

Kinetic Derivation of Fluid Dynamics for Mixtures

Gabriel Denicol

ITP - Frankfurt University

D. H. Rischke

T. Koide

P. Huovinen

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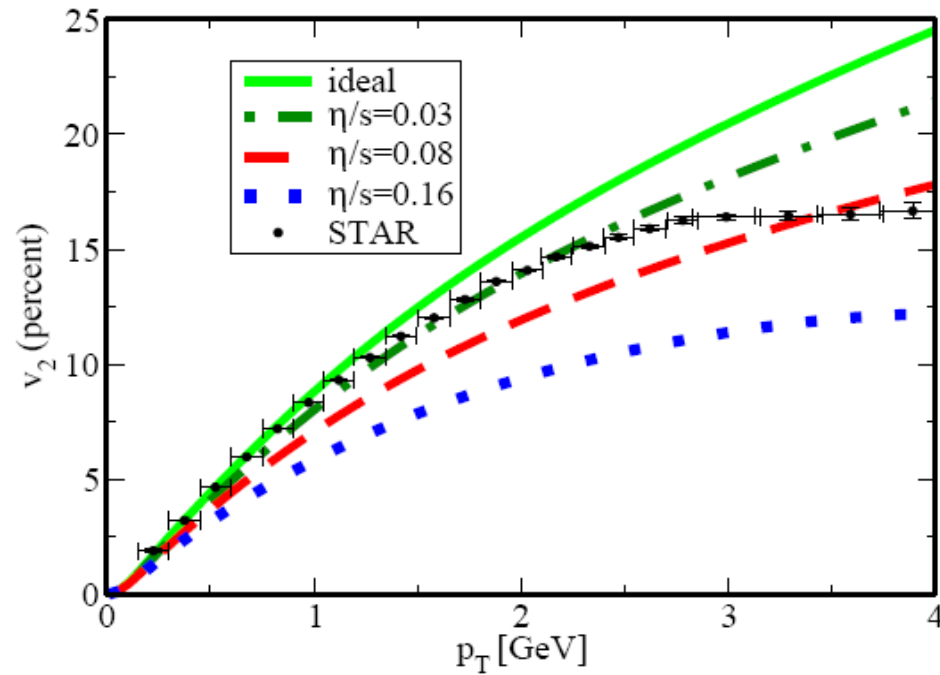


Contents

- Motivation/Introduction
- Approach
- Kinetic Corrections
- Conclusions/Perspectives

Motivation

- The significant effect of the shear viscosity



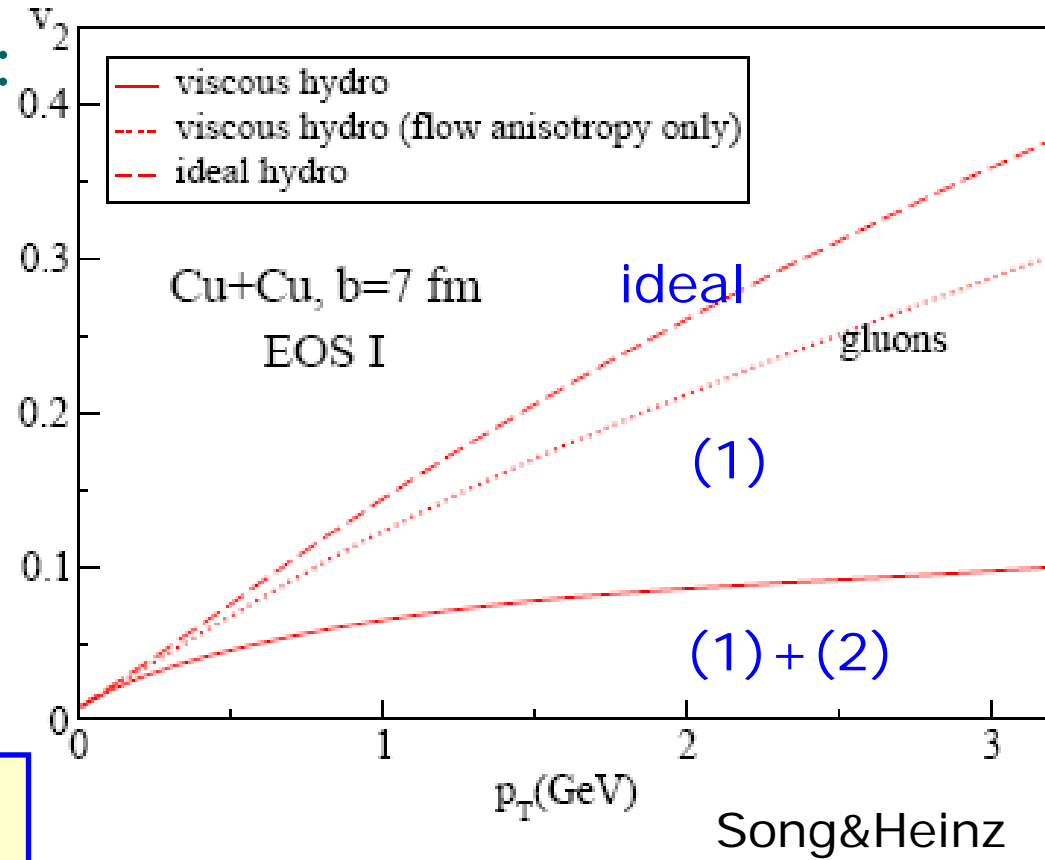
Romatschke&Romatschke

Effects of Viscosity

Two different effects :

- 1) Flow of the fluid
- 2) Momentum distribution function

$$f_{(i)} = f_{0(i)} + \delta f_{(i)}$$



Several Approaches

De Groot's book

- **BGK**

Bhatnagar, Gross and Kook (1954)
Marle (1964)

- **Chapman-Enskog-Hilbert**

Hilbert (1912), Chapman (1916) and Enskog (1917)
Marle (1964)

- **Variational Methods**

Robinson, Bernstein (1952), Van Leeuwen (1971)

- **Moments Method**

Maxwell (1867), Grad (1949), Mintzner (1965)
Chernikov (1963), Stewart (1969), Marle (1966/69)
Israel & Stewart (1978)

Several Approaches

Usually applied in
HIC

- Moments Method

Maxwell (1867), Grad (1949) , Mintzner(1965)

Chernikov (1963), Stewart (1969), Marle (1966/69)

Israel&Stewart(1978)

Several Approaches

However, the method was constructed
for a single component fluid

Usually applied in
HIC

- Moments Method

Maxwell (1867), Grad (1949) , Mintzner(1965)

Chernikov (1963), Stewart (1969), Marle (1966/69)

Israel&Stewart(1978)



What we investigate

- Moments Method for a mixture
- What are the differences ?

Basics

Boltzmann Equation

$$K_{(i)}^{\mu} \partial_{\mu} f_{(i)} = C[\{f\}]$$

Conserved Currents

$$T^{\mu\nu} = \sum_i T_{(i)}^{\mu\nu} \quad N_r^{\mu} = \sum_i q_{(i)}^r N_{(i)}^{\mu}$$

Currents for each particle specie

$$T_{(i)}^{\mu\nu} = \int d\omega_{(i)} K_{(i)}^{\mu} K_{(i)}^{\nu} f_{(i)}$$
$$N_{(i)}^{\mu} = \int d\omega_{(i)} K_{(i)}^{\mu} f_{(i)}$$

$$d\omega_{(i)} \equiv \frac{d^3 K}{(2\pi)^3 K_{(i)}^0}$$

Basics

The separation

$$K_{(i)}^{\mu} = u^{\mu} E_{(i)} + \Delta^{\mu\alpha} K_{\alpha(i)}$$

$$E_{(i)} \equiv u^{\mu} K_{\mu(i)}$$

General form of the currents

$$T_{(i)}^{\mu\nu} = \varepsilon_{(i)} u^{\mu} u^{\nu} - \Delta^{\mu\nu} (P_{0(i)} + \Pi_{(i)}) + 2q_{(i)}^{(\mu} u^{\nu)} + \pi_{(i)}^{\mu\nu}$$

$$N_{(i)}^{\mu} = n_{(i)} u^{\mu} + v_{(i)}^{\mu}$$

Matching Conditions

$$\varepsilon_{(i)} = \varepsilon_{0(i)} \quad n_{(i)} = n_{0(i)}$$

Basics

The separation

$$K_{(i)}^{\mu} = u^{\mu} E_{(i)} + \Delta^{\mu\alpha} K_{\alpha(i)}$$

$$E_{(i)} \equiv u^{\mu} K_{\mu(i)}$$

General form of the currents

$$T_{(i)}^{\mu\nu} = \varepsilon_{(i)} u^{\mu} u^{\nu} - \Delta^{\mu\nu} (P_{0(i)} + \Pi_{(i)}) + 2q_{(i)}^{(\mu} u^{\nu)} + \pi_{(i)}^{\mu\nu}$$

$$N_{(i)}^{\mu} = n_{(i)} u^{\mu} + v_{(i)}^{\mu}$$

Bulk

Energy
Difusion

Shear

Particle
Difusion

for each specie

Irreversible Currents

$$\Pi_{(i)} = -\frac{1}{3} m_{(i)}^2 \int d\omega_{(i)} \delta f_{(i)}$$

$$\pi_{(i)}^{\mu\nu} = \int d\omega_{(i)} \Delta^{\mu\nu\alpha\beta} K_{\alpha(i)} K_{\beta(i)} \delta f_{(i)}$$

$$q_{(i)}^{\mu} = \int d\omega_{(i)} E_{(i)} \Delta^{\mu\nu} K_{\nu(i)} \delta f_{(i)}$$

$$v_{(i)}^{\mu} = \int d\omega_{(i)} \Delta^{\mu\nu} K_{\nu(i)} \delta f_{(i)}$$

$$\Delta^{\alpha\beta} = g^{\alpha\beta} - u^{\alpha} u^{\beta}$$

$$\Delta^{\mu\nu\alpha\beta} = \Delta^{<\mu\nu} \Delta^{\alpha\beta>}$$

↓
Symmetric
+
Traceless

Local Eq. distribution function

$$f_{0(i)} = \left(\exp(\beta_0 E_{(i)} - \alpha_{0(i)}) \pm 1 \right)^{-1}$$

Deviation

$$\delta f_{(i)} = f_{(i)} - f_{0(i)}$$

Total Currents

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} (P_0 + \Pi) + 2q^{(\mu} u^{\nu)} + \pi^{\mu\nu}$$

$$N_r^\mu = n_r u^\mu + v_r^\mu$$



$$\Pi = \sum_{(i)} \Pi_{(i)}$$

$$\pi^{\mu\nu} = \sum_{(i)} \pi_{(i)}^{\mu\nu}$$

$$q^\mu = \sum_{(i)} q_{(i)}^\mu$$

$$v_r^\mu = \sum_{(i)} q_{(i)}^r v_{(i)}^\mu$$

$$T^{\mu\nu} = \sum_i T_{(i)}^{\mu\nu}$$
$$N_r^\mu = \sum_i q_{(i)}^r N_{(i)}^\mu$$

Diffusion relative to the net number density

Grad's Method (by Grad)

Expand the distribution function in terms of a complete set

$$f_{(i)} = f_{0(i)} \sum_n \frac{1}{n!} a_{(n)}^{i_1 \dots i_n} H_{(n)}^{i_1 \dots i_n} \quad H_{(n)}^{i_1 \dots i_n} \rightarrow \text{Hermite Polynomials}$$



$$a_{(n)}^{i_1 \dots i_n} = \int d^3 pf H_{(n)}^{i_1 \dots i_n}$$

Third Order Approximation → stop at n=3

$a_{(2)}^{i_1 i_2}$ $a_{(3)}^{i_1 i_2 i_3}$
 20 moments!

13 Moments Approx.

$$a_{(3)}^{i_1 i_2 i_3} = \frac{1}{5} \left(a_{(3)}^{i_1} \delta^{i_2 i_3} + a_{(3)}^{i_2} \delta^{i_1 i_3} + a_{(3)}^{i_3} \delta^{i_1 i_2} \right)$$

(no Bulk)

Grad's Method (by Grad)

Expand the distribution function in terms of a complete set

$$f_{(i)} = f_{0(i)} \sum_n \frac{1}{n!} a_{(n)}^{i_1 \dots i_n} H_{(n)}^{i_1 \dots i_n} \quad H_{(n)}^{i_1 \dots i_n} \rightarrow \text{Hermite Polynomials}$$



$$a_{(n)}^{i_1 \dots i_n} = \int d^3 p f H_{(n)}^{i_1 \dots i_n}$$

Third Order Ap

Not so easy in a relativistic regime

$a_{(2)}^{i_1 i_2}$ $a_{(3)}^{i_1 i_2 i_3}$
 20 moments!

13 Moments Ap

$$+ a_{(3)}^{i_2 i_3} + a_{(3)}^{i_2} \delta^{i_1 i_3} + a_{(3)}^{i_3} \delta^{i_1 i_2}$$

(no Bulk)

Grad's Method (by IS)

Assume

$$f_{(i)} = \left(\exp(y_{(i)}) \pm 1 \right)^{-1}$$

The following expansion is applied

$$y_{(i)} - y_{0(i)} = \varepsilon(E_{(i)}) + \varepsilon_{\mu}(E_{(i)})\Delta^{\mu\nu}K_{\nu(i)} + \varepsilon_{\mu\nu}(E_{(i)})\Delta^{\mu\nu\alpha\beta}K_{\alpha(i)}K_{\beta(i)}$$

Expand in Taylor

$$\varepsilon(E_{(i)}) = \varepsilon_{0(i)} + \varepsilon_{1(i)}E_{(i)} + \varepsilon_{2(i)}E_{(i)}^2$$

$$\varepsilon_{\mu}(E_{(i)}) = \varepsilon_{0\mu(i)} + \varepsilon_{1\mu(i)}E_{(i)}$$

$$\varepsilon_{\mu\nu}(E_{(i)}) = \varepsilon_{0\mu\nu}$$

14 Moments Approx.
(finite Bulk)

Grad's Method (by IS)

Assume

$$f_{(i)} = \left(\exp(y_{(i)}) \pm 1 \right)^{-1}$$

Expand in terms of irreducible tensors

$$y_{(i)} - y_{0(i)} = \varepsilon_{(i)} + \varepsilon_{\mu(i)} K_{(i)}^\mu + \varepsilon_{\mu\nu(i)} K_{(i)}^\mu K_{(i)}^\nu + \varepsilon_{\mu\nu\alpha\beta(i)} (E_{(i)})^{\mu\nu\alpha\beta} K_{\alpha(i)} K_{\beta(i)}$$

$$y_{(i)} - y_{0(i)} = \varepsilon_{(i)} + \varepsilon_{\mu(i)} K_{(i)}^\mu + \varepsilon_{\mu\nu(i)} K_{(i)}^\mu K_{(i)}^\nu$$

$\varepsilon_{(i)}$

$$\varepsilon_{\mu}(E_{(i)}) = \varepsilon_{\mu(i)}$$

$$\varepsilon_{\mu\nu}(E_{(i)}) = \varepsilon_{0\mu\nu}$$

Finite Bulk Approx.

(Finite Bulk)

Grad's Method (by IS)

The expansion variables are determined using

Definition of Currents

Matching Cond.

$$u_{\mu} u_{\nu} (T^{\mu\nu} - T_0^{\mu\nu}) = 0$$

$$u_{\mu} (N^{\mu} - N_0^{\mu}) = 0$$

$$v_{(i)}^{\mu} = \Delta^{\mu\nu} N_{\nu(i)}$$

$$q_{(i)}^{\mu} = u^{\alpha} \Delta^{\mu\beta} T_{\alpha\beta(i)}$$

$$\pi_{(i)}^{\mu\nu} = \Delta^{\mu\nu\alpha\beta} T_{\alpha\beta(i)}$$

$$\Pi_{(i)} = -\frac{1}{3} \Delta_{\alpha\beta} (T_{(i)}^{\alpha\beta} - T_{0(i)}^{\alpha\beta})$$

$$f_{(i)} = (\exp(y_{(i)} \pm 1))^{-1}$$

And expanding

$$f_{(i)} \approx f_{0(i)} + f_{0(i)} (1 \mp f_{0(i)}) (y_{(i)} - y_{0(i)})$$

Grad's Method (by IS)

$$\epsilon_{(0)} J_{00(i)} + \epsilon_{(1)} J_{10(i)} + \epsilon_{(2)} J_{20(i)} = -\frac{3\Pi_{(i)}}{m_{(i)}^2},$$

$$\epsilon_{(0)} J_{10(i)} + \epsilon_{(1)} J_{20(i)} + \epsilon_{(2)} J_{30(i)} = 0,$$

$$\epsilon_{(0)} J_{20(i)} + \epsilon_{(1)} J_{30(i)} + \epsilon_{(2)} J_{40(i)} = 0,$$

$$\Delta^{\mu\nu} \epsilon_{\mu(0)} J_{21(i)} + \Delta^{\mu\nu} \epsilon_{\mu(1)} J_{31(i)} + \Delta^{\mu\nu} \epsilon_{\mu(2)} J_{41(i)} = -n_{(i)}^\nu$$

$$\Delta^{\mu\nu} \epsilon_{\mu(0)} J_{31(i)} + \Delta^{\mu\nu} \epsilon_{\mu(1)} J_{41(i)} + \Delta^{\mu\nu} \epsilon_{\mu(2)} J_{51(i)} = -q_{(i)}^\nu$$

$$2\Delta^{\mu\nu\lambda\sigma} \epsilon_{\lambda\sigma(0)} J_{42(i)} = \pi_{(i)}^{\mu\nu}$$

Definition of L.R.F

We use the Landau picture

$$u_\nu (T^{\mu\nu} - T_0^{\mu\nu}) = 0$$

Then,

$$q_\nu = \sum_i q_{\nu(i)} = 0$$

But,

$$q_{\nu(i)} \neq 0$$

The energy flow for each particle specie
cannot be neglected

$$q_{\nu(i)} \quad v_{\nu(i)}$$

Solution

$$f_{(i)}(p) = f_{0(i)} \left[1 + (1 \mp f_{0(i)}) \left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p_{(i)}^{\mu} + \varepsilon_{\mu\nu(i)} p_{(i)}^{\mu} p_{(i)}^{\nu} \right) \right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon_{(i)}^{\mu} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q_{(i)}^{\mu} + D_{2(i)} v_{(i)}^{\mu}$$

$$\varepsilon_{(i)}^{\mu\nu} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu} u^{\nu} \right) \Pi_{(i)} + 2B_{1(i)} u^{(\mu} q_{(i)}^{\nu)} + 2B_{3(i)} u^{(\mu} v_{(i)}^{\nu)} + B_{2(i)} \pi_{(i)}^{\mu\nu}$$

Solution

$$f_{(i)}(p) = f_{0(i)} \left[1 + (1 \mp f_{0(i)}) \left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p_{(i)}^{\mu} + \varepsilon_{\mu\nu(i)} p_{(i)}^{\mu} p_{(i)}^{\nu} \right) \right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon_{(i)}^{\mu} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q_{(i)}^{\mu} + D_{2(i)} v_{(i)}^{\mu}$$

$$\varepsilon_{(i)}^{\mu\nu} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu} u^{\nu} \right) \Pi_{(i)} + 2B_{1(i)} u^{(\mu} q_{(i)}^{\nu)} + 2B_{3(i)} u^{(\mu} v_{(i)}^{\nu)} + B_{2(i)} \pi_{(i)}^{\mu\nu}$$

$$B_{2(i)} = \frac{1}{2J_{42(i)}}$$



shear viscosity

$$J_{nq(i)} = \frac{1}{(2q-1)!!} \int d\omega_{(i)} E_{(i)}^{n-2q} K^{2q} f_{0(i)} \left(1 - a f_{0(i)} \right)$$

Solution

$$f_{(i)}(p) = f_{0(i)} \left[1 + (1 \mp f_{0(i)}) \left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p_{(i)}^{\mu} + \varepsilon_{\mu\nu(i)} p_{(i)}^{\mu} p_{(i)}^{\nu} \right) \right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon_{(i)}^{\mu} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q_{(i)}^{\mu} + D_{2(i)} v_{(i)}^{\mu}$$

$$\varepsilon_{(i)}^{\mu\nu} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu} u^{\nu} \right) \Pi_{(i)} + 2B_{1(i)} u^{(\mu} q_{(i)}^{\nu)} + 2B_{3(i)} u^{(\mu} v_{(i)}^{\nu)} + B_{2(i)} \pi_{(i)}^{\mu\nu}$$

$$B_{2(i)} = \frac{1}{2J_{42(i)}}$$



shear viscosity

Boltzmann Gas



$$B_{2(i)} = \frac{1}{2(\varepsilon_{(i)} + P_{(i)}) T^2}$$

$$J_{nq(i)} = \frac{1}{(2q-1)!!} \int d\omega_{(i)} E_{(i)}^{n-2q} K^{2q} f_{0(i)} (1 - af_{0(i)})$$

Comment

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{1}{2(\varepsilon_{(i)} + P_{(i)})T^2} \pi_{\mu\nu(i)} K_{(i)}^\mu K_{(i)}^\nu$$

Solution

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{\sum_j \pi_{\mu\nu(j)}}{2T^2 \sum_j (\varepsilon_{(j)} + P_{(j)})} K_{(i)}^\mu K_{(i)}^\nu$$

used so far

$$\frac{\pi_{\mu\nu(i)}}{\varepsilon_{(i)} + P_{(i)}} \approx \frac{\sum_j \pi_{\mu\nu(j)}}{\sum_j (\varepsilon_{(j)} + P_{(j)})}$$

True ?

Comment

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{1}{2(\varepsilon_{(i)} + P_{(i)})T^2} \pi_{\mu\nu(i)} K_{(i)}^\mu K_{(i)}^\nu$$

Solution

$$\delta f_{(i)}^{shear} = f_{0(i)} \frac{\sum_j \pi_{\mu\nu(j)}}{2T^2 \sum_j (\varepsilon_{(j)} + P_{(j)})} K_{(i)}^\mu K_{(i)}^\nu$$

used so far

$$\frac{\pi_{\mu\nu(i)}}{\varepsilon_{(i)} + P_{(i)}} \approx \frac{\sum_j \pi_{\mu\nu(j)}}{\sum_j (\varepsilon_{(j)} + P_{(j)})}$$

True ?

$$\frac{\pi_{\mu\nu(i)}}{\pi_{\mu\nu}} \approx \frac{\eta_{(i)}}{\eta}$$

Solution

$$f_{(i)}(p) = f_{0(i)} \left[1 + (1 \mp f_{0(i)}) \left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p_{(i)}^{\mu} + \varepsilon_{\mu\nu(i)} p_{(i)}^{\mu} p_{(i)}^{\nu} \right) \right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon_{(i)}^{\mu} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q_{(i)}^{\mu} + D_{2(i)} v_{(i)}^{\mu}$$

$$\varepsilon_{(i)}^{\mu\nu} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu} u^{\nu} \right) \Pi_{(i)} + 2B_{1(i)} u^{(\mu} q_{(i)}^{\nu)} + 2B_{3(i)} u^{(\mu} v_{(i)}^{\nu)} + B_{2(i)} \pi_{(i)}^{\mu\nu}$$

$$D_{1(i)} = \frac{-J_{31(i)}}{J_{31(i)} J_{31(i)} - J_{41(i)} J_{21(i)}}$$

Energy diffusion

$$B_{1(i)} = \frac{J_{21(i)}}{J_{31(i)} J_{31(i)} - J_{41(i)} J_{21(i)}}$$

$$D_{2(i)} = \frac{J_{41(i)}}{J_{31(i)} J_{31(i)} - J_{41(i)} J_{21(i)}}$$

Particle diffusion

$$B_{3(i)} = \frac{-J_{31(i)}}{J_{31(i)} J_{31(i)} - J_{41(i)} J_{21(i)}}$$

Solution

$$f_{(i)}(p) = f_{0(i)} \left[1 + (1 \mp f_{0(i)}) \left(\varepsilon_{(i)} + \varepsilon_{\mu(i)} p_{(i)}^{\mu} + \varepsilon_{\mu\nu(i)} p_{(i)}^{\mu} p_{(i)}^{\nu} \right) \right]$$

$$\varepsilon_{(i)} = E_{0(i)} \Pi_{(i)}$$

$$\varepsilon_{(i)}^{\mu} = D_{0(i)} \Pi_{(i)} u^{\mu} + D_{1(i)} q_{(i)}^{\mu} + D_{2(i)} v_{(i)}^{\mu}$$

$$\varepsilon_{(i)}^{\mu\nu} = B_{0(i)} \left(\Delta^{\mu\nu} - 3u^{\mu} u^{\nu} \right) \Pi_{(i)} + 2B_{1(i)} u^{(\mu} q_{(i)}^{\nu)} + 2B_{3(i)} u^{(\mu} v_{(i)}^{\nu)} + B_{2(i)} \pi_{(i)}^{\mu\nu}$$

$$D_{0(i)} = -3C_{2(i)} B_{0(i)}, \quad E_{0(i)} = -3C_{1(i)} B_{0(i)}$$

-1

$$B_{0(i)} = \frac{-1}{3C_{1(i)} J_{21(i)} + 3C_{2(i)} J_{31(i)} + 3J_{41(i)} + 5J_{42(i)}}$$

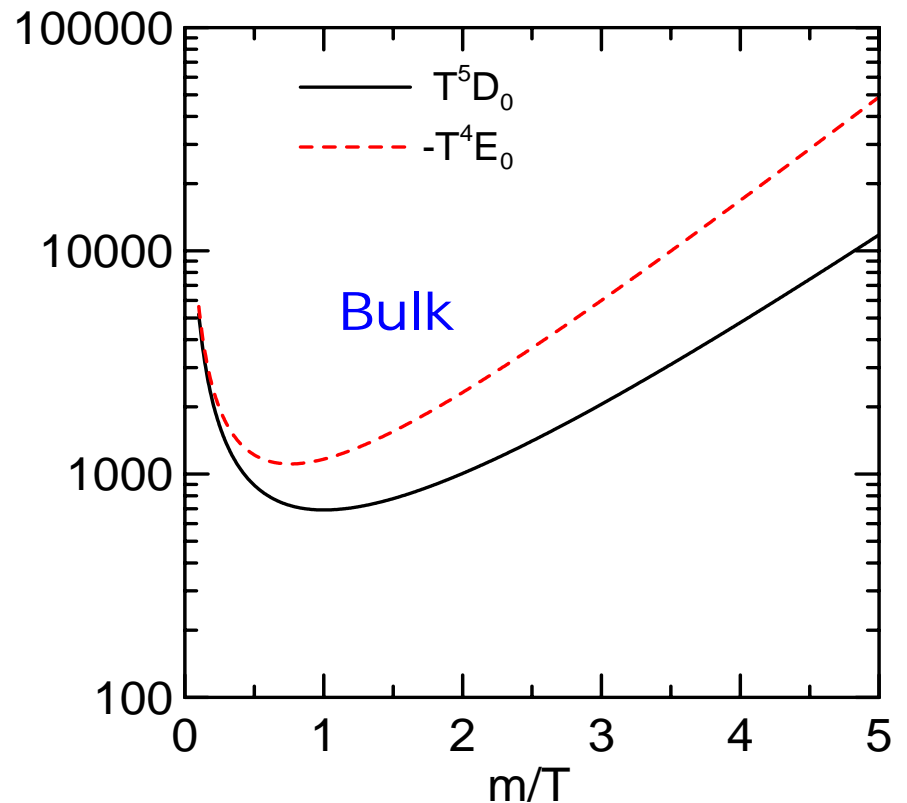
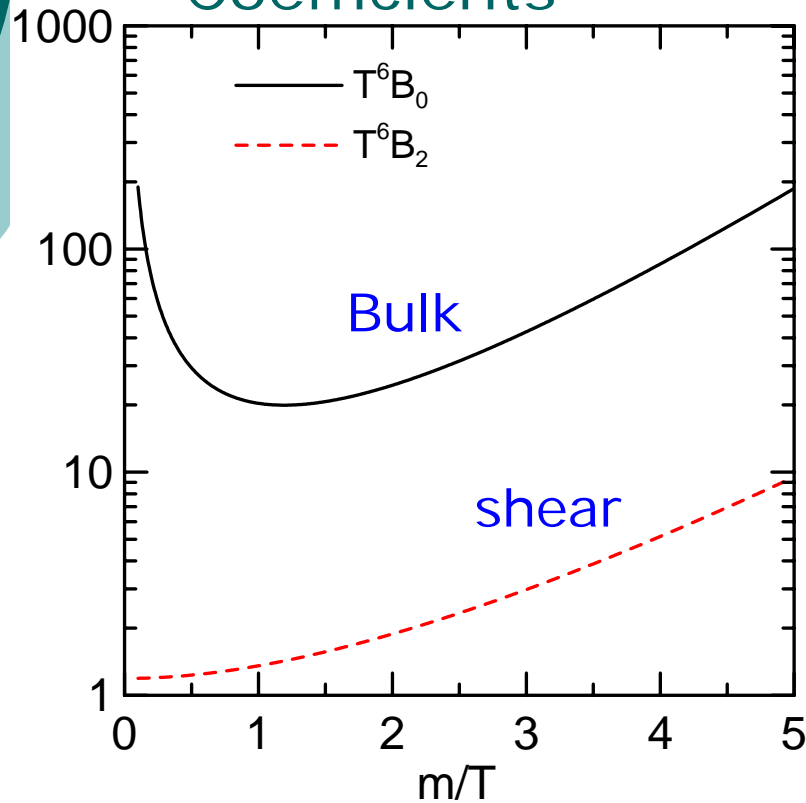
$$C_{1(i)} = -m_{(i)}^2 - 4 \frac{J_{31(i)} J_{30(i)} - J_{41(i)} J_{20(i)}}{J_{30(i)} J_{10(i)} - J_{20(i)} J_{20(i)}}$$

$$C_{2(i)} = 4 \frac{J_{31(i)} J_{20(i)} - J_{41(i)} J_{10(i)}}{J_{30(i)} J_{10(i)} - J_{20(i)} J_{20(i)}}$$

bulk viscosity

Grad's Method

Coefficients



$$B_0, D_0, E_0 \gg B_2$$

Bulk Viscosity

We obtain (for each specie),

$$\delta f_{(i)}(p) = f_{0(i)}(1 \mp f_{0(i)}) \Pi_{(i)} \left(E_{0(i)} + D_{0(i)} E_{p(i)} + B_{0(i)} \left(m_{(i)}^2 - 4E_{p(i)}^2 \right) \right)$$

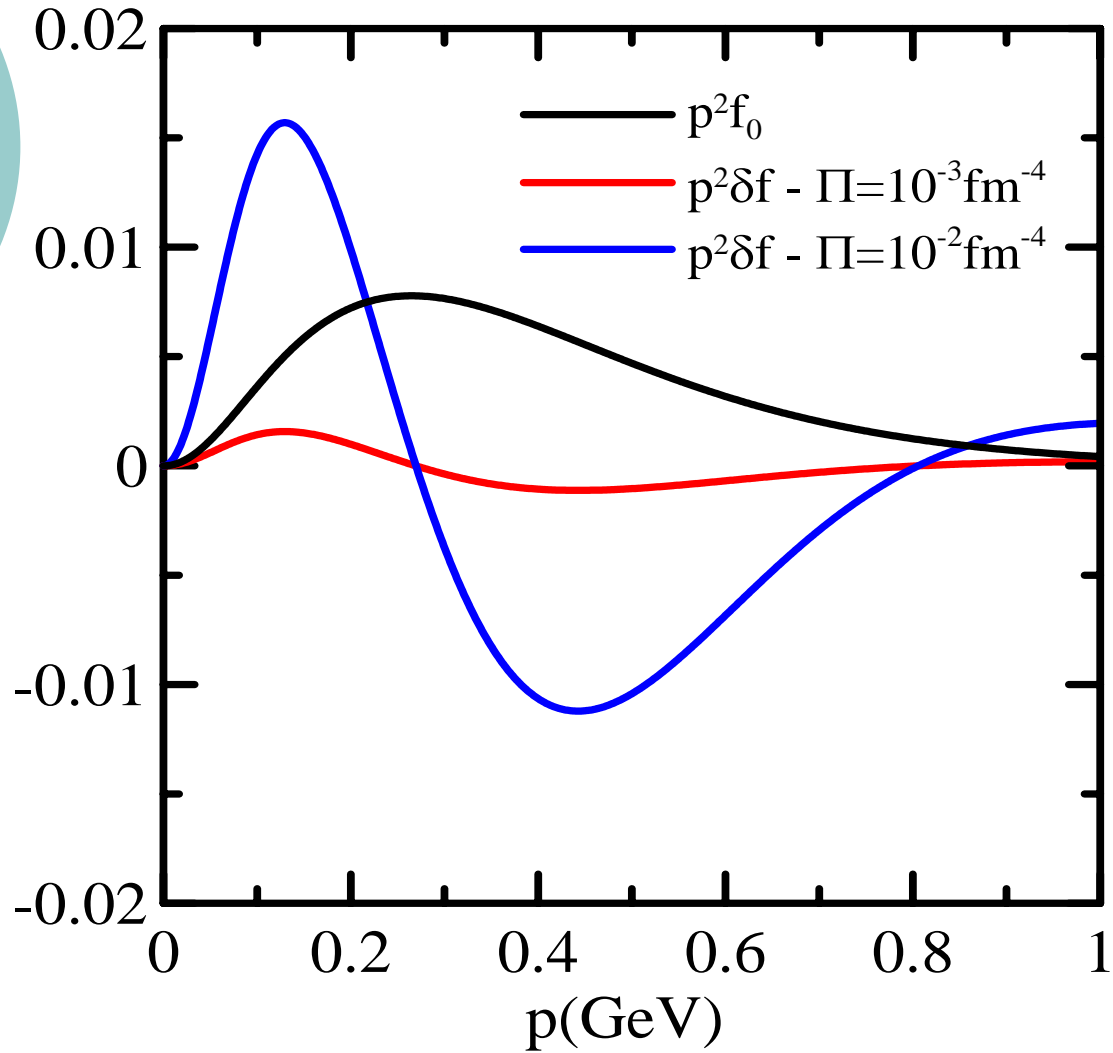
We assume,

$$\frac{\Pi_{(i)}}{\Pi} = \frac{\zeta_{(i)}}{\zeta}$$

with

$$\frac{\zeta_{(i)}}{\zeta} = \frac{s_{(i)}}{s} \sim 0.05$$

Bulk Visc. $p_{(i)}^2 \delta f_{(i)}$



$$\int d^3 p \delta f_{(i)} = 0$$

Equations – What changes ?

Example - Bulk

$$\begin{aligned} & \frac{d\Pi_{(i)}}{d\tau} + \Pi_{(i)} \nabla_{\alpha} u^{\alpha} + \frac{m_{(i)}^2}{3} \int \frac{d^3 k_{(i)}}{(2\pi)^3 \tau_{(i)}^2} C_{(i)}^{(1)} [\{f\}] \\ = & -\beta_{(i)} \partial_{\alpha} u^{\alpha} + \frac{d}{d\tau} \left(\frac{\varepsilon_{(i)} - 3P_{(i)}}{3} \right) - \frac{m_{(i)}^2 \lambda_{\Pi(i)}}{3} \Pi_{(i)} \nabla_{\alpha} u^{\alpha} \\ & + \frac{m_{(i)}^2}{3} \nabla^{\nu} \left(\lambda_{n(i)} n_{\nu(i)} + \lambda_{q(i)} q_{\nu(i)} \right) + \frac{m_{(i)}^2 \lambda_{\pi(i)}}{3} \pi_{\nu\alpha(i)} \nabla^{\nu} u^{\alpha} \end{aligned}$$

Equations – What changes ?

Example - Bulk

$$\begin{aligned}
 & \frac{d\Pi_{(i)}}{d\tau} + \Pi_{(i)} \nabla_{\alpha} u^{\alpha} + \frac{m_{(i)}^2}{3} \int \frac{d^3 k_{(i)}}{(2\pi)^3 \tau_{(i)}^2} C_{(i)}^{(1)} [\{f\}] \\
 = & \underbrace{-\beta_{(i)} \partial_{\alpha} u^{\alpha}} + \underbrace{\frac{d}{d\tau} \left(\frac{\varepsilon_{(i)} - 3P_{(i)}}{3} \right)} - \underbrace{\frac{m_{(i)}^2 \lambda_{\Pi(i)}}{3} \Pi_{(i)} \nabla_{\alpha} u^{\alpha}} \\
 & + \frac{m_{(i)}^2}{3} \nabla^{\nu} (\lambda_{n(i)} n_{\nu(i)} + \lambda_{q(i)} q_{\nu(i)}) + \frac{m_{(i)}^2 \lambda_{\pi(i)}}{3} \pi_{\nu\alpha(i)} \nabla^{\nu} u^{\alpha}
 \end{aligned}$$

Still have to simplify ...

Equations – What changes ?

Example - Bulk

$$\begin{aligned}
 & \frac{d\Pi_{(i)}}{d\tau} + \Pi_{(i)} \nabla_{\alpha} u^{\alpha} + \frac{m_{(i)}^2}{3} \int \frac{d^3 k_{(i)}}{(2\pi)^3 \tau_{(i)}^2} C_{(i)}^{(1)} [\{f\}] \\
 = & -\beta_{(i)} \partial_{\alpha} u^{\alpha} + \frac{d}{d\tau} \left(\frac{\varepsilon_{(i)} - 3P_{(i)}}{3} \right) - \frac{m_{(i)}^2 \lambda_{\Pi(i)}}{3} \Pi_{(i)} \nabla_{\alpha} u^{\alpha} \\
 & + \frac{m_{(i)}^2}{3} \nabla^{\nu} (\lambda_{n(i)} n_{\nu(i)} + \lambda_{q(i)} q_{\nu(i)}) + \frac{m_{(i)}^2 \lambda_{\pi(i)}}{3} \pi_{\nu\alpha(i)} \nabla^{\nu} u^{\alpha}
 \end{aligned}$$

1) The term $\Lambda_{(i)} \Pi_{(i)} + \sum_j \Lambda_{(i)(j)} \Pi_{(j)}$.

Equations – What changes ?

Example - Bulk

Truncated Eq.

$$\frac{d\Pi_{(i)}}{d\tau} + \Lambda_{(i)}\Pi_{(i)} + \sum_j \Lambda_{(i)(j)}\Pi_{(j)} = -\beta_{(i)}\partial_\alpha u^\alpha$$

1) The term $\Lambda_{(i)}\Pi_{(i)} + \sum_j \Lambda_{(i)(j)}\Pi_{(j)}$.

Equations – What changes ?

Example - Bulk

$$\begin{aligned} & \frac{d\Pi_{(i)}}{d\tau} + \Pi_{(i)} \nabla_{\alpha} u^{\alpha} + \frac{m_{(i)}^2}{3} \int \frac{d^3 k_{(i)}}{(2\pi)^3 \tau_{(i)}^2} C_{(i)}^{(1)} [\{f\}] \\ = & -\beta_{(i)} \partial_{\alpha} u^{\alpha} + \frac{d}{d\tau} \left(\frac{\varepsilon_{(i)} - 3P_{(i)}}{3} \right) - \frac{m_{(i)}^2 \lambda_{\Pi(i)}}{3} \Pi_{(i)} \nabla_{\alpha} u^{\alpha} \\ & + \frac{m_{(i)}^2}{3} \nabla^{\nu} (\lambda_{n(i)} n_{\nu(i)} + \lambda_{q(i)} q_{\nu(i)}) + \frac{m_{(i)}^2 \lambda_{\pi(i)}}{3} \pi_{\nu\alpha(i)} \nabla^{\nu} u^{\alpha} \end{aligned}$$

2) Eqs. will depend on $\Pi_{(i)}, q_{(i)}^{\mu}, \pi_{(i)}^{\mu\nu}, v_{(i)}^{\mu}$

~~$\Pi, q^{\mu}, \pi^{\mu\nu}, v_r^{\mu}$ (possible?)~~

Conclusions

- Corrections to the equilibrium distribution function will depend on the irreversible currents for each particle species
- Same will happen for the equations of motion

Problem is not as simple as it
looks

