Beyond second order theories of relativistic dissipative fluids

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Motivation

- Development of causal theories of relativistic dissipative fluid dynamics for heavy ion collisions is an important achievement.
- It has inspired many authors to apply its methodology in describing observables from heavy ion collision experiments and to compare to transport models
- AM & D.H. Rischke, U. Heinz & H. Song, D. Teaney & Dusling, P. Romatschke & M. Luzum, R. Baier, S. Baas, M. Bleicher, P. Huovinen & D. Molnar, T. Kodama & T. Koide, S. Pratt & J. Vredevoogd, C. Greiner, A. El, Z. Xu, H. Niemi, & E. Molnar, I. Bouras, G. Torrieri, G. Denicol, M. Martinez, H. Peterson, T. Hirano, C. Nonaka, A. Monnai, R. Lacey ,... String Theorists

Motivation

- causal theories are mainly furnished by imposing relativity and entropy principles up to second order in dissipative fluxes
- we refrained from exploiting subsequent orders because the methodology is long and cumbersome; also because the contribution of these higher order terms to the solution of the problem at hand might be negligible.
- However, the exploitation of these higher order terms in dissipative fluxes is desirerable for the following resons

Motivation

- $\cdots$ a second order theory is necessary to link more closely the relativistic case to the non-relativistic one; we are happy just to know that the principle of relativity is not violated
- $\cdots$ imposing the conditions up to order N we get the N-th order equations. These equations depends also on the lower order terms. Thus the conditions to the solution at higher order constrain the lower order expressions.
- we will present the results at third order because it is at this order were all three main dissipative fluxes for a single fluid component couple and this coupling is very interesting to study in detail

Notation and conventions

- The metric tensor $g^{\alpha\beta} = \text{diag}(1, -1, -1, -1)$;
- The four velocity of the fluid is u^α and we have $u^\alpha u_\alpha = c^2 = 1$;
- Projection tensor is $\Delta^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ and we have $u_\beta \Delta^{\alpha\beta} = 0$.
- The comoving time derivative: $D \equiv u^\alpha \partial_\alpha$.
- The comoving space derivative: $\nabla^\alpha \equiv \Delta^{\alpha\beta} \partial_\alpha$.
- Parenthesis around some indices denote symmetrization :

$$A_{(\mu}B_{\nu)} = \frac{1}{2}\left(A_\mu B_\nu + A_\nu B_\mu\right) , \qquad A_{[\mu}B_{\nu]} = \frac{1}{2}\left(A_\mu B_\nu - A_\nu B_\mu\right) .$$

- The angular brackets around two indices denote skew-symmetrization: a tensor that is symmetric, traceless, and orthogonal to the fluid velocity:

$$S^{\langle\alpha\beta\rangle} = \left(\Delta^\alpha_\mu \Delta^\beta_\nu - \frac{1}{3}\Delta^{\alpha\beta}\Delta_{\mu\nu}\right) S^{\mu\nu}$$

14-Fields Theory of Relativistic Dissipative Fluid Dynamics :

The 14 fields are $p(n\,,\,\varepsilon)$, Π, u^α, q^α, $\pi^{\langle\alpha\beta\rangle}$ and they are governed by the following fields equations

$$
\begin{aligned}
\partial_\alpha N^\alpha &= 0 \\
\partial_\alpha T^{\alpha\beta} &= 0 \\
\partial_\gamma F^{\alpha\beta\gamma} &= P^{\alpha\beta}
\end{aligned}
$$

For all thermodynamic processes the entropy principle holds

$$\partial_\alpha S^\alpha \geq 0$$

Representation of primary variables

$$
\begin{aligned}
N^{\alpha} &= n u^{\alpha} \\
T^{\alpha\beta} &= \varepsilon u^{\alpha} u^{\beta} - (p + \Pi)\Delta^{\alpha\beta} + 2 q^{(\alpha} u^{\beta)} + \pi^{\langle\alpha\beta\rangle} \\
P^{\alpha\beta} &= \mathcal{P}_{\Pi}\Pi(\Delta^{\alpha\beta} - 3u^{\alpha}u^{\beta}) + 2\mathcal{P}_{q} q^{(\alpha} u^{\beta)} + \mathcal{P}_{\pi}\pi^{\langle\alpha\beta\rangle} ,
\end{aligned}
$$

Using of u^{α} and $\Delta^{\alpha\beta}$ we can express local rest frame quantities in terms of N^{α} and $T^{\alpha\beta}$

$$
\begin{aligned}
n &= \left(N^{\alpha} N_{\alpha}\right)^{1/2}, \qquad u^{\alpha} = \frac{N^{\alpha}}{n}, \qquad \varepsilon = u_{\alpha} u_{\beta} T^{\alpha\beta} \\
p + \Pi &= -\frac{1}{3}\Delta_{\alpha\beta} T^{\alpha\beta}, \qquad q^{\alpha} = \Delta^{\alpha}_{\mu} u_{\nu} T^{\mu\nu} \\
\pi^{\langle\alpha\beta\rangle} &= \left(\Delta^{\alpha}_{\mu}\Delta^{\beta}_{\nu} - \frac{1}{3}\Delta^{\alpha\beta}\Delta_{\mu\nu}\right) T^{\mu\nu}
\end{aligned}
$$

The coefficients $\mathcal{P}_{F}$ are related to transport coefficients for corresponding flux F

Zeroth order: Equilibrium

The equilibrium values of N^α, $T^{\alpha\beta}$, $F^{\alpha\beta\gamma}$ and S^α are calculated as moments of the Jütner-type equilibrium distribution function

$$F^{\alpha\beta\gamma} = \frac{1}{2}\mathcal{F}_1^0 g^{(\alpha\beta}u^{\gamma)} + \frac{1}{2}\mathcal{F}_2^0\left(g^{(\alpha\beta}u^{\gamma)} - 2u^\alpha u^\beta u^\gamma\right)$$

$$S^\alpha = \mathcal{S}_1^0 u^\alpha$$

where here and below the calligraphic coefficients are functions of (ε, n) and thus determined by the equation of state.

First order

$$
\begin{aligned}
F^{\alpha\beta\gamma\,(1)} &= \mathcal{F}_1^1 \Pi \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\
&\quad + \mathcal{F}_2^1 \left(\Delta^{(\alpha\beta} q^{\gamma)} - 5 u^{(\alpha} u^\beta q^{\gamma)} \right) \\
&\quad + \mathcal{F}_3^1 \pi^{(\langle\alpha\beta\rangle} u^{\gamma)} \\
S^{\alpha\,(1)} &= \mathcal{S}_1^1 \Pi u^\alpha + \mathcal{S}_2^1 q^\alpha
\end{aligned}
$$

Second order

$$
\begin{aligned}
F^{\alpha\beta\gamma\,(2)} = \; & \mathcal{F}_1^2 \Pi^2 \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\
& + \mathcal{F}_2^2 \left(q^\nu q_\nu \Delta^{(\alpha\beta} u^{\gamma)} - 3 u^{(\alpha} q^\beta q^{\gamma)} \right) \\
& + \mathcal{F}_3^2 q^\nu q_\nu \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\
& + \mathcal{F}_4^2 \left(3 u^\alpha \pi^{2\langle\beta\gamma\rangle} - \pi^{2\langle\nu\nu\rangle} u^\alpha u^\beta u^\gamma \right) \\
& + \mathcal{F}_5^2 \pi^{2\langle\nu\nu\rangle} \left(\Delta^{(\alpha\beta} u^{\gamma)} - u^\alpha u^\beta u^\gamma \right) \\
& + \mathcal{F}_6^2 \left(q^{(\alpha} \pi^{\langle\beta\gamma\rangle)} - 2 u^{(\alpha} u^\beta \pi^{\langle\gamma)\nu\rangle} q_\nu \right) \\
& + \mathcal{F}_7^2 \left(\Delta^{(\alpha\beta} \pi^{\langle\gamma)\nu\rangle)} q_\nu - 5 u^\alpha u^\beta \pi^{\langle\gamma)\nu\rangle} q_\nu \right) \\
& + \mathcal{F}_8^2 \Pi u^{(\alpha} \pi^{\langle\beta\gamma\rangle)} + \mathcal{F}_9^2 \Pi \left(\Delta^{(\alpha\beta} q^{\gamma)} - 5 q^{(\alpha} u^\beta u^{\gamma)} \right)
\end{aligned}
$$

Second order

$$S^{\alpha\,(2)} = \left(\mathcal{S}_1^2\Pi^2 - \mathcal{S}_2^2 q^\nu q_\nu + \mathcal{S}_3^2 \pi^{2\langle\nu\nu\rangle}\right)u^\alpha + \mathcal{S}_4^2 \Pi q^\alpha + \mathcal{S}_5^2 \pi^{\langle\alpha\nu\rangle} q_\nu$$

Third order

$$
\begin{aligned}
S^{\alpha\,(3)} \;=\;& \Big(\mathcal{S}_1^3 \Pi^3 - \mathcal{S}_2^3 \Pi q_\nu q^\nu + \mathcal{S}_3^3 \Pi \pi^{2\langle\nu\nu\rangle} + \mathcal{S}_4^3 q_\nu q_\mu \pi^{\langle\mu\nu\rangle} \\
& + \mathcal{S}_5^3 \pi^{3\langle\nu\nu\rangle} \Big) u^\alpha \\
& + \Big(\mathcal{S}_6^3 \Pi^2 + \mathcal{S}_7^3 q_\nu q^\nu + \mathcal{S}_8^3 \pi^{2\langle\nu\nu\rangle} \Big) q^\alpha \\
& + \mathcal{S}_9^3 \Pi \pi^{\langle\alpha\nu\rangle} q_\nu + \mathcal{S}_{10}^3 \pi^{2\langle\alpha\nu\rangle} q_\nu
\end{aligned}
$$

Dissipative fluxes: Zeroth order: Equilibrium

$$\begin{aligned}
\Pi &= \Pi_{Eq.} = 0 \\
q^{\alpha} &= q^{\alpha}_{Eq.} = 0 \\
\pi^{\langle\alpha\beta\rangle} &= \pi^{\langle\alpha\beta\rangle}_{Eq.} = 0
\end{aligned}$$

Dissipative fluxes: First order

$$\Pi^{(1)} = \Pi_E = -\zeta \nabla_\alpha u^\alpha$$

$$q^{\alpha\,(1)} = q_E^\alpha = \kappa T \Delta^{\alpha\mu} \left(\frac{\nabla_\alpha T}{T} - a_\alpha \right)$$

$$\pi^{\langle\alpha\beta\rangle\,(1)} = \pi_E^{\langle\alpha\beta\rangle} = 2\eta \Delta^{\alpha\mu} \Delta^{\beta\nu} \nabla_{\langle\alpha} u_{\beta\rangle}$$

Dissipative fluxes: Second order

$$\Pi^{(2)} = \Pi_{MIS} = -\zeta\left[2\mathcal{S}_1^2\dot{\Pi} + \mathcal{S}_4^2\nabla_\alpha q^\alpha\right]$$
$$-\zeta\left[\Pi(\dot{\mathcal{S}}_1^2 + \mathcal{S}_1^2\nabla_\alpha u^\alpha) + q^\alpha(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2 a_\alpha)\right]$$

$$q^{\mu\,(2)} = q^\mu_{MIS} = \kappa T\Delta^{\alpha\mu}\left[2\mathcal{S}_2^2\dot{q}_\alpha + \mathcal{S}_4^2\nabla_\alpha\Pi + \mathcal{S}_5^2\nabla^\beta\pi_{\langle\alpha\beta\rangle}\right]$$
$$+\kappa T\Delta^{\alpha\mu}\left[q_\alpha(\dot{\mathcal{S}}_2^2 + \mathcal{S}_2^2\nabla_\nu u^\nu) + \Pi(\nabla_\alpha\mathcal{S}_4^2 - \mathcal{S}_4^2 a_\alpha)\right.$$
$$\left.+\pi_{\langle\alpha\beta\rangle}(\nabla^\beta\mathcal{S}_5^2 - \mathcal{S}_5^2 a^\beta)\right]$$

$$\pi^{\langle\mu\nu\rangle\,(2)} = \pi^{\langle\mu\nu\rangle}_{MIS} = 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[2\mathcal{S}_3^2\dot{\pi}_{\langle\alpha\beta\rangle} + \mathcal{S}_5^2\nabla_{\langle\alpha}q_{\beta\rangle}\right]$$
$$+2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\left[\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^2 + \mathcal{S}_3^2\nabla_\lambda u^\lambda)\right.$$
$$\left.+q_{\langle\alpha}(\nabla_{\beta\rangle}\mathcal{S}_5^2 - \mathcal{S}_5^2 a_{\beta\rangle})\right]$$

Dissipative fluxes: Third order: Bulk equation

$$
\begin{aligned}
\Pi^{(3)} = & -\zeta\Big[3\mathcal{S}_1^3\dot{\Pi} + 2\mathcal{S}_2^3\dot{q}_\lambda q^\lambda + 2\mathcal{S}_3^3\dot{\pi}_{\langle\alpha\beta\rangle}\pi^{\langle\alpha\beta\rangle} \\
& + \mathcal{S}_6^3(\Pi\nabla_\alpha q^\alpha + q^\alpha\nabla_\alpha\Pi) + \mathcal{S}_9^3(\pi^{\langle\alpha\beta\rangle}\nabla_\alpha q_\beta + q_\beta\nabla_\alpha\pi^{\langle\alpha\beta\rangle})\Big] \\
& -\zeta\Big[\Pi^2(\dot{\mathcal{S}}_1^3 + \mathcal{S}_1^3\nabla_\alpha u^\alpha) - q^\alpha q_\alpha(\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3\nabla_\alpha u^\alpha) \\
& + \pi^{2\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_\alpha u^\alpha) \\
& + \Pi q^\alpha(\nabla_\alpha\mathcal{S}_6^3 - \mathcal{S}_6^3 a_\alpha) + \pi^{\langle\alpha\beta\rangle}q_\beta(\nabla_\alpha\mathcal{S}_9^3 - \mathcal{S}_9^3 a_\alpha)\Big]
\end{aligned}
$$

Dissipative fluxes: Third order: Heat equation

$$
\begin{aligned}
q^{\mu\,(3)} \;=\; & \kappa T \Delta^{\alpha\mu} \Big[-\mathcal{S}_2^3 (2\Pi \dot{q}_\alpha + q_\alpha \dot{\Pi}) + \mathcal{S}_4^3 (2\dot{q}^\beta \pi_{\langle\alpha\beta\rangle} + q^\beta \dot{\pi}_{\langle\alpha\beta\rangle}) \\
& + 2\mathcal{S}_6^3 \Pi \nabla_\alpha \Pi - 2\mathcal{S}_7^3 q^\beta \nabla_\alpha q_\beta + \mathcal{S}_9^3 (\Pi \nabla^\beta \pi_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle} \nabla^\beta \Pi) \\
& + 2\mathcal{S}_{10}^3 \pi_{\langle\beta\nu\rangle} \nabla_\alpha \pi^{\langle\beta\nu\rangle} \Big] \\
& + \kappa T \Delta^{\alpha\mu} \Big[\Pi q_\alpha (\dot{\mathcal{S}}_2^3 + \mathcal{S}_2^3 \nabla_\nu u^\nu) + q^\beta \pi_{\langle\alpha\beta\rangle} (\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3 \nabla_\nu u^\nu) \\
& + (\Pi^2 \nabla_\alpha \mathcal{S}_6^3 - q^\lambda q_\lambda \nabla_\alpha \mathcal{S}_7^3 + \pi^{2\langle\lambda\lambda\rangle} \nabla_\alpha \mathcal{S}_8^3) \\
& + \Pi \pi_{\langle\alpha\beta\rangle} (\nabla^\beta \mathcal{S}_9^3 - \mathcal{S}_9^3 a^\beta) + \pi^2_{\langle\alpha\beta\rangle} (\nabla^\beta \mathcal{S}_{10}^3 - \mathcal{S}_{10}^3 a^\beta) \\
& + \mathcal{S}_7^3 q_\alpha q^\lambda a_\lambda \Big]
\end{aligned}
$$

Dissipative fluxes: Third order: Shear equation

$$
\begin{aligned}
\pi^{\langle\mu\nu\rangle\,(3)} = \; & 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\mathcal{S}_3^3(2\Pi\dot{\pi}_{\langle\alpha\beta\rangle} + \pi_{\langle\alpha\beta\rangle}\dot{\Pi}) + 2\mathcal{S}_4^3\dot{q}_{\langle\alpha}q_{\beta\rangle} \\
& + 3\mathcal{S}_5^3\dot{\pi}_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle} + \mathcal{S}_8^3\pi_{\langle\alpha\beta\rangle}\nabla_\lambda q^\lambda + \mathcal{S}_9^3(\Pi\nabla_{\langle\alpha}q_{\beta\rangle} + q_{\langle\alpha}\nabla_{\beta\rangle}\Pi) \\
& + \mathcal{S}_{10}^3(q_{(\beta}\nabla^\lambda\pi_{\langle\alpha)\lambda\rangle} + \pi_{\langle\lambda(\alpha\rangle}\nabla^\lambda q_{\beta)})\Big] \\
& + 2\eta\Delta^{\alpha\mu}\Delta^{\beta\nu}\Big[\Pi\pi_{\langle\alpha\beta\rangle}(\dot{\mathcal{S}}_3^3 + \mathcal{S}_3^3\nabla_\lambda u^\lambda) \\
& + q_{\langle\alpha}q_{\beta\rangle}(\dot{\mathcal{S}}_4^3 + \mathcal{S}_4^3\nabla_\lambda u^\lambda) + \pi_{\langle\alpha\lambda\rangle}\pi^{\langle\lambda}_{\beta\rangle}(\dot{\mathcal{S}}_5^3 + \mathcal{S}_5^3\nabla_\lambda u^\lambda) \\
& + \pi_{\langle\alpha\beta\rangle}q^\lambda(\nabla_\lambda\mathcal{S}_8^3 - \mathcal{S}_8^3 a_\lambda) + \Pi q_{\langle\beta}(\nabla_{\alpha\rangle}\mathcal{S}_9^3 - \mathcal{S}_9^3 a_{\alpha\rangle}) \\
& + \pi_{\langle\alpha\lambda}q^\lambda(\nabla_{\beta\rangle}\mathcal{S}_{10}^3 - \mathcal{S}_{10}^3 a_{\beta\rangle})\Big]
\end{aligned}
$$

Conclusions

- We can extend causal theories of relativistic dissipative fluid dynamics to N-th order and here we have reported the results up to 3rd order with respect to entropy 4-current
- The equations are now highly coupled via space gradients and time gradients of both equilibrim quantities and dissipative fluxes
- The equations depends also on the lower order terms
- It is desirable to include higher order and in particular third order corrections (where the marriage of the three dissipative fluxes takes place)