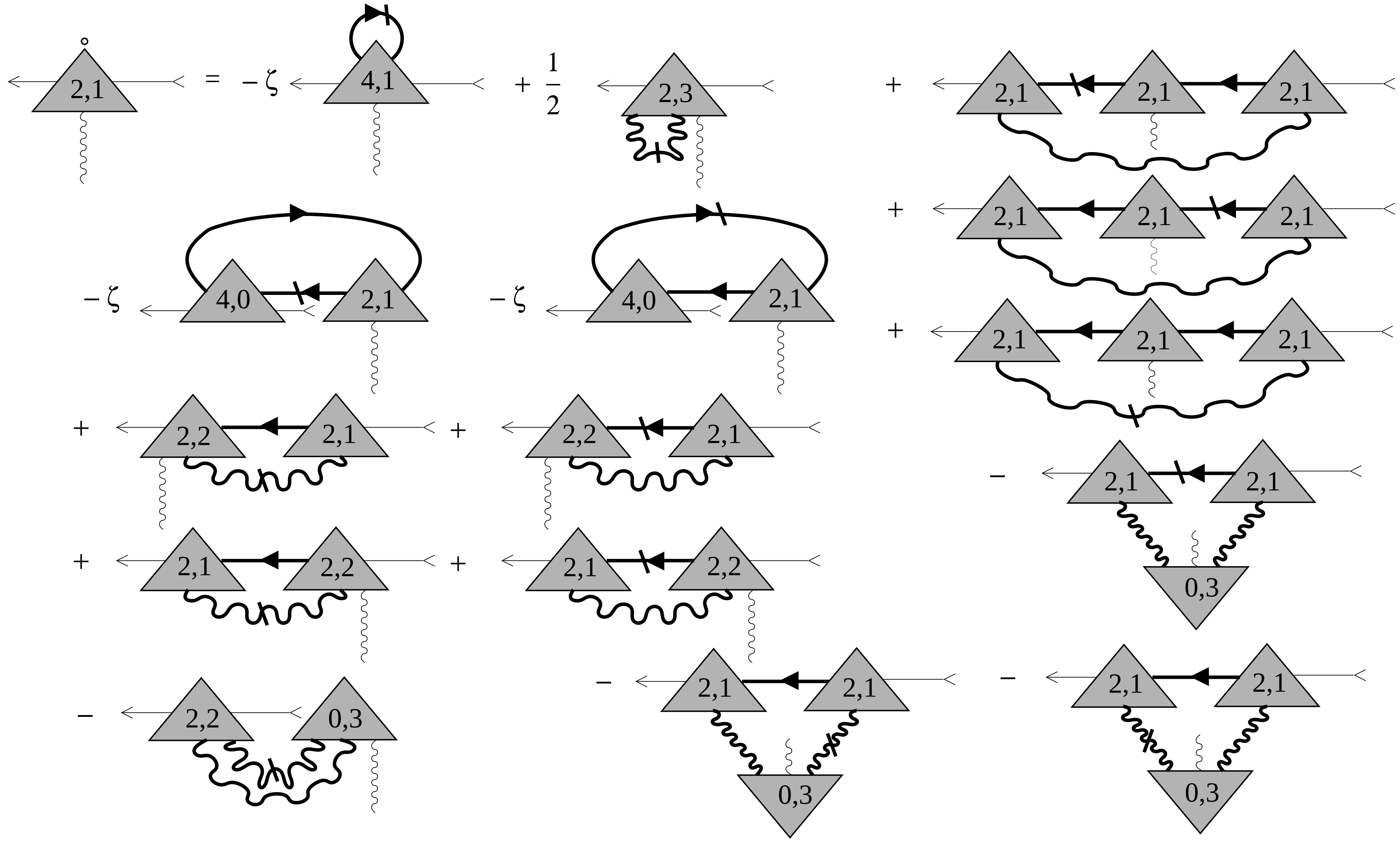


- flow equation for self-energy:



RG eqs. for phys. corr. func. (cont.)

- vertex corrections:



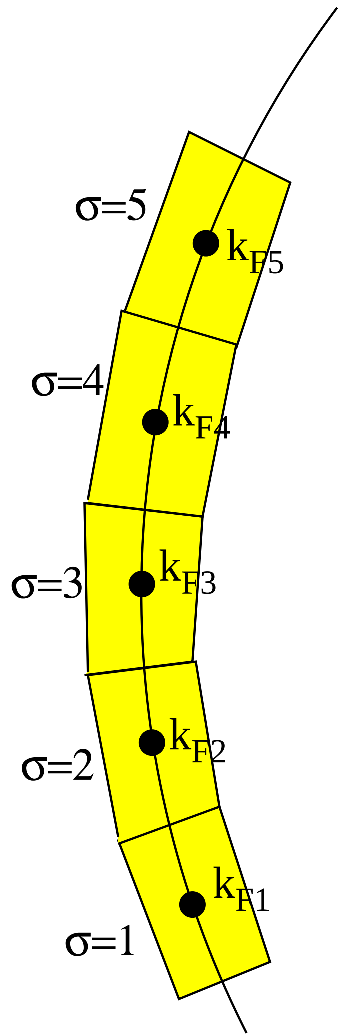
Rescaling, classification of vertices

- dimensionless bosonic momenta and frequencies:

$$\bar{\mathbf{q}} = \bar{\mathbf{k}}/\Lambda \ , \ \bar{\varepsilon} = \bar{\omega}/\bar{\Omega}_\Lambda \ , \ \bar{\Omega}_\Lambda \propto \Lambda^{z_\varphi} \ .$$

- for fermionic momenta use patching construction:

$$\mathbf{q} = (\mathbf{k} - \mathbf{k}_{F,\sigma})/\Lambda \ , \ \varepsilon = \omega/\Omega_\Lambda \ , \ \Omega_\Lambda \propto \Lambda^{z_\psi} \ .$$



- rescaling of fields, including anomalous rescaling:

$$\psi_{K\sigma} = \left(\frac{Z}{\Lambda^D \Omega_\Lambda^2} \right)^{1/2} \tilde{\psi}_{Q\sigma} \ , \quad \varphi_{\bar{K}\sigma} = \left(\frac{\bar{Z}}{\Lambda^D \Omega_\Lambda \nu_0} \right)^{1/2} \tilde{\varphi}_{\bar{Q}\sigma} \ .$$

- anomalous dimensions, $\Lambda = \Lambda_0 e^{-l}$:

$$\eta_l = -\partial_l \ln Z \ , \ \bar{\eta}_l = -\partial_l \ln \bar{Z} .$$

- rescaled vertices with flow equations:

$$\partial_l \tilde{\Gamma}_l^{(2n,m)} = \left[D^{(2n,m)} - n\eta_l - \frac{m}{2}\bar{\eta}_l - \sum_{i=1}^{n} \left(Q_i' \frac{\partial}{\partial Q_i'} + Q_i \frac{\partial}{\partial Q_i} \right) - \sum_{i=1}^{m} \bar{Q}_i \frac{\partial}{\partial \bar{Q}_i} \right] \tilde{\Gamma}_l^{(2n,m)} + \dot{\tilde{\Gamma}}_l^{(2n,m)} ,$$

- with:

$$Q \frac{\partial}{\partial Q} \equiv \mathbf{q} \cdot \nabla_{\mathbf{q}} + z_\psi \, \varepsilon \frac{\partial}{\partial \varepsilon} , \quad \bar{Q} \frac{\partial}{\partial \bar{Q}} \equiv \bar{\mathbf{q}} \cdot \nabla_{\bar{\mathbf{q}}} + z_\varphi \, \bar{\varepsilon} \frac{\partial}{\partial \bar{\varepsilon}} .$$

- scaling dimension of vertices:

$$D^{(2n,m)} = \begin{cases} (1-n)D + z_{\text{min}} - (D+z_\varphi)m/2 & \text{for } n \geq 1 \\ (D+z_\varphi)(1-m/2) & \text{for } n = 0 \end{cases}$$

- for Tomonaga-Luttinger model:

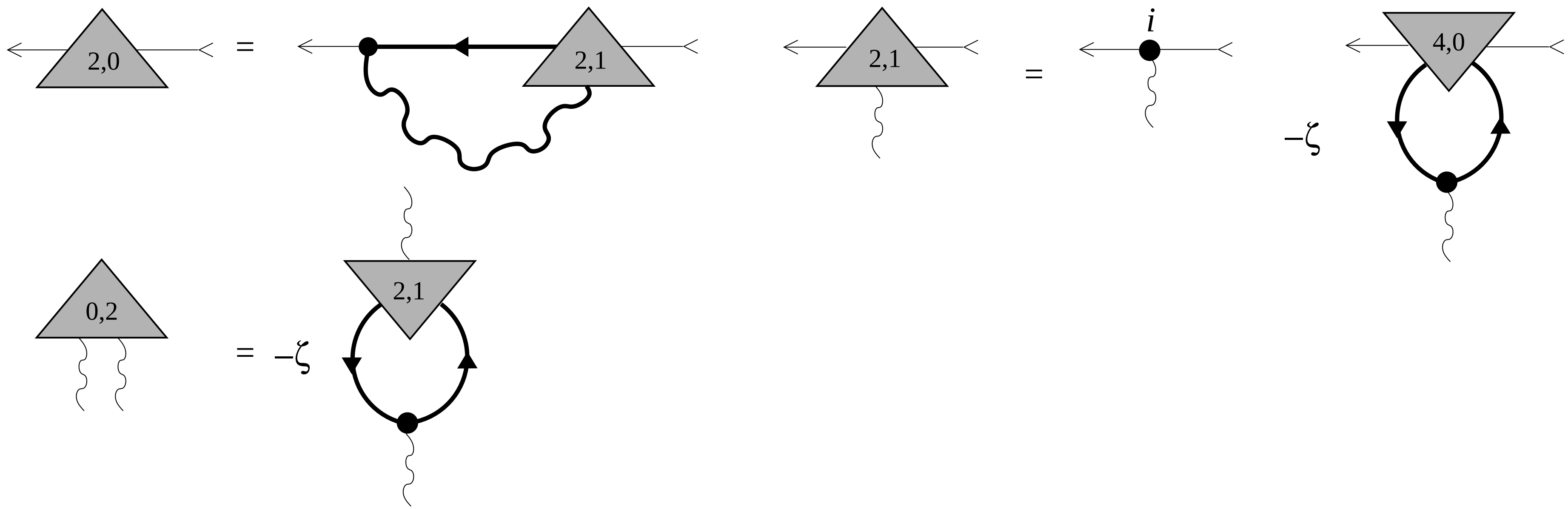
$$D = 1, \; z_\psi = z_\varphi = 1 \quad \Rightarrow \quad D^{(2n,m)} = 2 - n - m$$

Schwinger Dyson equation,skeleton expansion

- infinitesimal shifts in the integration variables Φ_α,
  Schwinger-Dyson equation:

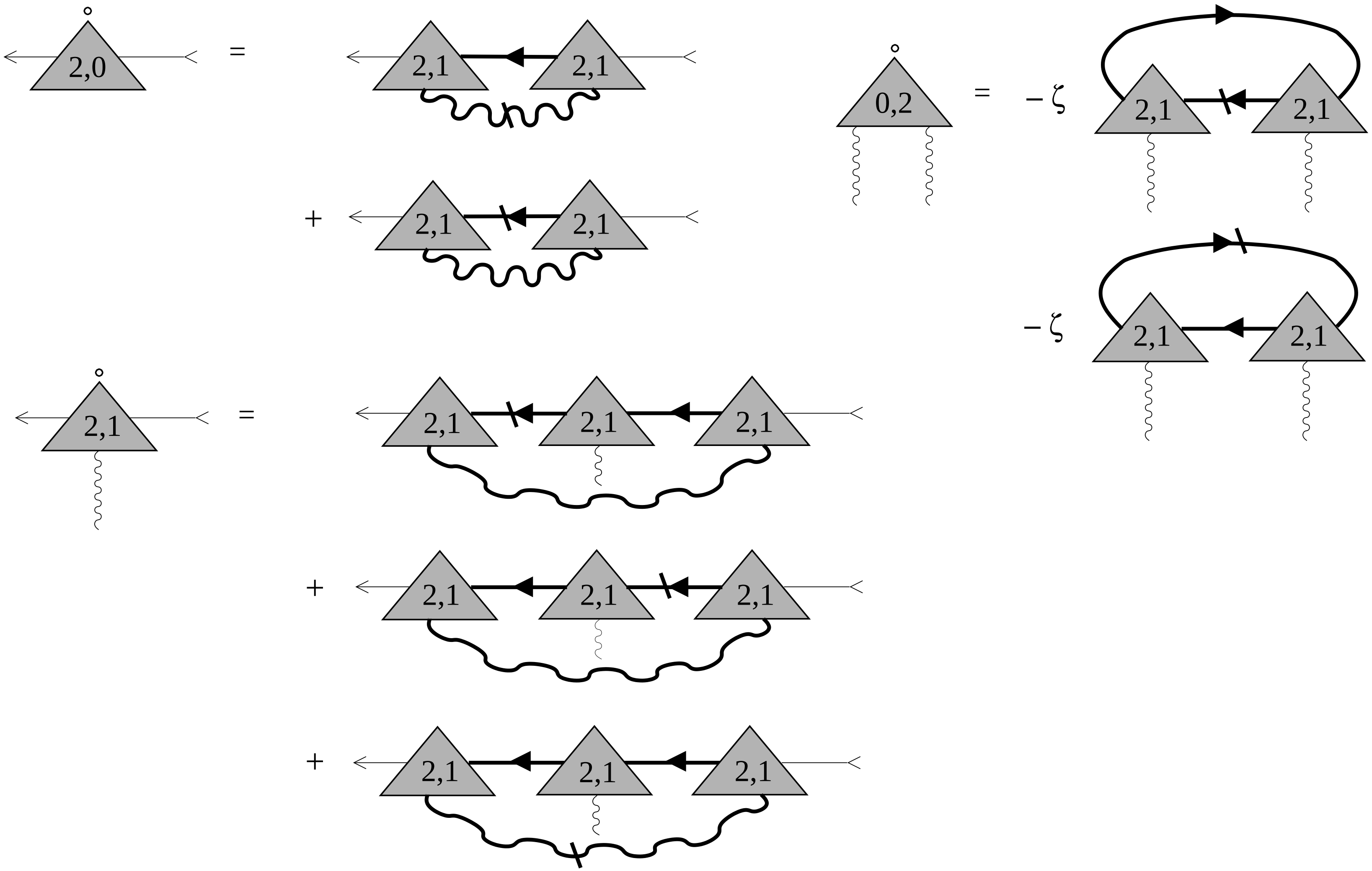
$$\left(\zeta_\alpha J_\alpha - \frac{\delta S}{\delta \Phi_\alpha}\left[\frac{\delta}{\delta J_\alpha}\right]\right)\mathcal{G}[J_\alpha] = 0$$

- translate to equation for Γ and expand in powers of fields:



A simple truncation scheme

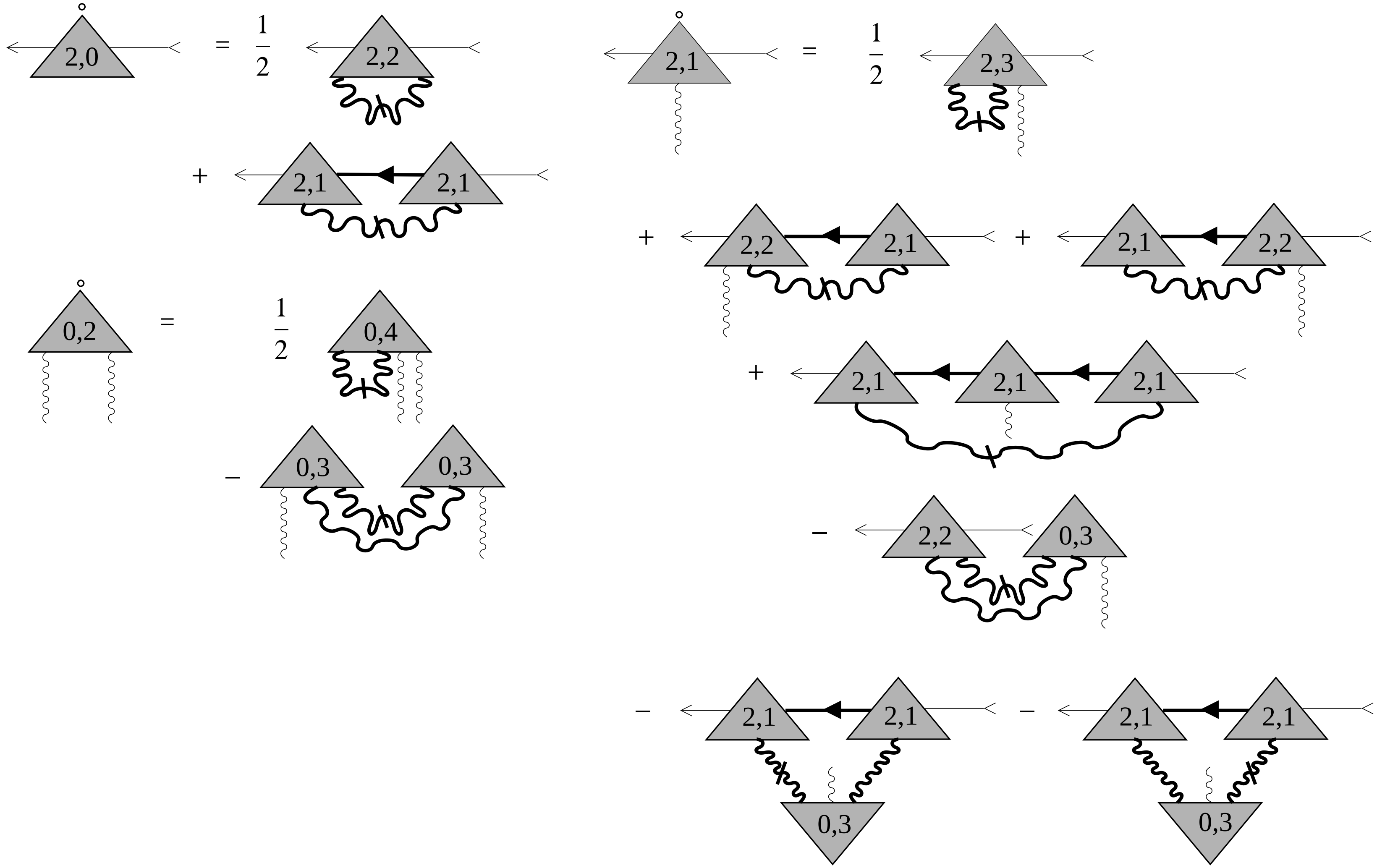
- keep only leading skeleton elements:



- numerical solution?

Interaction cutoff scheme

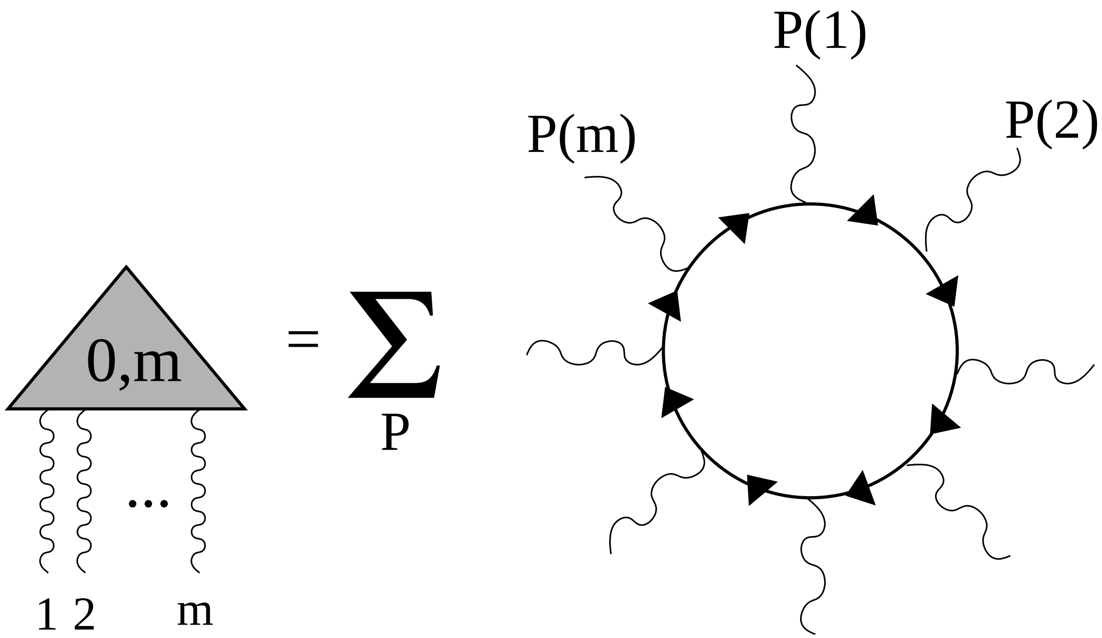
- from now on: only interaction cutoff, exact flow equations:



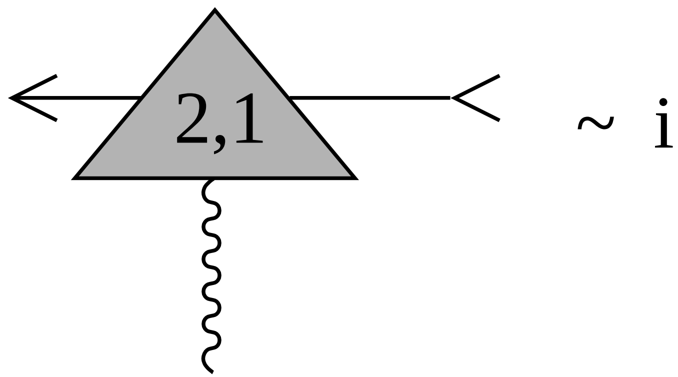
- hierarchy with respect to no. of Fermi lines
  # on r.h.s of flow equation $\leq$ # on l.h.s.

Initial condition

- Fermi loops



- bare vertex:



- all other vertices vanish

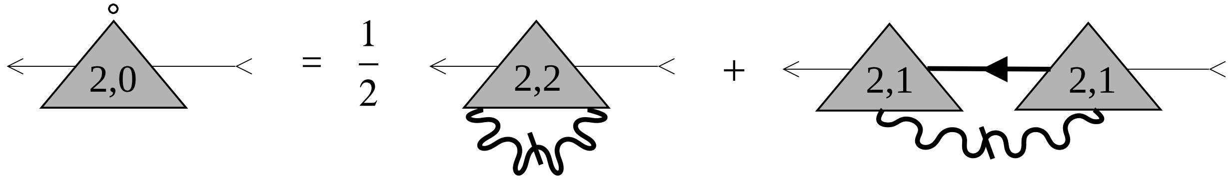
- linear dispersion: closed-loop theorem

$$\Gamma^{(0,m)}\Big|_{\Lambda=\Lambda_0} = 0\,, \qquad m > 2\,.$$

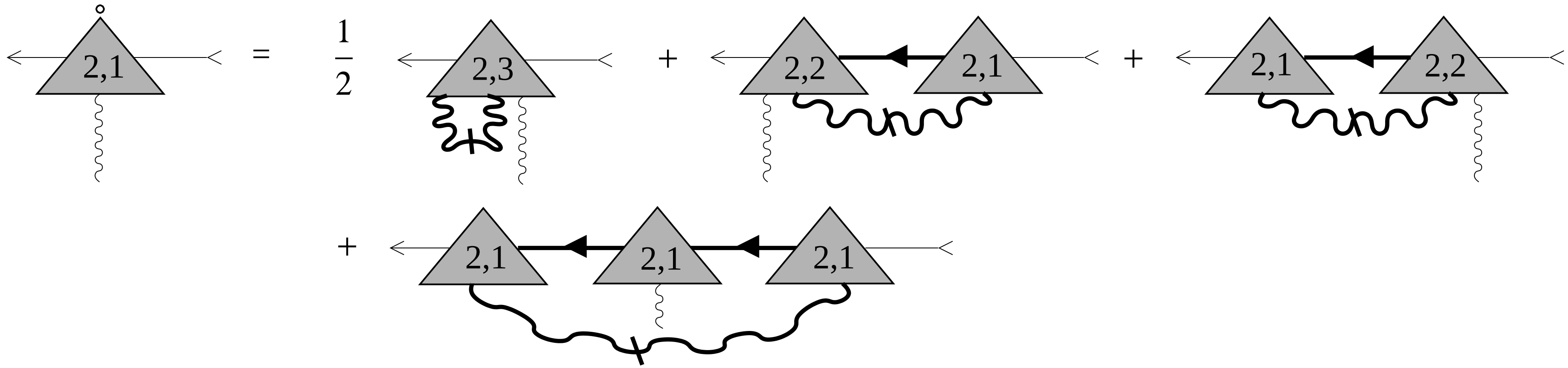
$\Rightarrow$ pure boson vertices don't flow

$$\partial_\Lambda \Gamma^{(0,m)} = 0\,.$$

Flow without higher boson vertices



⇓ attach an additional Boson leg in all possible ways



⇓

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- simple structure, solve complete hierarchy?

Ward identities

- Action in real space and imaginary time ($X = (\tau, \mathbf{r})$):

$$
\begin{aligned}
S[\bar\psi,\psi,\varphi] &= S_0[\bar\psi,\psi] + S_0[\varphi] + S_1[\bar\psi,\psi,\varphi] \\
S_0[\bar\psi,\psi] &= \sum_\sigma \int_X \bar\psi_\sigma(X)\partial_\tau\psi_\sigma(X) + \sum_\sigma \int d\tau \int d^Dr\, d^Dr'\, \bar\psi_\sigma(\tau,\mathbf{r})\xi_\sigma(\mathbf{r}-\mathbf{r}')\psi_\sigma(\tau,\mathbf{r}') \\
S_1[\bar\psi,\psi,\varphi] &= i\sum_\sigma \int_X \bar\psi_\sigma(X)\psi_\sigma(X)\varphi_\sigma(X)
\end{aligned}
$$

- local gauge transformation: $\psi_\sigma(X) = e^{i\alpha_\sigma(X)}\tilde\psi_\sigma$
- expand generating functional to first order in α:

$$
0 = \int_K \left\{ [i\bar\omega - \xi_{\mathbf{k}+\bar{\mathbf{k}},\sigma} + \xi_{\mathbf{k}\sigma}] \frac{\delta^{(2)}\mathcal{G}}{\delta j_{K\sigma}\delta \bar{j}_{K+\bar{K}\sigma}} + \bar{j}_{K+\bar{K}\sigma}\frac{\delta\mathcal{G}}{\delta \bar{j}_{K\sigma}} - j_{K\sigma}\frac{\delta\mathcal{G}}{\delta j_{K+\bar{K}\sigma}} \right\}
$$

- Linearize dispersion: $\xi_{\mathbf{k}+\bar{\mathbf{k}},\sigma} - \xi_{\mathbf{k}\sigma} \rightarrow \mathbf{v}_{F,\sigma}\cdot\bar{\mathbf{k}}$,
  use Dyson-Schwinger equation, master Ward identity:

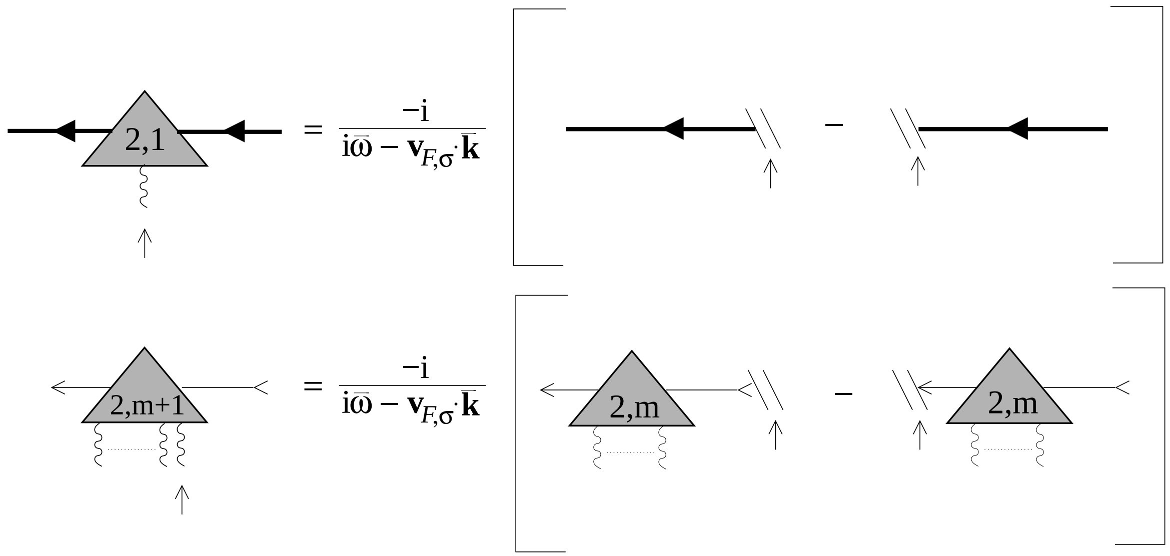
$$
0 = (i\bar\omega - \mathbf{v}_{F,\sigma}\cdot\bar{\mathbf{k}})\left[\frac{\delta\Gamma}{\delta\varphi_{\bar{K}\sigma}} - i\int_K \bar\psi_{K+\bar{K}\sigma}\psi_{K\sigma}\right] + i\int_K \left[\psi_{K\sigma}\frac{\delta\Gamma}{\delta\psi_{K+\bar{K}\sigma}} - \bar\psi_{K+\bar{K}\sigma}\frac{\delta\Gamma}{\delta\bar\psi_{K\sigma}}\right]
$$

WIs as solution of flow equations

- Ward identity for vertex correction:

$$G(K+\bar{K})\Gamma^{(2,1)}(K+\bar{K};K;\bar{K})G(K) = \frac{-i}{i\bar{\omega} - \mathbf{v}_{F,\sigma}\cdot\mathbf{k}}\Big[G(K+\bar{K}) - G(K)\Big]$$

- diagrammatically:



consistent with flow equations:

- ids trivially fulfilled initially
- insert into flow for $\Gamma^{(2,m+1)}$
- intermediate 'breaks' cancel out
- difference of flow of $\Gamma^{(2,m)}$ remains
- ids are conserved

□

$\Rightarrow$ WIs are valid at every stage of RG flow

Exact solution of TL model

- Ward ids in flow $\Rightarrow$ closed flow equation for self-energy:

$$\partial_\Lambda \Sigma_\sigma(K) = G_\sigma^{-2}(K) \int_{\bar{K}} \frac{\dot{F}_{\sigma\sigma}(\bar{K})}{(i\bar{\omega} - \mathbf{v}_{F,\sigma} \cdot \bar{\mathbf{k}})^2} \left[G_\sigma(K) - G_\sigma(K + \bar{K}) \right]$$

- linear integro-differential equation for Green function:

$$\partial_\Lambda G_\sigma(K) = \int_{\bar{K}} \underbrace{\frac{\dot{F}_{\sigma\sigma}(\bar{K})}{(i\bar{\omega} - \mathbf{v}_{F,\sigma} \cdot \bar{\mathbf{k}})^2}}_{=:H_\Lambda(\bar{K})} \left[G_\sigma(K) - G_\sigma(K + \bar{K}) \right]$$

- solve by Fourier transformation:

$$G_\sigma(i\omega, k) = \int dx \int d\tau \, G_\sigma(\tau, x) \, e^{i\omega\tau - ikx}$$

- flow equation in real space:

$$\Big[\partial_\Lambda + H_\Lambda(X) - H_\Lambda(0)\Big] G_\sigma(X) = 0\,,$$

- solution as in bosonization:

$$\boxed{G_\sigma(X) = G_{\sigma,0}(X)\, e^{Q_\sigma(X)}}$$

- with Debye-Waller factor

$$Q_\sigma(X) = -S_\sigma(X) + S_\sigma(0)$$

- and

$$S_\sigma(X) = -\int_0^{\Lambda_0} d\Lambda' H_{\Lambda'}(X) = \int_{\bar{K}} \frac{\theta(|\bar{k}| < \Lambda_0) F_{\sigma\sigma}(\bar{K})}{(i\bar{\omega} - \mathbf{v}_{F,\sigma}\cdot\bar{\mathbf{k}})^2} \cos(\bar{\omega}\tau - \bar{\mathbf{k}}\cdot\mathbf{x})$$

Truncation schemes

- for TL-model $\eta_t^{\varphi} = 0$
- simplest approximation: keep only relevant and marginal terms on r.h.s. of rescaled flow equation
- coupling constants:

$$\tilde{\mu} = \tilde{G}^{-1}\big|_{Q=0}\,, \quad \tilde{v} = \partial_{\bar{q}}\tilde{G}^{-1}\big|_{Q=0}\,, \quad \tilde{\lambda} = \tilde{\Gamma}^{(2,1)}(0)\,.$$

- $\tilde{\mu}$ has to be fine tuned, $\tilde{v}$ and $\tilde{\lambda}$ don't flow.
- anomalous dimension as in exact solution:

$$\eta = -\partial_{i\varepsilon}\tilde{\Sigma}\big|_{Q=0} = \frac{\bar{g}_0^2}{2\sqrt{1+\bar{g}_0}\,[\sqrt{1+\bar{g}_0}+1]^2}\,, \qquad \bar{g}_0 = \frac{g_0}{2\pi v_F}\,.$$

- integrate to obtain physical self-energy:

$$\Sigma_\alpha(K) = -\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}\int_{-k_c}^{k_c}\frac{d\bar{k}}{2\pi}\,\frac{F_{\alpha\alpha}(i\bar{\omega},\bar{k})}{i(\omega+\bar{\omega})-\alpha v_F(k+\bar{k})+\tilde{\mu}v_F|\bar{k}|}\left(\frac{k_c}{|\bar{k}|}\right)^{\eta}$$

Conclusions

Summary:

- Introduce collective variables in fRG from the very beginning
- new RG schemes due to:
  - 1LI vertices
  - momentum transfer cutoff in the interaction
- exact solution of TL model is recovered, ingredients:
  - closed loop theorem as initial condition
  - Ward identities to close flow equations

Outlook:

- truncation schemes when Ward ids are not valid
- decoupling in other channels (work in progress)
- broken symmetry
- renormalization of the Fermi surface (S.Ledowski, A. Ferraz, P.K, cond-mat/0412620)