Complete second-order dissipative relativistic fluid dynamics

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Preliminaries (I)

Tensor decomposition of net charge current and energy-momentum tensor:

1. **Net charge current:** $$N^\mu = n\, u^\mu + \nu^\mu$$

   u^μ fluid 4-velocity, $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

   $g_{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$ (West coast!!) metric tensor,

   $n \equiv u^\mu N_\mu$ net charge density in fluid rest frame

   $\nu^\mu \equiv \Delta^{\mu\nu} N_\nu$ diffusion current (flow of net charge relative to u^μ), $\nu^\mu u_\mu = 0$

   $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ projector onto 3-space orthogonal to u^μ, $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:** $$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \Pi)\, \Delta^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}$$

   $\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$ energy density in fluid rest frame

   p pressure in fluid rest frame

   Π bulk viscous pressure, $p + \Pi \equiv -\frac{1}{3}\, \Delta^{\mu\nu} T_{\mu\nu}$

   $q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$ heat flux current (flow of energy relative to u^μ), $q^\mu u_\mu = 0$

   $\pi^{\mu\nu} \equiv T^{<\mu\nu>}$ shear stress tensor, $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0\,,\ \pi^\mu{}_\mu = 0$

   $a^{(\mu\nu)} \equiv \frac{1}{2}\,(a^{\mu\nu} + a^{\nu\mu})$ symmetrized tensor

   $a^{<\mu\nu>} \equiv \left(\Delta_\alpha{}^{(\mu} \Delta^{\nu)}{}_\beta - \frac{1}{3}\, \Delta^{\mu\nu} \Delta_{\alpha\beta}\right) a^{\alpha\beta}$ symmetrized, traceless spatial projection

Preliminaries (II)

Fluid dynamical equations:

1. **Net charge (e.g., baryon, strangeness, etc.) conservation:**

$$\boxed{\partial_\mu N^\mu = 0} \iff \boxed{\dot{n} + n\,\theta + \partial \cdot \nu = 0}$$

$\dot{a} \equiv u^\mu \partial_\mu a$ convective (comoving) derivative

(fluid rest frame, $u^\mu_{\rm RF} \equiv g^\mu{}_0 \implies$ time derivative, $\dot{a}_{\rm RF} \equiv \partial_t a$)

$\theta \equiv \partial_\mu u^\mu$ expansion scalar

2. **Energy-momentum conservation:**

$$\boxed{\partial_\mu T^{\mu\nu} = 0} \iff \text{energy conservation:}$$

$$\boxed{u_\nu\,\partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\,\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0}$$

acceleration equation:

$$\Delta^{\mu\nu}\,\partial^\lambda T_{\nu\lambda} = 0 \iff$$

$$(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q^\nu\partial_\nu u^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ 3-gradient (spatial gradient in fluid rest frame)

Preliminaries (III)

Problem:

5 equations, but 15 unknowns (for given u^μ): ϵ , p , n , Π , ν^μ (3) , q^μ (3) , $\pi^{\mu\nu}$ (5)

Solution:

1. clever choice of frame (Eckart, Landau,...): eliminate ν^μ or q^μ

   $\Longrightarrow$ does not help! Promotes u^μ to dynamical variable!

2. ideal fluid limit: all dissipative terms vanish, $\Pi = \nu^\mu = q^\mu = \pi^{\mu\nu} = 0$

   $\Longrightarrow$ 6 unknowns: ϵ , p , n , u^μ (3) (not quite there yet...)

   $\Longrightarrow$ fluid is in local thermodynamical equilibrium

   $\Longrightarrow$ provide equation of state (EOS) $p(\epsilon, n)$ to close system of equations

3. provide additional equations for dissipative quantities

   $\Longrightarrow$ dissipative relativistic fluid dynamics

   (a) First-order theories: e.g. generalization of Navier-Stokes (NS) equations to the relativistic case (Eckart, Landau-Lifshitz)

   (b) Second-order theories: e.g. Israel-Stewart (IS) equations

Preliminaries (IV)

Navier-Stokes (NS) equations:

1. **bulk viscous pressure:** $\boxed{\Pi_{\rm NS} = -\zeta\,\theta}$

   ζ bulk viscosity

2. **heat flux current:** $\boxed{q^\mu_{\rm NS} = \frac{\kappa}{\beta}\frac{n}{\beta(\epsilon+p)}\nabla^\mu\alpha}$

   $\beta \equiv 1/T$ inverse temperature,

   $\alpha \equiv \beta\,\mu$, μ chemical potential,

   κ thermal conductivity

3. **shear stress tensor:** $\boxed{\pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}}$

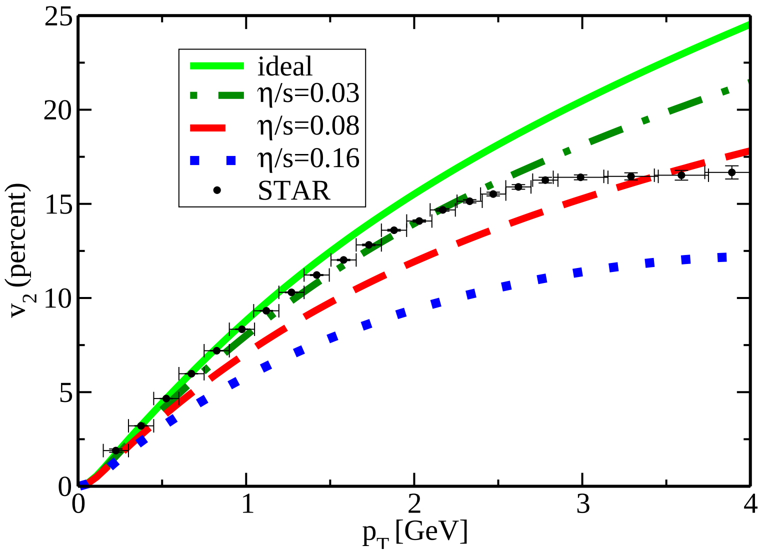
   η shear viscosity,

   $\sigma^{\mu\nu} = \nabla^{<\mu}u^{\nu>}$ shear tensor

$\Longrightarrow$ algebraic expressions in terms of thermodynamic and fluid variables

$\Longrightarrow$ simple... but: unstable and acausal equations of motion!!

Motivation (I)

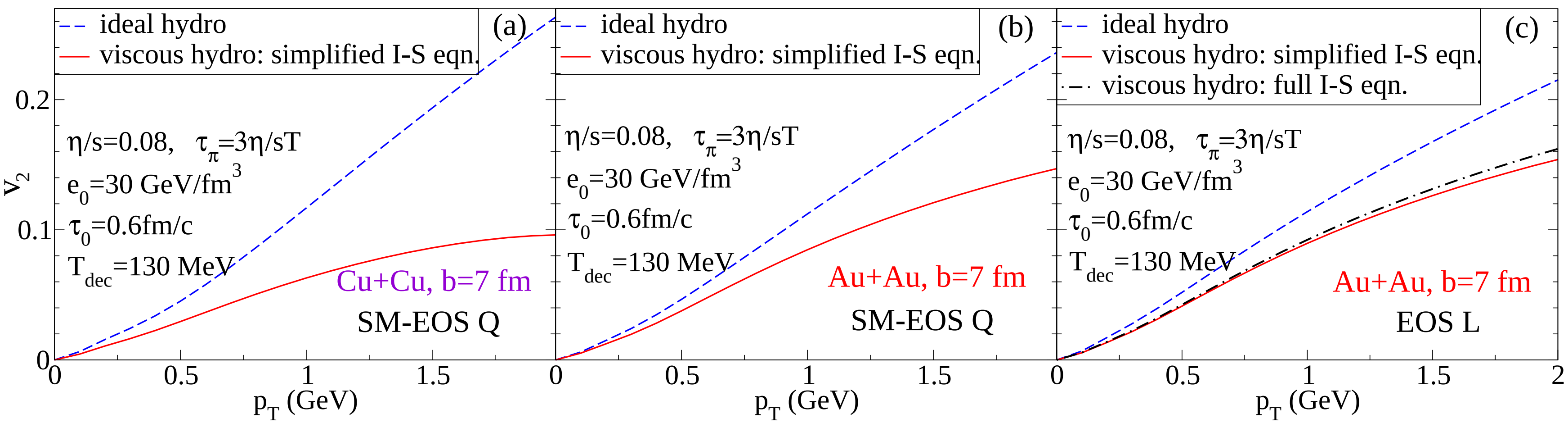


P. Romatschke, U. Romatschke, PRL 99 (2007) 172301

Au+Au @ $\sqrt{s} = 200$ AGeV

charged particles, min. bias

H. Song, U.W. Heinz, PRC 78 (2008) 024902



Motivation (II)

Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

"Simplified" IS equations: e.g. shear stress tensor

$$\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\text{NS}}$$

- $\Longrightarrow$ **dynamical (instead of algebraic) equations for dissipative terms!**
- $\Longrightarrow$ **$\pi^{\mu\nu}$ relaxes to its NS value $\pi^{\mu\nu}_{\text{NS}}$ on the time scale τ_π**
- $\Longrightarrow$ **stable and causal fluid dynamical equations of motion!**

"Full" IS equations:

$$\tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{\text{NS}} - \frac{\eta}{2\beta}\, \pi^{\mu\nu}\, \partial_\lambda \left(\frac{\tau_\pi}{\eta}\, \beta\, u^\lambda \right) + 2\, \tau_\pi\, \pi_\lambda{}^{<\mu} \omega^{\nu>\lambda}$$

$$\omega^{\mu\nu} \equiv \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta - \partial_\beta u_\alpha \right) \qquad \textbf{vorticity}$$

Motivation (III)

$\Longrightarrow$ **Difference between "simplified" and "full" IS equations: the latter include higher-order terms?**

For instance, if $\dfrac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1\,, \quad \tau_\pi\, \omega^{\mu\nu} \sim \delta \ll 1 \quad \Longrightarrow \quad \tau_\pi\, \omega_\lambda{}^{<\mu} \pi^{\nu>\lambda}\, \dfrac{1}{\epsilon} \sim \delta^2$

$\Longrightarrow$ **Goals:**

1. **What are the correct equations of motion for the dissipative quantities?**
   $\Longrightarrow$ **develop consistent power counting scheme**
2. **Generalization to $\mu \neq 0$ (relevant for FAIR physics!)**
   $\Longrightarrow$ **include heat flux q^μ**
3. **Generalization to non-conformal fluids (relevant near T_c!)**
   $\Longrightarrow$ **include bulk viscous pressure Π**

Results (I)

Power counting:

3 length scales: 2 microscopic, 1 macroscopic

- **thermal wavelength** $\lambda_{\rm th} \sim \beta \equiv 1/T$
- **mean free path** $\ell_{\rm mfp} \sim (\langle \sigma \rangle n)^{-1}$

  $\langle \sigma \rangle$ **averaged cross section,** $n \sim T^3 = \beta^{-3} \sim \lambda_{\rm th}^{-3}$
- **length scale over which macroscopic fluid fields vary** $L_{\rm hydro}$, $\partial_\mu \sim L_{\rm hydro}^{-1}$

Note: since $\eta \sim (\langle \sigma \rangle \lambda_{\rm th})^{-1}$ $\Longrightarrow$ $$\boxed{\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{1}{\langle \sigma \rangle n} \frac{1}{\lambda_{\rm th}} \sim \frac{\lambda_{\rm th}^3}{\langle \sigma \rangle \lambda_{\rm th}} \sim \frac{\lambda_{\rm th}^3}{\langle \sigma \rangle \lambda_{\rm th}} \sim \frac{\eta}{s}}$$

s **entropy density,** $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\rm th}^{-3}$

$\Longrightarrow$ $\dfrac{\eta}{s}$ **solely determined by the 2 microscopic length scales!**

Note: similar argument holds for $\dfrac{\zeta}{s}$, $\dfrac{\kappa}{\beta\, s}$

Results (II)

3 regimes:

- dilute gas limit $\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{\eta}{s} \gg 1 \iff \langle \sigma \rangle \ll \lambda_{\rm th}^2 \implies$ weak-coupling limit
- viscous fluids $\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{\eta}{s} \sim 1 \iff \langle \sigma \rangle \sim \lambda_{\rm th}^2$
  interactions happen on the scale $\lambda_{\rm th} \implies$ moderate coupling
- ideal fluid limit $\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{\eta}{s} \ll 1 \iff \langle \sigma \rangle \gg \lambda_{\rm th}^2 \implies$ strong-coupling limit

gradient (derivative) expansion: $$\ell_{\rm mfp}\, \partial_\mu \sim \frac{\ell_{\rm mfp}}{L_{\rm hydro}} \equiv K \sim \delta \ll 1$$

K Knudsen number

$\implies$ equivalent to $k\, \ell_{\rm mfp} \ll 1$, k typical momentum scale

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

$\implies$ separation of macroscopic fluid dynamics (large scale $\sim L_{\rm hydro}$)
from microscopic particle dynamics (small scale $\sim \ell_{\rm mfp}$)

Results (III)

Primary quantities: $\epsilon\,,\ p\,,\ n\,,\ s$ $\Longleftrightarrow$ **Dissipative quantities:** $\Pi\,,\ q^\mu\,,\ \pi^{\mu\nu}$

If $K \sim \ell_{\rm mfp}\,\partial_\mu \sim \delta \ll 1$ **, then** $\dfrac{\Pi}{\epsilon} \sim \dfrac{q^\mu}{\epsilon} \sim \dfrac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$

Dissipative quantities are small compared to primary quantities

$\Longrightarrow$ **small deviations from local thermodynamical equilibrium!**

Note: statement independent of value of $\dfrac{\zeta}{s}\,,\ \dfrac{\kappa}{\beta\, s}\,,\ \dfrac{\eta}{s}$ **!**

Proof: **Gibbs relation:** $\epsilon + p = Ts + \mu n \Longrightarrow \beta\,\epsilon \sim s$ **!**

Estimate dissipative terms by their Navier-Stokes values:

$$\Pi \sim \Pi_{\rm NS} = -\zeta\,\theta\,,\quad q^\mu \sim q^\mu_{\rm NS} = \frac{\kappa}{\beta}\,\frac{n}{\beta(\epsilon+p)}\,\nabla^\mu\alpha\,,\quad \pi^{\mu\nu} \sim \pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}$$

$$\Longrightarrow\ \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\,\epsilon}\,\beta\,\theta \sim -\frac{\zeta}{s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\theta \sim \ell_{\rm mfp}\,\partial_\mu u^\mu \sim \delta\,,$$

$$\frac{q^\mu}{\epsilon} \sim \frac{\kappa}{\beta}\,\frac{1}{\beta\,\epsilon}\,\frac{n}{\beta(\epsilon+p)}\,\beta\,\nabla^\mu\alpha \sim \frac{\kappa}{\beta\,s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\nabla^\mu\alpha \sim \ell_{\rm mfp}\,\nabla^\mu\alpha \sim \delta\,,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2\,\frac{\eta}{\beta\,\epsilon}\,\beta\,\sigma^{\mu\nu} \sim 2\,\frac{\eta}{s}\,\frac{\beta}{\lambda_{\rm th}}\,\frac{\lambda_{\rm th}}{\ell_{\rm mfp}}\,\ell_{\rm mfp}\,\sigma^{\mu\nu} \sim \ell_{\rm mfp}\,\nabla^{<\mu}u^{\nu>} \sim \delta\,,\quad \text{q.e.d.}$$

Results (IV)

IS equations:

$$\begin{aligned}
\tau_\Pi \, \dot{\Pi} \; + \; \Pi \; &= \; \Pi_{\rm NS} + \tau_{\Pi q} \, q \cdot \dot{u} - \ell_{\Pi q} \, \partial \cdot q \\
& \quad - \tau_\Pi \, \frac{\hat{\zeta}_1}{\zeta} \, \Pi^2 \; - \; \tau_\Pi \, \frac{\hat{\zeta}_2 \, \beta}{\kappa} \, q \cdot q \; - \; \tau_\Pi \, \frac{\hat{\zeta}_3}{2\,\eta} \, \pi^{\mu\nu} \pi_{\mu\nu} \\
\tau_q \, \Delta^{\mu\nu} \dot{q}_\nu \; + \; q^\mu \; &= \; q^\mu_{\rm NS} - \tau_{q\Pi} \, \Pi \, \dot{u}^\mu \; - \tau_{q\pi} \, \pi^{\mu\nu} \, \dot{u}_\nu \\
& \quad + \ell_{q\Pi} \, \nabla^\mu \Pi \; - \; \ell_{q\pi} \, \Delta^{\mu\nu} \, \partial^\lambda \pi_{\nu\lambda} \; + \; \tau_q \, \omega^{\mu\nu} \, q_\nu \\
& \quad - \tau_q \, \frac{\hat{\kappa}_1}{\zeta} \, q^\mu \, \Pi \; - \; \tau_q \, \frac{\hat{\kappa}_2}{2\,\eta} \, \pi^{\mu\nu} \, q_\nu \\
\tau_\pi \, \dot{\pi}^{<\mu\nu>} \; + \; \pi^{\mu\nu} \; &= \; \pi^{\mu\nu}_{\rm NS} + 2\, \tau_{\pi q} \, q^{<\mu} \dot{u}^{\nu>} + 2\, \ell_{\pi q} \, \nabla^{<\mu} q^{\nu>} \; + \; 2\, \tau_\pi \, \pi_\lambda{}^{<\mu} \omega^{\nu>\lambda} \\
& \quad - 2\, \tau_\pi \, \frac{\hat{\eta}_1}{2\,\eta} \, \pi_\lambda{}^{<\mu} \pi^{\nu>\lambda} \; - \; 2\, \tau_\pi \, \frac{\hat{\eta}_2 \, \beta}{\kappa} \, q^{<\mu} q^{\nu>} \; - \; 2\, \tau_\pi \, \frac{\hat{\eta}_3}{\zeta} \, \Pi \, \pi^{\mu\nu}
\end{aligned}$$

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

A. Muronga, PRC 76 (2007) 014909 (and parts of $\hat{\zeta}_1$, $\hat{\kappa}_1$, $\hat{\eta}_3$)

B. Betz, D. Henkel, DHR, Prog. Part. Nucl. Phys. 62 (2009) 556

B. Betz, T. Koide, H. Niemi, DHR, in preparation

Results (V)

Remarks:

1. **Structure of second-order terms follows exclusively from Lorentz covariance**
2. **Coefficients can be computed from kinetic theory and Grad's 14-moment method** B. Betz, H. Niemi, T. Koide, DHR, in preparation
3. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100: **second-order fluid dynamics for conformal fluids (AdS/CFT correspondence)**
   $\Longrightarrow$ **second-order terms (in flat space):**
   $$4\,\lambda_1\,\sigma_\lambda{}^{<\mu}\sigma^{\nu>\lambda} - 2\,\lambda_2\,\sigma_\lambda{}^{<\mu}\omega^{\nu>\lambda} + \lambda_3\,\omega_\lambda{}^{<\mu}\omega^{\nu>\lambda}$$
   to 2nd order: may use $\pi^{\mu\nu} \simeq \pi^{\mu\nu}_{\rm NS} = 2\,\eta\,\sigma^{\mu\nu}$ **to replace** $\sigma^{\mu\nu}$ **with** $\pi^{\mu\nu}$
   $\Longrightarrow$ **in kinetic theory:** $\lambda_1 \equiv -\tau_\pi\,\eta\,\hat{\eta}_1\ ,\quad \lambda_2 \equiv -2\tau_\pi\eta\ ,\quad \lambda_3 \equiv 0$
   Note: second-order terms from collision integral $\Longrightarrow$ $\hat{\eta}_1 \neq 1$!
   cf. M.A. York, G.D. Moore, arXiv:0811.0729

Results (VI)

4. **P. Romatschke, arXiv:0906.4787:**
   second-order fluid dynamics for nonconformal fluids without conserved charges
   $\Longrightarrow$ second-order terms (in flat space):
   $4\,\xi_1\,\sigma^{\mu\nu}\sigma_{\mu\nu} + \xi_2\,\theta^2 + \xi_3\,\omega^{\mu\nu}\omega_{\mu\nu} + \xi_4\,\nabla^\mu \ln s \nabla_\mu \ln s$
   to 2nd order: may use $\pi^{\mu\nu} \simeq 2\,\eta\,\sigma^{\mu\nu}$, $\Pi \simeq -\zeta\,\theta$
   and 0th order acceleration eq. $\dot{u}^\mu \simeq \dfrac{\nabla^\mu p}{\epsilon + p} = c_s^2 \nabla^\mu \ln s$
   $\Longrightarrow$ in kinetic theory: $\xi_1 \equiv -\frac{1}{2}\,\tau_\Pi\,\eta\,\hat{\zeta}_3$, $\xi_2 \equiv -\tau_\Pi\,\zeta\,\hat{\zeta}_1$, $\xi_3 \equiv \xi_4 \equiv 0$
5. Coefficients $\hat{\zeta}_1$, $\hat{\zeta}_2$, $\hat{\zeta}_3$, $\hat{\kappa}_1$, $\hat{\kappa}_2$, $\hat{\eta}_1$, $\hat{\eta}_2$, $\hat{\eta}_3$ are (complicated) dimensionless functions of α , β
6. Viscosities and thermal conductivity ζ , η , κ , relaxation times τ_Π , τ_q , τ_π , coefficients $\tau_{\Pi q}$, $\tau_{q\Pi}$, $\tau_{q\pi}$, $\tau_{\pi q}$, $\ell_{\Pi q}$, $\ell_{q\Pi}$, $\ell_{q\pi}$, $\ell_{\pi q}$ are (complicated) functions of α , β, divided by tensor coefficients of second moment of collision integral:
   $\sim \chi_i(\alpha,\beta)/\langle\sigma\rangle \to 0$ as cross section $\sigma \to \infty$ ("strong coupling limit"!)
   $\Longrightarrow$ $\Pi = q^\mu = \pi^{\mu\nu} \to 0$ ideal fluid limit!

Results (VII)

7. **IS equations are formally independent of calculational frame (Eckart, Landau,...), but ...**
8. **Values of coefficients are frame dependent! We have analyzed:**

   (a) **Eckart (N) or (net) charge frame:** $\boxed{\nu^\mu = 0\,,\ \epsilon = \epsilon_0\,,\ n = n_0}$

   $\epsilon_0\,,\ n_0$: **energy density and charge density in local thermodyn. equilibrium**

   (b) **Landau (E) or energy frame:** $\boxed{q^\mu = 0\,,\ \epsilon = \epsilon_0\,,\ n = n_0}$

   Note: in IS equations $\quad q^\mu \equiv -\dfrac{\epsilon + p}{n}\,\nu^\mu$

   (c) **Tsumura-Kunihiro-Ohnishi (TKO) frame:** $\boxed{\nu^\mu = 0\,,\ \epsilon = \epsilon_0 - 3\,\Pi\,,\ n = n_0}$

   We have checked agreement with the results of IS for most coefficients computed by IS...
9. **R.h.s.: all terms except NS terms are of second order,** $\sim \delta^2$

   $\Longrightarrow$ $t < \tau_\Pi \sim \tau_q \sim \tau_\pi$: **dissipative terms relax towards their NS values,**

   $t > \tau_\Pi \sim \tau_q \sim \tau_\pi$: **last terms on r.h.s. and NS terms on l.h.s. largely cancel, second-order terms govern evolution!**

Conclusions and open problems

1. **Derived Israel-Stewart (IS) equations from kinetic theory via Grad's 14-moment method $\Longrightarrow$ new second-order terms!**

2. **Results consistent with**

   R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

   M.A. York, G.D. Moore, arXiv:0811.0729

3. **Coefficients of terms in IS equations are frame dependent**

   $\Longrightarrow$ **have (not yet completely) been computed in various frames (Eckart, Landau, TKO)**

4. **Generalization to a system of various particle species (done: quarks, antiquarks, gluons), various conserved charges**

   cf. M. Prakash, M. Prakash, R. Venugopalan, G. Welke, Phys. Rept. 227 (1993) 321

   G. Denicol, DHR, in preparation

5. **Numerical implementation**

   E. Molnar, H. Niemi, DHR, in preparation

Derivation: Grad's 14-moment method (I)

1. history: derivation of IS equations from kinetic theory
   - H. Grad, Commun. Pure Appl. Math. 2 (1949) 381
   - J.M. Stewart, Proc. Roy. Soc. London Ser. A 357 (1977) 59
   - W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341
   - DHR, PhD thesis, 1992 (unpublished)
   - R. Baier, P. Romatschke, U.A. Wiedemann, PRC 73 (2006) 064903
   - A. Muronga, PRC 76 (2007) 014909, 014910

2. single-particle distribution function in local thermodynamical equilibrium:

$$f_0 = (e^{-y_0} + a)^{-1}\,, \quad y_0 = \alpha_0 - \beta_0\, K \cdot u$$

$a = \pm 1, 0$ for fermions/bosons, Boltzmann particles

local: $\alpha_0 \equiv \alpha_0(X)\,,\ \beta_0 \equiv \beta_0(X)\,,\ u^\mu \equiv u^\mu(X)\,,\ X^\mu = (t, \vec{x})$

single-particle distribution function in local non-equilibrium:

$$f = (e^{-y} + a)^{-1}\,, \quad y = \alpha - \beta K \cdot u - K \cdot v + K^\mu K^\nu w_{\mu\nu}$$

$v \cdot u = 0\,,\ w^\mu{}_\mu = 0$

Derivation: Grad's 14-moment method (II)

3. assume small deviations from equilibrium:

$$y - y_0 = \alpha - \alpha_0 - (\beta - \beta_0)\, K \cdot u - K \cdot v + K^\mu K^\nu w_{\mu\nu} \sim \delta \ll 1$$

$$\Longleftrightarrow \quad \alpha - \alpha_0 \sim \beta - \beta_0 \sim v^\mu \sim w^{\mu\nu} \sim \delta$$

$\Longrightarrow$ expand f around f_0 to linear order in $y - y_0$:

$$f \simeq f_0 + f_0(1 - a\, f_0)(y - y_0) + O(\delta^2)$$

4. compute N^μ and $T^{\mu\nu}$ as moments of f:

$$N^\mu = \int \mathrm{d}\tilde{K}\, K^\mu\, f = I_0^\mu + (\alpha - \alpha_0)\, J_0^\mu - (\beta - \beta_0)\, J_0^{\mu\nu} u_\nu - J_0^{\mu\nu} v_\nu + J_0^{\mu\nu\lambda} w_{\nu\lambda}$$
$$T^{\mu\nu} = \int \mathrm{d}\tilde{K}\, K^\mu K^\nu\, f = I_0^{\mu\nu} + (\alpha - \alpha_0)\, J_0^{\mu\nu} - (\beta - \beta_0)\, J_0^{\mu\nu\lambda} u_\lambda - J_0^{\mu\nu\lambda} v_\lambda + J_0^{\mu\nu\lambda\rho} w_{\lambda\rho}$$

where $\int \mathrm{d}\tilde{K} \equiv \frac{g}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{k}}{E}$, $E = \sqrt{k^2 + m^2}$, g no. of internal d.o.f.'s, and

$$I_0^{\alpha_1 \cdots \alpha_n} \equiv \int \mathrm{d}\tilde{K}\, K^{\alpha_1} \cdots K^{\alpha_n}\, f_0\,,$$
$$J_0^{\alpha_1 \cdots \alpha_n} \equiv \int \mathrm{d}\tilde{K}\, K^{\alpha_1} \cdots K^{\alpha_n}\, f_0(1 - a f_0)$$

Note: $I_0^\mu \equiv N_0^\mu$ net charge current in local thermodyn. equilibrium

$I_0^{\mu\nu} \equiv T_0^{\mu\nu}$ energy-momentum tensor in local thermodyn. equilibrium

Derivation: Grad's 14-moment method (III)

5. tensor decomposition of $I_0^{\alpha_1\cdots\alpha_n}$ up to $n=3$ and of $J_0^{\alpha_1\cdots\alpha_n}$ up to $n=5$:

$$I_0^\mu = I_{10}\, u^\mu\,, \qquad J_0^\mu = J_{10}\, u^\mu\,,$$

$$I_0^{\mu\nu} = I_{20}\, u^\mu u^\nu + I_{21}\, \Delta^{\mu\nu}\,, \qquad J_0^{\mu\nu} = J_{20}\, u^\mu u^\nu + J_{21}\, \Delta^{\mu\nu}\,,$$

$$I_0^{\mu\nu\lambda} = I_{30}\, u^\mu u^\nu u^\lambda + 3\, I_{31}\, u^{(\mu}\Delta^{\nu\lambda)} \qquad J_0^{\mu\nu\lambda} = J_{30}\, u^\mu u^\nu u^\lambda + 3\, J_{31}\, u^{(\mu}\Delta^{\nu\lambda)}$$

$$J_0^{\mu\nu\lambda\rho} = J_{40}\, u^\mu u^\nu u^\lambda u^\rho + 6\, J_{41}\, u^{(\mu}u^\nu\Delta^{\lambda\rho)} + 3\, J_{42}\, \Delta^{\mu(\nu}\Delta^{\lambda\rho)}$$

$$J_0^{\mu\nu\lambda\rho\sigma} = J_{50}\, u^\mu u^\nu u^\lambda u^\rho u^\sigma + 10\, J_{51}\, u^{(\mu}u^\nu u^\lambda\Delta^{\rho\sigma)} + 15\, J_{52}\, u^{(\mu}\Delta^{\nu\lambda}\Delta^{\rho\sigma)}$$

Note: $I_{10} \equiv n_0\,,\ I_{20} \equiv \epsilon_0\,,\ I_{21} \equiv -p_0$

In general:
$$I_{nq} \equiv \frac{1}{(2q+1)!!}\int \mathrm{d}\tilde{K}\, E^{n-2q}(-k^2)^q\, f_0$$
$$J_{nq} \equiv \frac{1}{(2q+1)!!}\int \mathrm{d}\tilde{K}\, E^{n-2q}(-k^2)^q\, f_0(1-af_0)$$

6. insert tensor decomposition for $I_0^{\alpha_1\cdots\alpha_n}$, $J_0^{\alpha_1\cdots\alpha_n}$, as well as

$$w_{\mu\nu} = w\, \left(u_\mu u_\nu - \tfrac{1}{3}\,\Delta_{\mu\nu}\right) + 2\, w_{(\mu}u_{\nu)} + \tilde{w}_{\mu\nu}$$

$$w_\mu u^\mu = \tilde{w}_{\mu\nu}u^\nu = \tilde{w}_{\mu\nu}u^\mu = 0\,;\ \tilde{w}_{<\mu\nu>} = \tilde{w}_{\mu\nu}$$

into expansion for N^μ, $T^{\mu\nu}$, cf. 4.

Derivation: Grad's 14-moment method (IV)

7. choose specific frame, i.e., impose matching conditions (cf. Results (VII), 8.)

$$\Longrightarrow \quad \boxed{\alpha - \alpha_0 = w\,\chi_\alpha(\alpha,\beta)\,,\ \beta - \beta_0 = w\,\chi_\beta(\alpha,\beta)\,,\ v^\mu = w^\mu \chi_v(\alpha,\beta)}$$

8. re-insert into tensor decomposition for N^μ , $T^{\mu\nu}$

$$\Longrightarrow \quad \boxed{w = \Pi\,\chi_\Pi(\alpha,\beta)\,, \quad w^\mu = q^\mu\,\chi_q(\alpha,\beta)\,, \quad \tilde{w}^{\mu\nu} = \pi^{\mu\nu}\,\chi_\pi(\alpha,\beta)}$$

9. compute third moment of f:

$$\boxed{\begin{aligned} S^{\mu\nu\lambda} &= \int \mathrm{d}\tilde{K}\, K^\mu K^\nu K^\lambda\, f \\ &= S_1\, u^\mu u^\nu u^\lambda + 3\, S_2\, u^{(\mu}\Delta^{\nu\lambda)} + 3\,\psi_1\, q^{(\mu}u^\nu u^{\lambda)} + 3\,\psi_2\, q^{(\mu}\Delta^{\nu\lambda)} + 3\,\psi_3\,\pi^{(\mu\nu}u^{\lambda)} \end{aligned}}$$

$$S_1 = I_{30} + \psi_4\,\Pi\,,\ S_2 = I_{31} - \tfrac{1}{3}\psi_4\,\Pi\,, \quad \psi_i = \psi_i(\alpha,\beta)\,,\ i = 1,\ldots,4$$

10. compute second moment of collision integral $\mathcal{C}[f]$: to first order in δ

$$\boxed{C^{\nu\lambda} = \int \mathrm{d}\tilde{K}\, K^\nu K^\lambda\, \mathcal{C}[f] = A\,\Pi\,\left(u^\nu u^\lambda - \tfrac{1}{3}\,\Delta^{\nu\lambda}\right) + B\,\pi^{\nu\lambda} + 2\,C\,u^{(\nu}q^{\lambda)}}$$

$$A\,,\ B\,,\ C \sim \langle\sigma\rangle$$

Derivation: Grad's 14-moment method (V)

11. compute second moment of Boltzmann equation, $K \cdot \partial f = \mathcal{C}[f]$:

$\Longrightarrow$ $\partial_\mu S^{\mu\nu\lambda} = C^{\nu\lambda}$ or, tensor-decomposed:

$$\begin{aligned}
A\,\Pi - \psi_4\,\dot{\Pi} &= -\beta\,J_{41}\,\theta + \dot{I}_{30} + \dot{\psi}_4\,\Pi + \tfrac{5}{3}\,\psi_4\,\Pi\,\theta \\
&\quad + \psi_1\,\partial\cdot q - q\cdot\nabla\psi_1 - 2\,\psi_1\,q\cdot\dot{u} - 2\,\psi_3\,\pi^{\mu\nu}\sigma_{\mu\nu} \\
C\,q^\mu - \psi_1\,\Delta^{\mu\nu}q_\nu &= -\beta\,J_{41}\,\dot{u}^\mu + \nabla^\mu J_{31} - \tfrac{1}{3}\,\psi_4\,\nabla^\mu\Pi - \tfrac{1}{3}\,\Pi\,\nabla^\mu\psi_4 + \tfrac{5}{3}\,\psi_4\,\Pi\,\dot{u}^\mu \\
&\quad + (\psi_1 - 2\,\psi_2)\,q_\nu\sigma^{\mu\nu} - \psi_1\,q_\nu\omega^{\mu\nu} + q^\mu\left[\dot{\psi}_1 + \tfrac{1}{3}\,(4\,\psi_1 - 5\,\psi_2)\,\theta\right] \\
&\quad + \psi_3\,\Delta^{\mu\nu}\partial^\lambda\pi_{\nu\lambda} + \pi^{\mu\nu}\,(\nabla_\nu\psi_3 - \psi_3\,\dot{u}_\nu) \\
B\,\pi^{\mu\nu} - \psi_3\,\dot{\pi}^{<\mu\nu>} &= 2\,I_{31}\,\sigma^{\mu\nu} - \tfrac{2}{3}\,\psi_4\,\Pi\,\sigma^{\mu\nu} \\
&\quad + 2\,\psi_2\,\nabla^{<\mu}q^{\nu>} + 2\,q^{<\mu}\nabla^{\nu>}\psi_2 + 2\,(\psi_1 - \psi_2)\,q^{<\mu}\dot{u}^{\nu>} \\
&\quad + 2\,\psi_3\,\pi_\lambda{}^{<\mu}\sigma^{\nu>\lambda} - 2\,\psi_3\,\pi_\lambda{}^{<\mu}\omega^{\nu>\lambda} + \pi^{\mu\nu}\left(\dot{\psi}_3 + \tfrac{5}{3}\,\psi_3\,\theta\right)
\end{aligned}$$

12. substitute $\dot{I}_{30}\,,\ -\beta\,J_{41}\,\dot{u}^\mu\,,\ \nabla^\mu I_{31}\,,\ \dot{\psi}_i\,,\ \nabla^\mu\psi_i\,,\ i = 1,\ldots,4$
with the help of energy conservation and acceleration equation

$\Longrightarrow$ IS equations with explicit (frame-dependent) expressions for coefficients