Riemann problem and kinetic theory as numerical tests for relativistic hydrodynamics

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I. Bouras, E. Molnar, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner, D. H. Rischke, Phys. Rev. Lett. **103**, 032301 (2009)

E. Molnar, H. Niemi and D. H. Rischke, **arXiv:0705.2114 [hep-ph]**

What we do ...

- Solve Israel-Stewart equations numerically
- Test different numerical algorithms in the perfect fluid limit against analytic solution to the Riemann problem
- Compare viscous hydrodynamical solution with the numerical solution of the Boltzmann equation
- Identify the relevant Knudsen number
- Compare 2D algoritm to the 1D algorithm for the same problem.

Israel-Stewart equations

(without bulk viscosity)

$$D\pi^{\mu\nu} = \tfrac{1}{\tau_\pi}\left(\pi^{\mu\nu}_{NS} - \pi^{\mu\nu}\right) - 2\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - (\pi^{\lambda\mu}u^\nu + \pi^{\lambda\nu}u^\mu)Du_\lambda - \tfrac{1}{2}\pi^{\mu\nu}\left(\nabla_\lambda u^\lambda + D\ln\tfrac{\beta_2}{T}\right) - \tfrac{\alpha_1}{\beta_2}\nabla^{\langle\mu}q^{\nu\rangle} + \tfrac{a_1'}{\beta_2}q^{\langle\mu}Du^{\nu\rangle}$$

$$Dq^\mu = \frac{1}{\tau_q}\left(q^\mu_{NS} - q^\mu\right) - \omega^{\mu\lambda}q_\lambda - u^\mu q_\nu Du^\nu - \frac{1}{2}q^\mu\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) + \frac{\alpha_1}{\beta_1}(\partial_\lambda\pi^{\lambda\mu} + u^\mu\pi^{\lambda\nu}\partial_\lambda u_\nu) - \frac{a_1}{\beta_1}\pi^{\lambda\mu}Du_\lambda$$

In (1+1) dimensions only 2 independent dissipative quantities:

$$-2\pi^{xx}_{NS} = \pi_{NS} = -\frac{4}{3}\eta\theta \qquad\qquad \theta = \partial_t\gamma + \partial_z(\gamma v_z)$$

$$q^z_{NS} = \kappa_q\frac{T^2 n}{e+p}\gamma^2\underbrace{\left(v_z\frac{\partial_t\lambda}{\lambda} + \frac{\partial_z\lambda}{\lambda}\right)}_{\lambda = e^{\mu/T}}$$

Israel-Stewart equations in (1+1) dimensions
(*or rather (1+1)-D problem in (3+1)-D*)

$$D\pi = -\frac{1}{\tau_\pi}\left(\pi + \frac{4}{3}\eta\theta\right) - \frac{\pi}{2}\left(\theta + D\ln\frac{\beta_2}{T}\right) + \frac{2}{3}\frac{\alpha_1}{\beta_2}\left(v_z\partial_t q^z + \partial_z q^z - \frac{q^z v_z}{\gamma}\theta\right) + \frac{2}{3}\frac{a_1'}{\beta_2}(q^z\gamma^2)\left(\partial_t v_z + v_z\partial_z v_z\right)$$

$$Dq^z = -\frac{1}{\tau_q}\left[q^z + \kappa_q\frac{T^2 n}{e+p}\nabla^z\left(\frac{\mu}{T}\right)\right] - u^z q_\nu Du^\nu - \frac{1}{2}q^z\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) + \frac{\alpha_1}{\beta_1}(\partial_\lambda\pi^{\lambda z} + u^z\pi^{\lambda\nu}\partial_\lambda u_\nu)$$

This term is problematic if π >> q

Conservation laws:

$$\partial_t N^0 + \partial_z(v_z N^0) = \partial_z\left(\frac{q^z\, n}{\gamma^2(e+p)}\right)$$

$$\partial_t T^{00} + \partial_z(v_z T^{00}) = -\partial_z(v_z P_z)$$

$$\partial_t T^{0z} + \partial_z(v_z T^{0z}) = -\partial_z(P_z)$$

$$P_z = p(e,n) + \pi$$

Shear and bulk viscosity have same effect in (1+1) dimensions

Ultrarelativistic Boltzmann gas

Equation of State:

$$e = \lambda \frac{3g}{\pi^2} T^4 \qquad n = \lambda \frac{g}{\pi^2} T^3 \qquad p = nT \qquad g = 16$$

Coefficients in the IS equations:

$$\beta_2 = \frac{3}{4p} \qquad \alpha_1 = -\frac{1}{4p} \qquad a_1 = 0$$

$$\beta_1 = \frac{5}{4p} \qquad a_1' = 5\alpha_1$$

Assume isotropic 2-2 scatterings:

$$\eta = 0.8T/\sigma^{tr}$$

$$\kappa = 1.3/\sigma^{tr}$$

$$\sigma^{tr} = \frac{2}{3}\sigma$$

Cross section in kinetic theory is fixed in such way that η/s = constant

Numerical methods

Problem in numerical fluid dynamics:
first-order solutions: too much numerical diffusion (but stable)
Second-order solutions: too much numerical dispersion (but no diffusion)

SHASTA (Boris, Book, deVore, Zalesak ...)
1. Calculate low-order solution with strong numerical diffusion.
2. Remove numerical diffusion from the solution as much as possible without generating new structures into solution (Flux limiter).

Amount of residual numerical diffusion after step 2 can be controlled by adjusting antidiffusion coefficient A_{AD} = (0...1)

High resolution central schemes **NT** and **KT**
1. Second order methods
2. Stability is obtained by calculating spatial derivatives using minmod limiters.

These are tested only in the perfect fluid case

NT = Nessyahu & Tadmor
KT = Kurganov & Tadmor

Kinetic theory (BAMPS)

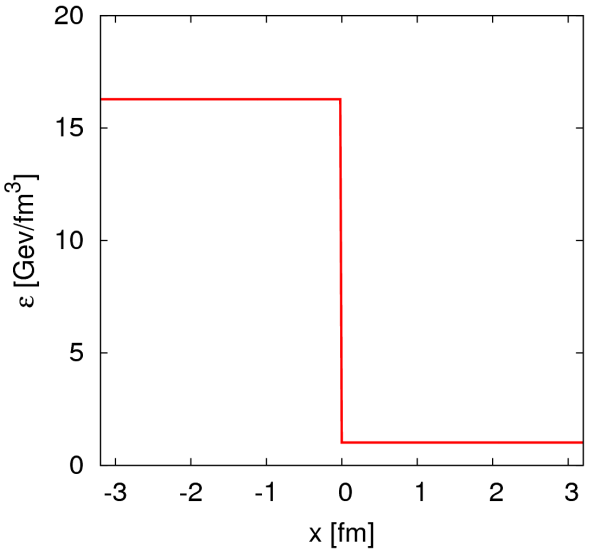
BAMPS (Xu & Greiner) solves numerically Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = C(x,p)$$

$$T^{\mu\nu} = \int \frac{d^3p}{E} p^\mu p^\nu f(x,p)$$

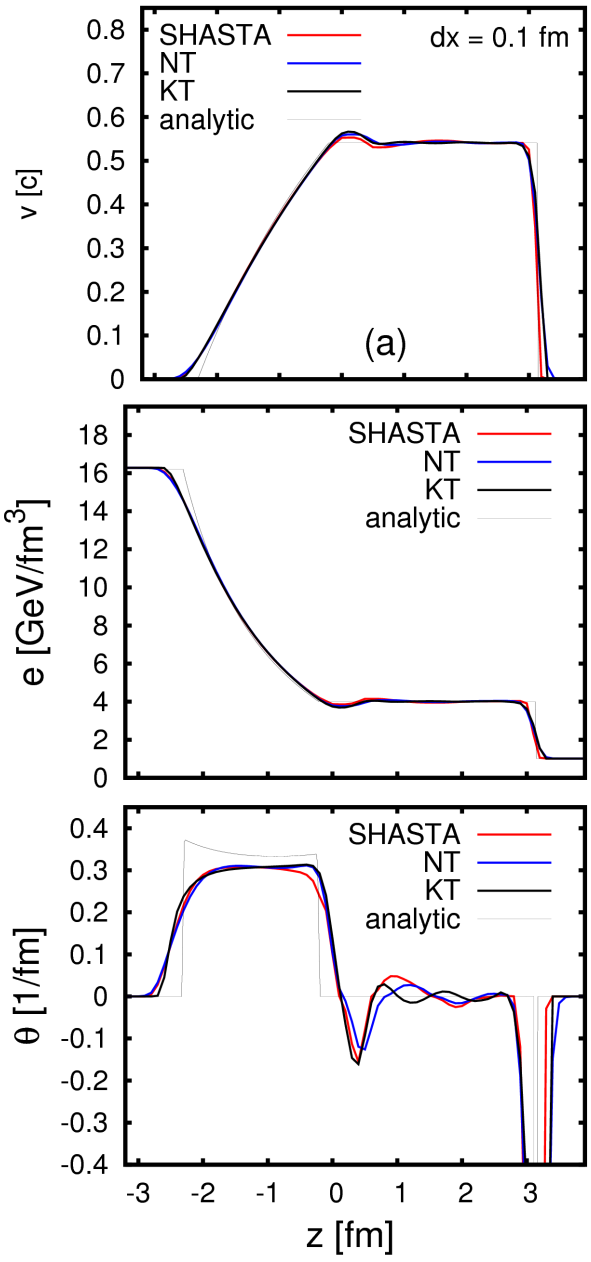
Details: see talk by I. Bouras

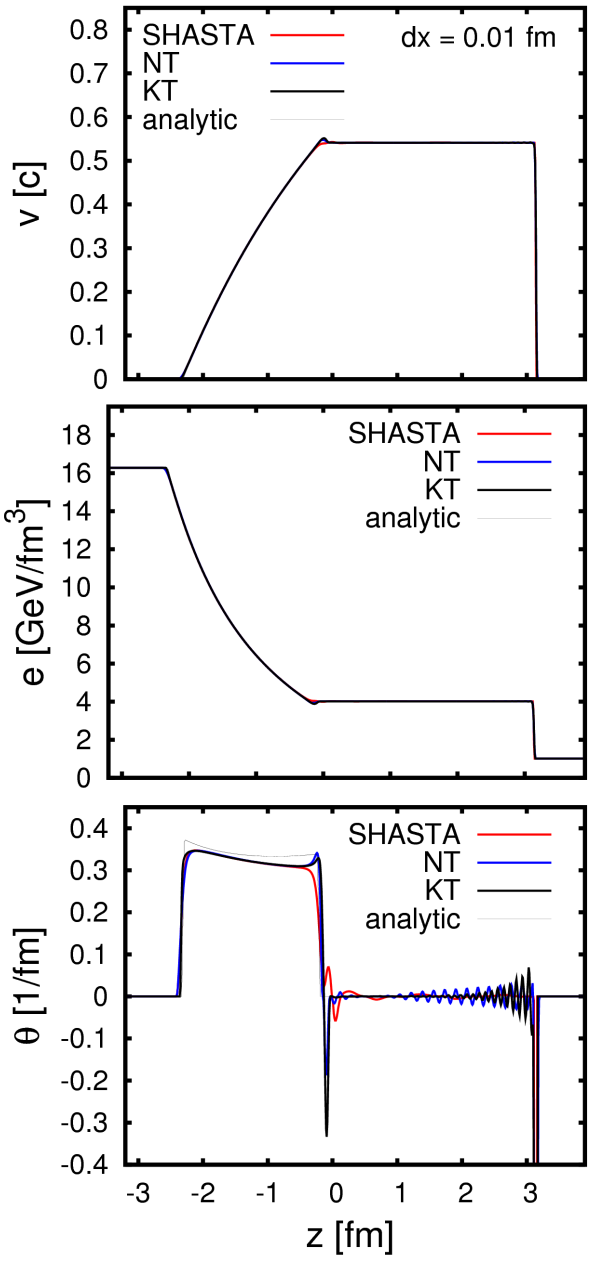
Riemann problem



- Two equilibrium states separated by membrane at t = 0 fm
- equilibrium → q = 0 π = 0
- Initially v = 0
- For perfect fluids problem has analytic solution.

SHASTA vs NT vs KT (perfect fluid)

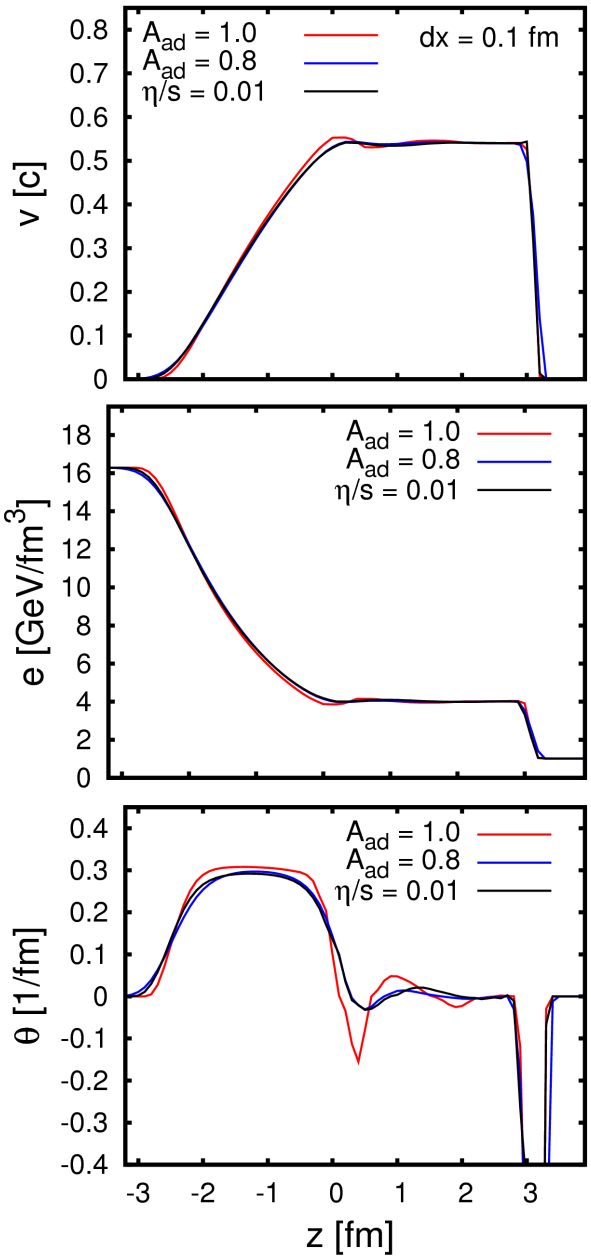


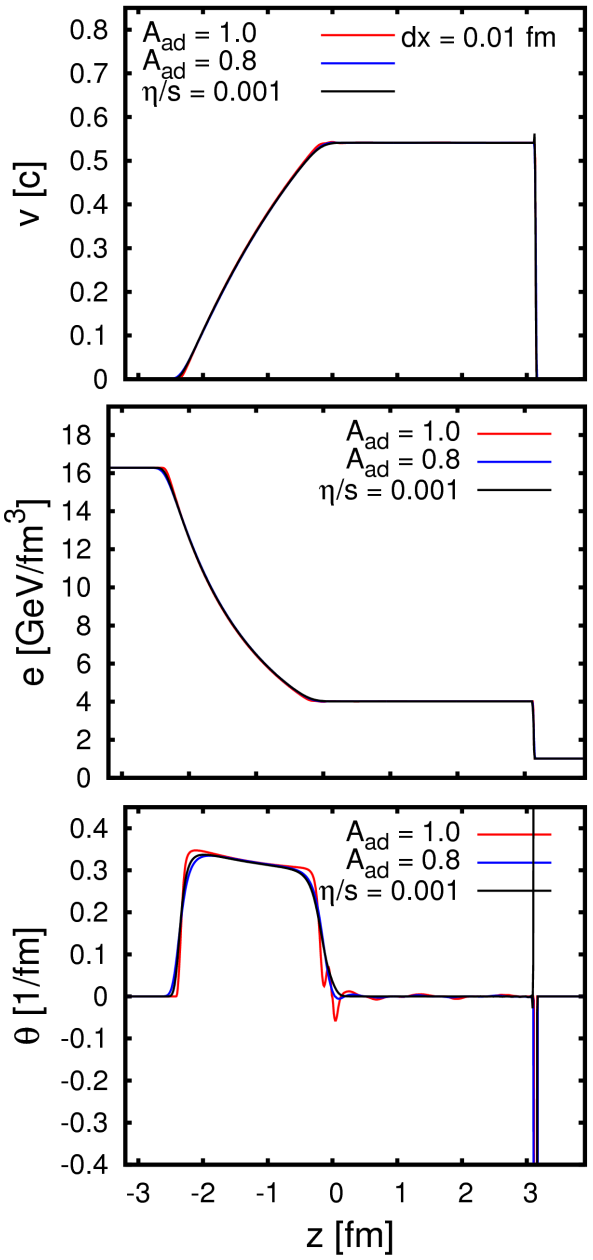


T0/T4 = 0.4 GeV/0.2 GeV
dt/dx = 0.4
t = 4 fm

- Analytic solution reproduced with sufficiently small dx
- Similar performance for all methods

Numerical diffusion in SHASTA

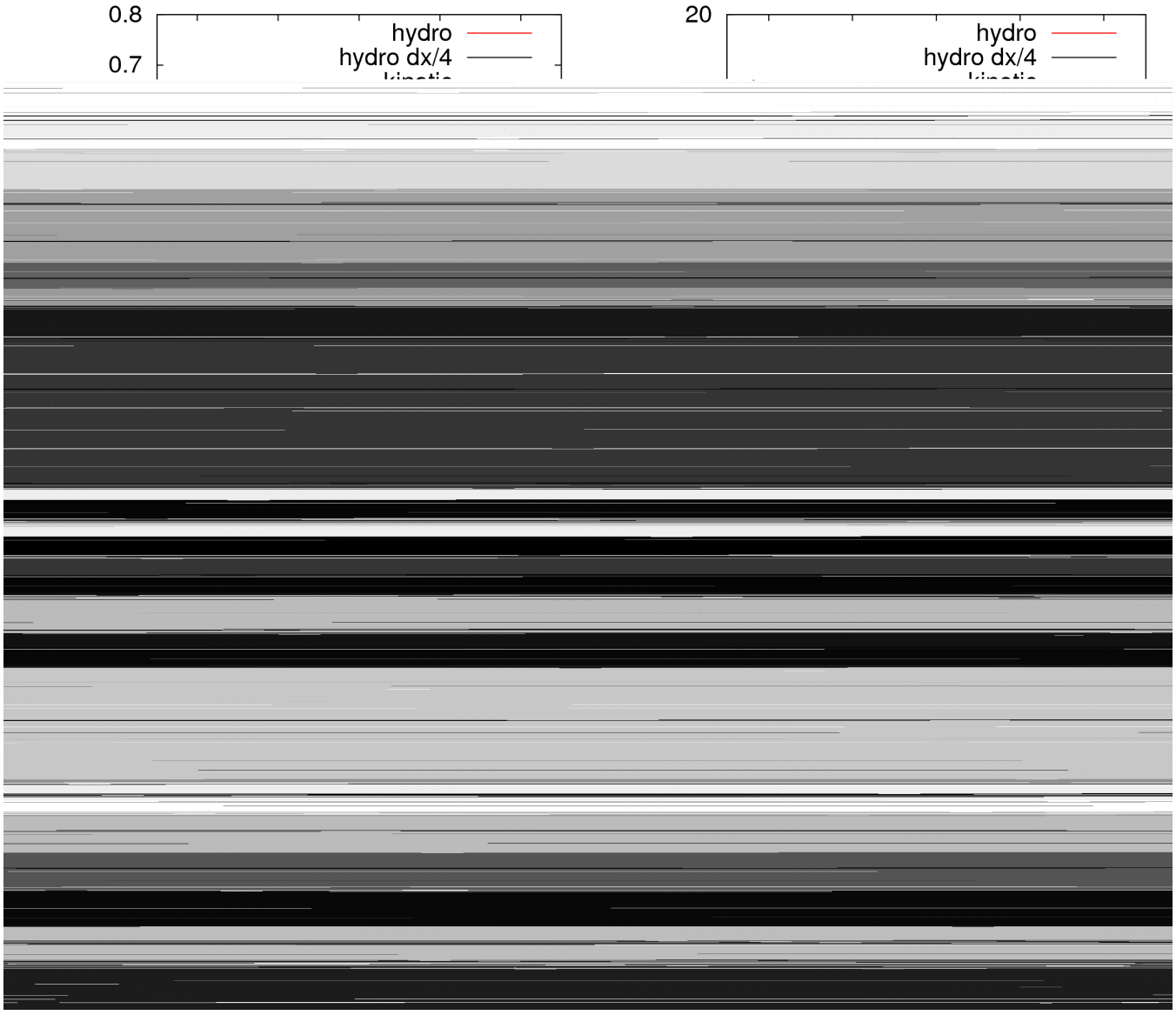




T0/T4 = 0.4 GeV/0.2 GeV
dt/dx = 0.4
t = 4 fm

- Reduction of the anti-diffusion coefficient results in only small increase in diffusion
- Amount of numerical diffusion scales with dx

HYDRO vs BAMPS: low viscosity

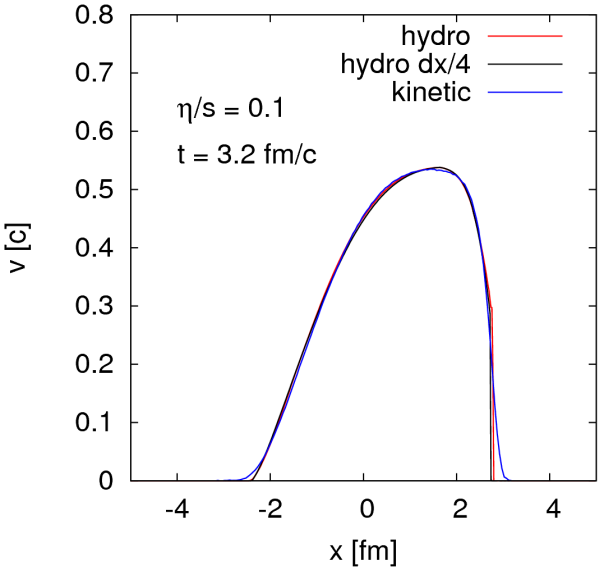


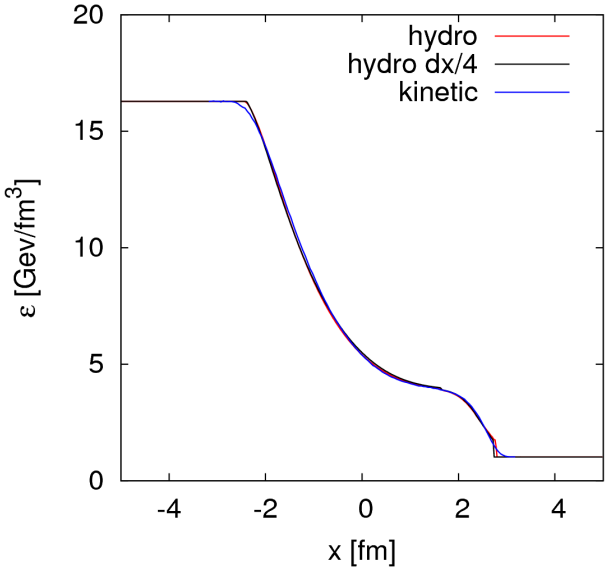
T0/T4 = 0.4 GeV/0.2 GeV
η/s = 0.01

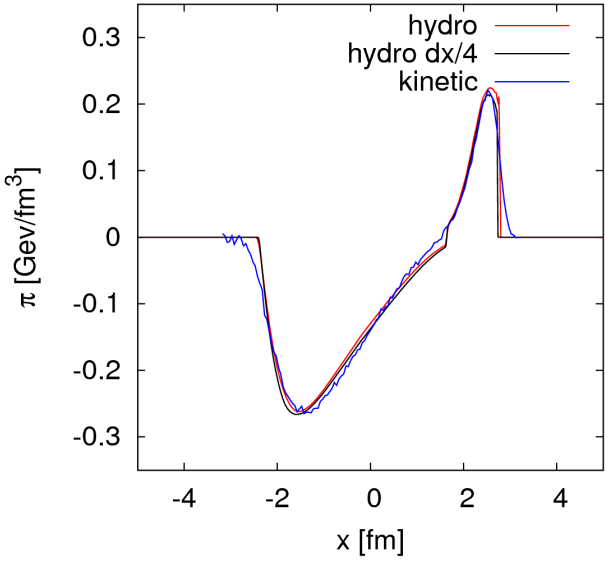
dx = 0.01 fm
dt/dx = 0.4
t = 3.2 fm

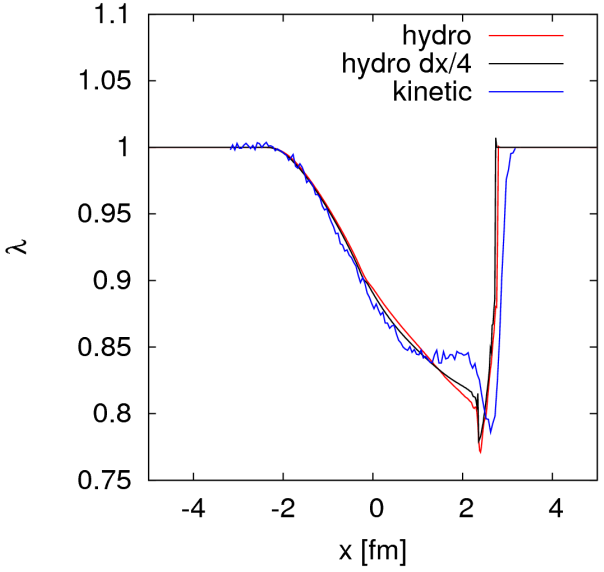
- Excellent agreement with the kinetic theory
- Hydro solution converges when dx → 0
- BAMPS converges when dx → m.f.p.
  (see talk by I. Bouras)

HYDRO vs BAMPS: moderate viscosity









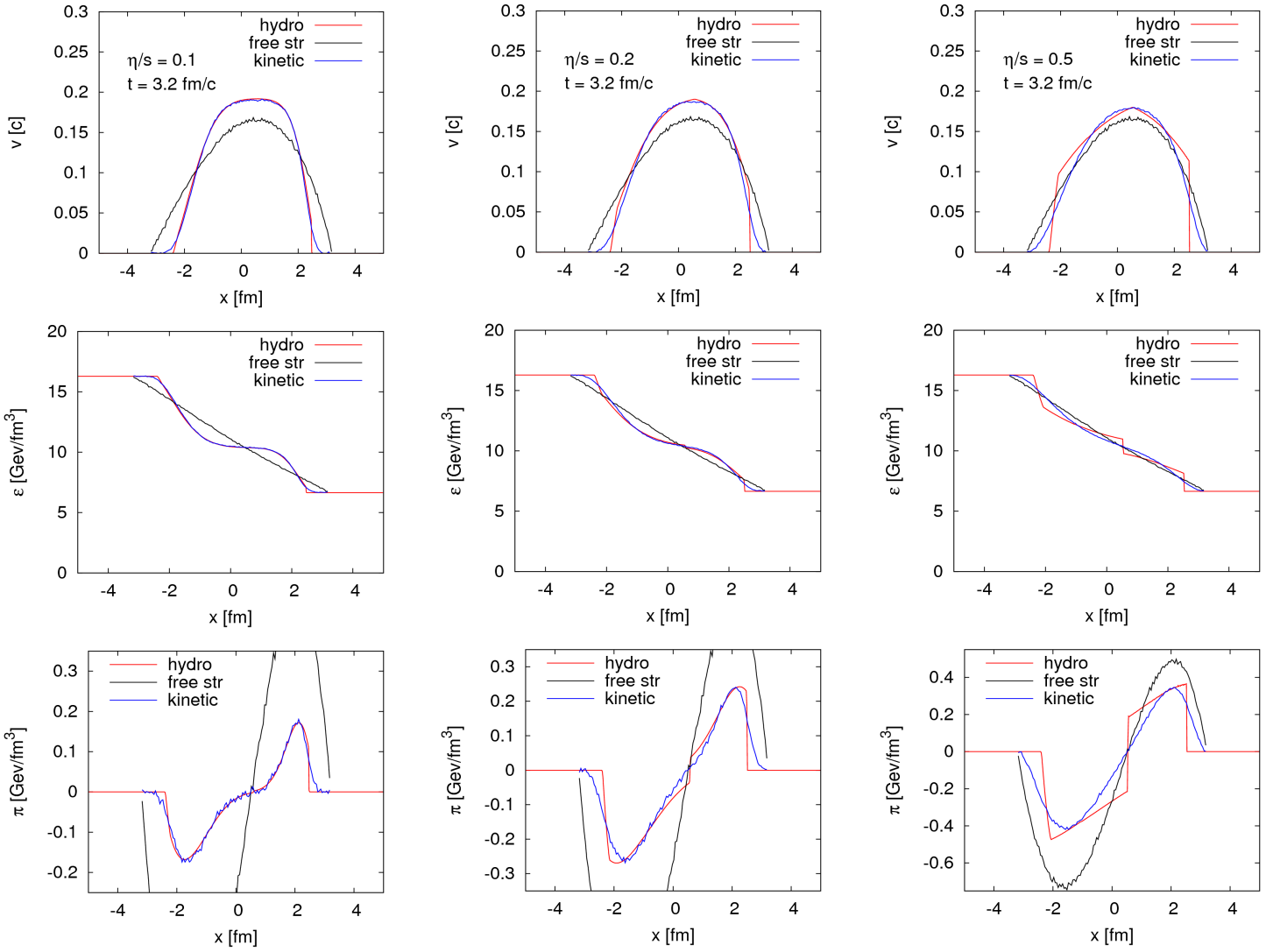
T0/T4 = 0.4 GeV/0.2 GeV
η/s = 0.1

dx = 0.01 fm
dt/dx = 0.4
t = 3.2 fm

- Agreement with the kinetic theory is still good
- Small sensitivity to the numerical parameters, especially near the shock front

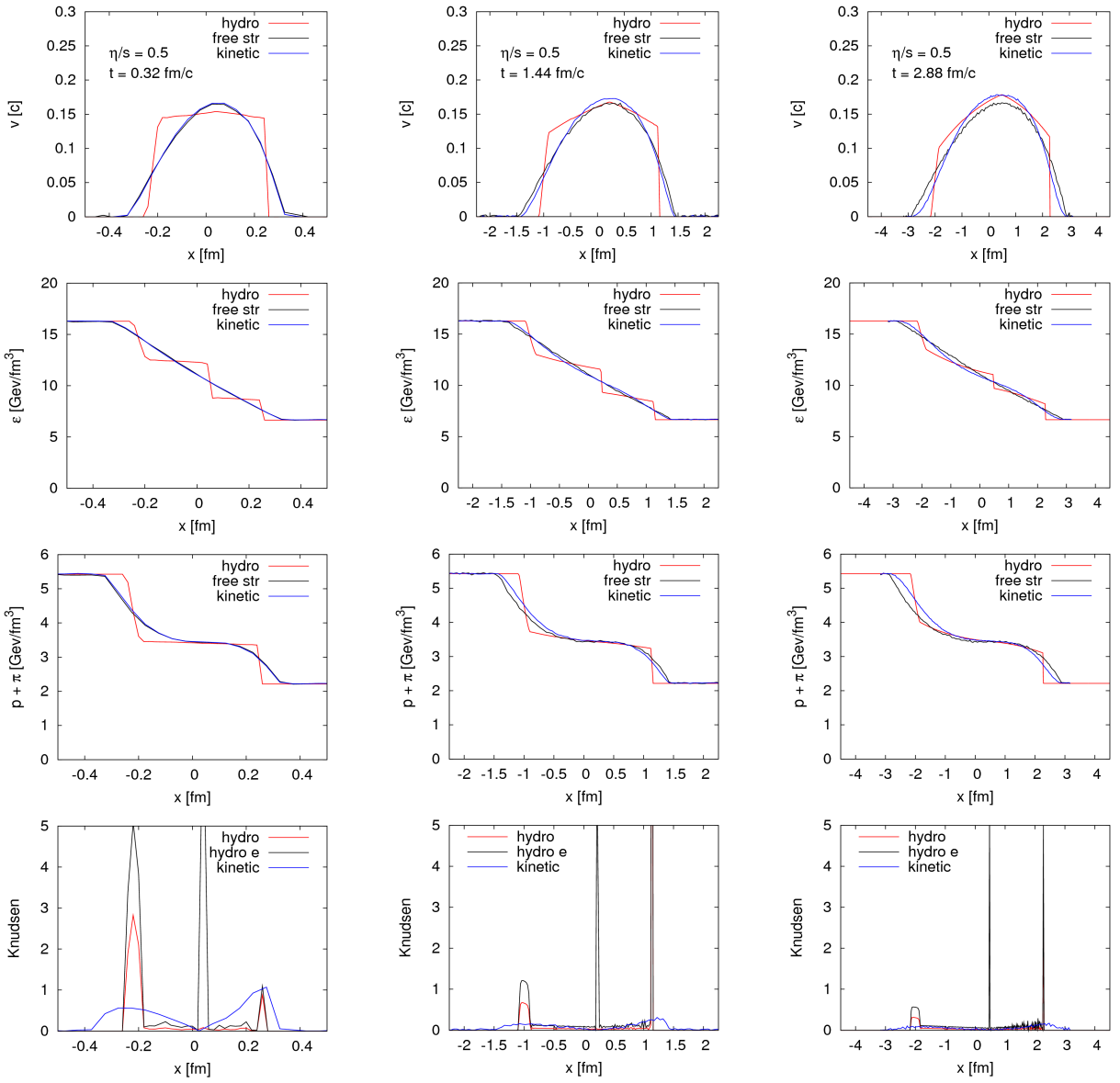
… and when it fails?

$\frac{\eta}{s} = 0.1$ $\frac{\eta}{s} = 0.2$ $\frac{\eta}{s} = 0.5$ $\frac{T_0}{T_4} = 0.4/0.32$



- Kinetic solution approaches free streaming when $\eta \to \infty$
- Hydrodynamic solution approaches something else
- Similar “double shocks” also seen by Koide et. al. (with complete different numerical method)

Time evolution of the failure



- At early times kinetic evolution is indistinguishable from free streaming.
- Viscous hydro evolution is driven by the gradients of effective pressure $p_{eff} = p + \pi$
- In hydro π grows until p_{eff} is constant. Simultaneously constant velocity profile develops
- Physically "reasonable" mechanism, but no support from kinetic theory.

Relevant Knudsen number?

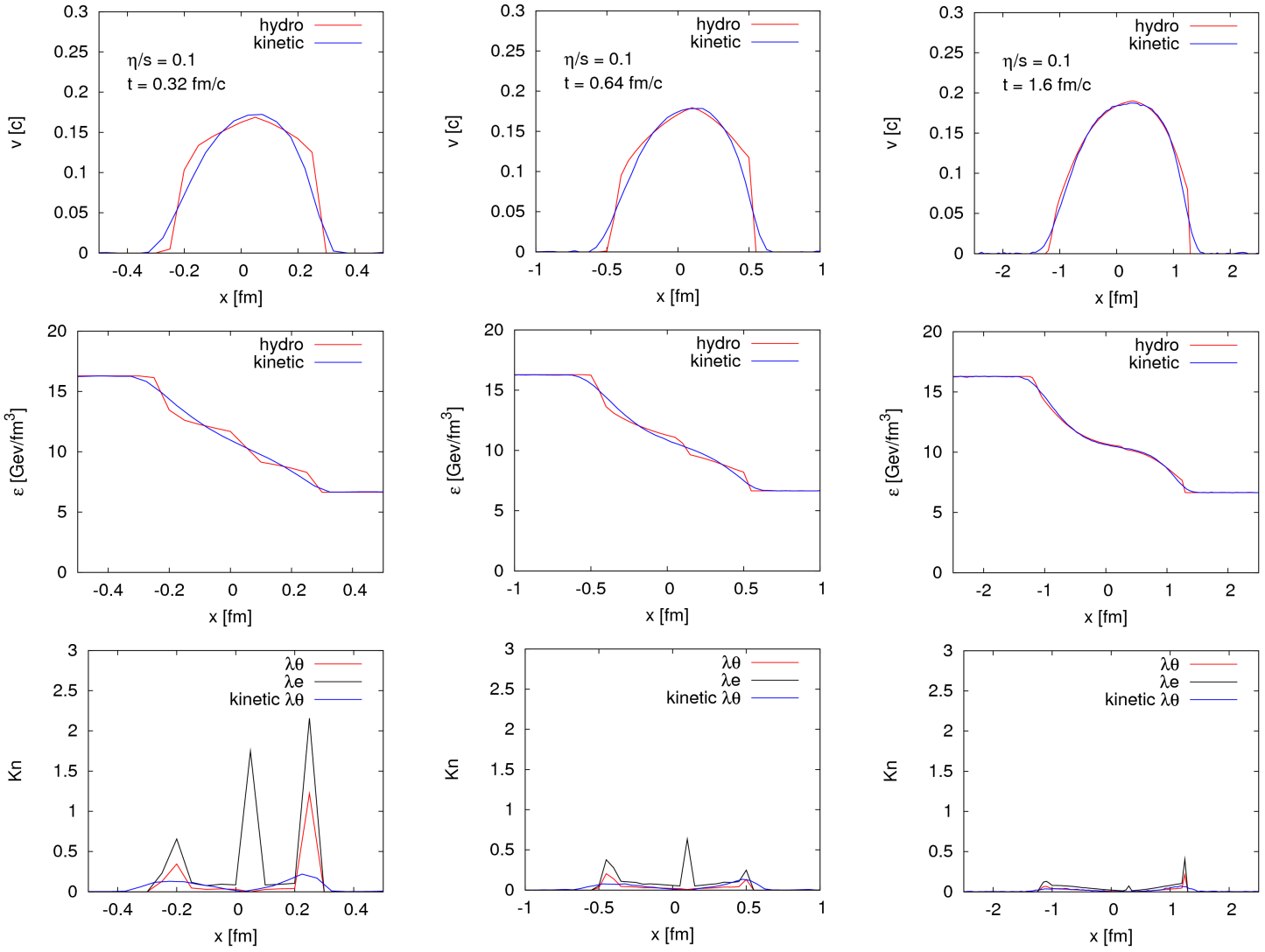
from π_{NS}: $K_n = \lambda_{mfp}\theta$
(remains small at z~0)

or $K_n = \frac{\lambda_{mfp}}{e}\sqrt{\nabla_\mu e \nabla^\mu e}$

(doesn't appear in IS eqs.)

Time evolution for η/s = 0.1

$$\frac{T_0}{T_4} = 0.4/0.32$$



- At early times Kn always large
- With small viscosity hydro solution approaches kinetic solution fast.

When we expect agreement

- IS eqs. can be thought as an expansion in Knudsen number $Kn = \frac{\lambda_{mfp}}{L}$
- All terms in IS eqs. are 2nd order in Kn (if $\pi \sim \pi_{NS}$)
- D. Molnar & P. Huovinen: Scaling flow → Kn < 0.5

$$D\pi^{\mu\nu} = \frac{1}{\tau_\pi}\left(\pi^{\mu\nu}_{NS} - \pi^{\mu\nu}\right) - 2\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - (\pi^{\lambda\mu}u^\nu + \pi^{\lambda\nu}u^\mu)Du_\lambda - \frac{1}{2}\pi^{\mu\nu}\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_2}{T}\right) - \frac{\alpha_1}{\beta_2}\nabla^{\langle\mu}q^{\nu\rangle} + \frac{a_1'}{\beta_2}q^{\langle\mu}Du^{\nu\rangle}$$

or in (1+1) dimensions:

$$D\pi = -\frac{1}{\tau_\pi}\left(\pi + \frac{4}{3}\eta\theta\right) - \frac{\pi}{2}\left(\theta + D\ln\frac{\beta_2}{T}\right) + \frac{2}{3}\frac{\alpha_1}{\beta_2}\left(v_z\partial_t q^z + \partial_z q^z - \frac{q^z v_z}{\gamma}\theta\right) + \frac{2}{3}\frac{a_1'}{\beta_2}(q^z\gamma^2)\left(\partial_t v_z + v_z\partial_z v_z\right)$$

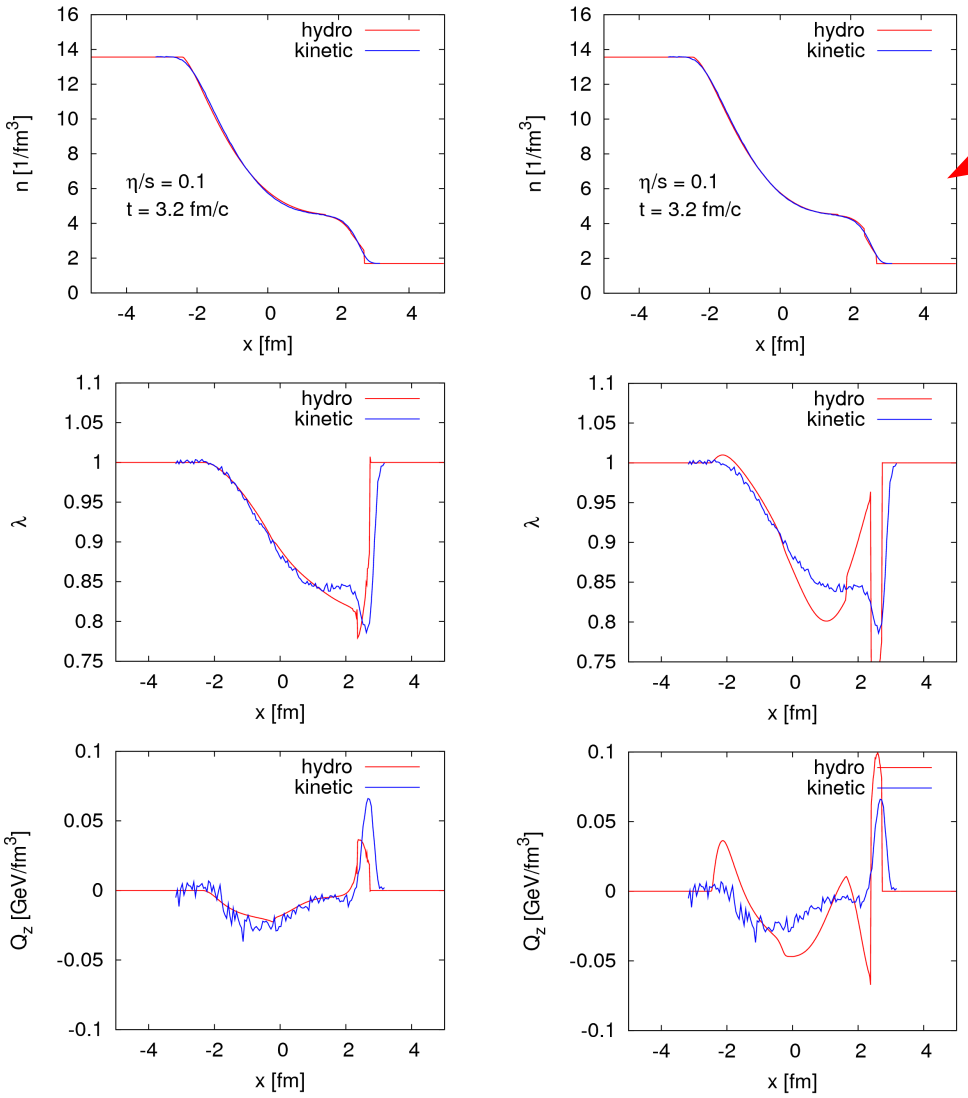
Macroscopic length scales:

$1/L = \theta$ or $1/L = \frac{1}{e}\sqrt{\nabla_\mu e \nabla^\mu e}$ → This appears only with q $Du^\mu \sim \frac{\nabla^\mu p}{e+p}$

need 3rd order theory with $\pi^{\mu\nu}Du^\alpha Du^\beta$?

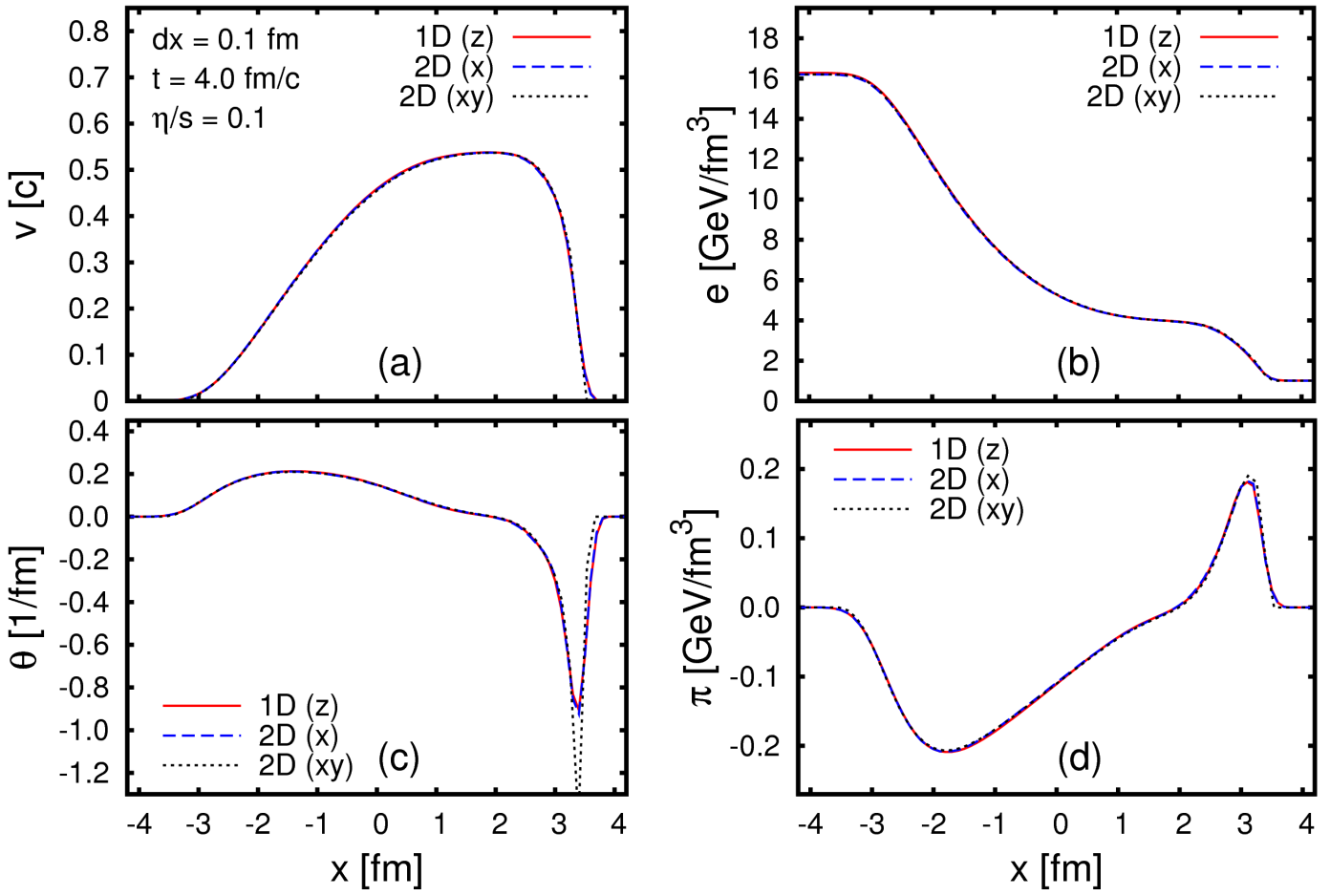
Heat

$$Dq^z = -\frac{1}{\tau_q}\left[q^z + \kappa_q \frac{T^2 n}{e+p}\nabla^z\left(\frac{\mu}{T}\right)\right] - u^z q_\nu Du^\nu - \frac{1}{2}q^z\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) + \frac{\alpha_1}{\beta_1}(\partial_\lambda \pi^{\lambda z} + u^z \pi^{\lambda\nu}\partial_\lambda u_\nu)$$



- Reasonable agreement with kinetic theory, **but** only if coupling between q and π is neglected
- If q << π last term can have big contribution to q even near the equilibrium
- However, the IS theory used here is not complete 2nd order theory: There exist more 2nd order terms that couple q and π. (talk by D. Rischke)
- Will these terms cure this problem?

(2+1) dimensional implementation



- Same (1+1) dimensional Riemann problem solved with (2+1) dimensional code
- Initial discontinuity orientated different ways in xy-plane: along the y-axis “2D (x)” or along the y = -x line “2D(xy)”
- Along the axis perfect agreement
- Slight differences in the xy direction.

Summary

- We have tested different numerical schemes in the perfect fluid limit.
- All methods give comparable results
- In low viscosity limit we have excellent agreement with the kinetic theory
- Breaking of the IS theory in high viscosity can be understood in terms of the Knudsen number
- Two dimensional code can reproduce one dimensional results with good accuracy