Relativistic magnetohydrodynamics: foundations, stability, and solutions

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Outline

A biased[1] review of relativistic MHD from a (heavy-ion) theoretical P.O.V

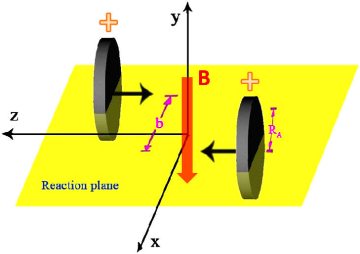
- Why should we study resistive MHD?
- Equilibrium and stability in second-order MHD
- (Mostly) analytical and numerical solutions

[1] Picture from https://medium.com/ml-and-automation/a-i-bias-a-thought-experiment-3ad6a5da74dc

Notations and conventions

- Natural units ($\hbar = c = k_B = 1$) and mostly minus metric sign convention, i.e., $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
- Covariant derivative D, Lie derivative $\mathcal{L}$
- Projector $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$, $\nabla_\mu \Phi = \Delta_\mu^\alpha D_\alpha \Phi$
- Dot notation $\dot{X} = u^\mu D_\mu X$
- Traceless symmetric projector of rank four reads $\Delta^{\mu\nu}_{\alpha\beta} := \Delta^{(\mu}_\alpha \Delta^{\nu)}_\beta - \Delta^{\mu\nu}\Delta_{\alpha\beta}/3 \to A^{\langle\mu\nu\rangle} := \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$

Why should we study resistive MHD?



Huang, 2016

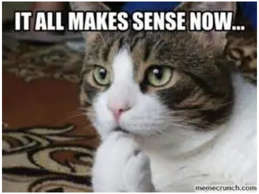
- EM fields comparable with QCD scale $eE \sim eB \sim \Lambda^2_{\rm QCD}$ [2]
- $\rightarrow$ studies on the effects of **strong magnetic fields** on different observables [3]
- ...assuming long-standing eB at QGP stage (when one can define T)

[2] $B \sim 10^{18} - 10^{20}$Gauss (see, e.g., [Huang (2016)] for a review)

[3] Some examples are cited in [Jaiswal et al. (2021)]

Let's assume $eB \sim m_\pi^2$ in the QGP stage

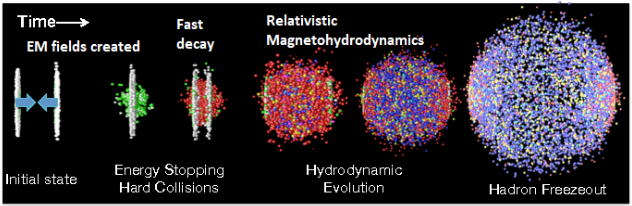
- $\implies$ Alfven speed[4] $\sim$ speed of sound
- EM fields compete with ∇P to modify the flow
- ...but hydrodynamics agrees well with flow observables
- Our current understanding of HICs suggests that $v_a^2 \ll v_s^2$ at mid-rapidity
- An estimate from the experimental data is $eB \leq 3 \times 10^{-3} m_\pi^2$ at late times [Müller and Schäfer (2018)]



But where did it go?

[4] $v_a^2 = \sqrt{2\sigma_P/(1 + 2\sigma_P + v_s^2)}$ with $\sigma_p = B^2/(2\varepsilon)$.

A quantitative understanding of the fate of EM fields requires realistic simulations based on a **robust** theory of **resistive dissipative magnetohydrodynamics**



- Early time fireball is almost an insulator → fast decay of EM fields
- At the QGP stage the fluid is conductive → MHD fits
- Do we need a resistive theory?

Ideal MHD

- Ohm's law $\mathbf{J} = \sigma_{\mathrm{e}} \mathbf{E}$
- Ideal MHD limit: $\sigma_{\mathrm{e}} \to \infty \implies \mathbf{E} \to 0$ (in the LRF)

Basic equations of ideal MHD

$$\overbrace{\partial_\mu T^{\mu\nu}}^{=T^{\mu\nu}_{\mathrm{f}}+T^{\mu\nu}_{\mathrm{em}}} = 0 \qquad \partial_\mu J^{\mu e} = 0 \qquad \overbrace{\epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho}}^{\text{3 d.o.f in } \mathbf{B}} = 0$$

EM energy-momentum tensor is well known

$$T^{\mu\nu}_{\mathrm{em}} = -F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

Covariant iMHD condition

$$F^{\mu\nu} u_\nu \equiv E^\mu = 0 \qquad \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta} \equiv B^\mu \overbrace{=}^{\text{LRF}} (0, \mathbf{B})$$

QGP's electrical conductivity

Is QGP highly conductive?

- LQCD: $\sigma_e \sim 0.02T$ (huge number $\sim 1000\sigma_e$ for copper)[5]
- Magnetic Reynolds number $R_m = Lu\sigma_e \sim 0.2T/(200\text{MeV})$
- ... is quite small $\rightarrow$ iMHD might be a poor approximation
- Magnetic susceptibility of QGP $\chi_m = \frac{\mathbf{M}}{\mathbf{B}} \sim 10^{-2}$ can be neglected [6]

[5] [Aarts and Nikolaev (2021)]
[6] Bali et al. (2014)

Basic equations of resistive MHD

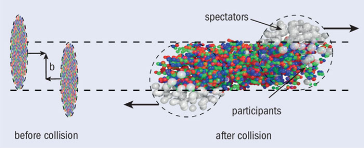
CRC-TR 211

Now we have 6 EM d.o.f:

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\alpha B^\beta$$

Maxwell equations

$$\epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho} = 0 \qquad \partial_\mu F^{\mu\nu} = J^\nu$$



$$J^\mu = \overbrace{J^\mu_{\text{ext}}}^{\text{Mostly spectators}} + J^\mu_{\text{f}} \qquad \partial_\mu T^{\mu\nu} = -\overbrace{F^{\nu\lambda} J_{\lambda,\text{ext}}}^{\text{Lorenz force}}$$

Currents Beyond iMHD

> "In a famous remark, Einstein once likened the left-hand side of his field equations to a marble palace, while the right-hand side was only a house of straw, "a formal condensation of all things whose comprehension in the sense of a field theory is still problematical""
>
> [Israel (1978)]

- $F_{\mu\nu}$ is exact (as the Einstein tensor is in GR) while J^μ is not (as $T^{\mu\nu}$ on the r.h.s of Einstein's equation)
- The energy-momentum tensor of the nonpolarizable matter doesn't change
- Gradient expansion of the current

$$J^\mu = J^\mu_{(0)} + J^\mu_{(1)} + \cdots$$

Charge diffusion

- In Landau's frame (single charge)

$$J^\mu = q\left(nu^\mu + V^\mu\right)$$

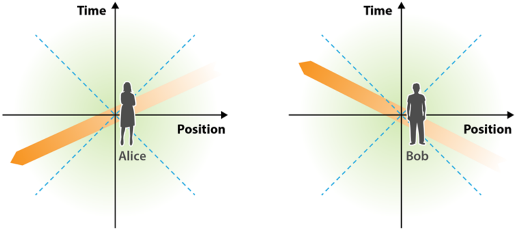
- Relativistic (Navier-Stokes) Ohm's law

$$qV^\mu = q\kappa\nabla^\mu\alpha + \sigma E^\mu$$

- Wiedemann–Franz law: $\sigma = q^2\kappa/T$ (κ is the particle diffusion coefficient)

The need for a better theory

Navier-Stokes diffusion equation is acausal and unstable



While one observer sees damped waves another one sees them growing[7]

[7] Picture from G. Denicol https://physics.aps.org/articles/v15/149

Approaches to better theories

- MIS theory from entropy current [Israel (1976)]

$$S^\mu = S^\mu_{\mathrm{NS}} - \mathcal{R} u^\mu \qquad \mathcal{R} \propto V^2$$

$$\partial_\mu S^\mu \geq 0 \implies \tau_V \dot{V} + V = V_{\mathrm{NS}}$$

- Denicol-Niemi-Molnar-Rischke (DNMR) theory: formulation of dissipative hydrodynamics based on kinetic theory [Denicol et al. (2012)]
- Bemfica-Discnosi-Noronha-Kovtun (BDNK) theory: *enhanced* gradient expansion [Bemfica et al. (2020)] [Hoult and Kovtun (2020)]

Examples of currently developed theories

- Gradient expansion [Hernandez and Kovtun (2017)]
- BDNK theory for polarized and non-resistive fluid [Armas and Camilloni (2022)]
- Second-order resistive dissipative MHD from Boltzmann-Vlasov equation (single charge) [Denicol et al. (2019)]

$$p.\partial f(x,p) + qF^{\mu\nu} p_\nu \frac{\partial}{\partial p^\mu} f = C[f]$$

- In the next part of this talk I focus on this theory which is a Maxwell-Israel-Stewart type one

Second-order resistive and dissipative MHD

Constitutive relations for a multi-component Maxwell-Israel-Stewart type theory

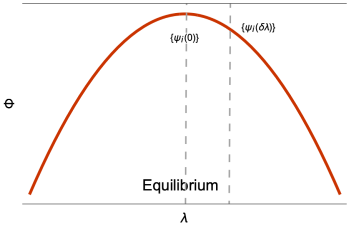
$$T_{\mathrm{f}}^{\mu\nu} = \varepsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\overbrace{N_A^\mu = n_A u^\mu + V_A^\mu}^{A = 1 \cdots l \text{ species index}} \qquad J^\mu = \overbrace{q^{eA} N_A^\mu}^{\text{electric charge of A}}$$

- Equations of motion for Π, $\pi^{\mu\nu}$ and V_A^μ that are (C.I) consistent with $D_\mu S^\mu \geq 0$
- ...and are (C.II) identically satisfied in equilibrium

Gibbs stability

The Gibbs stability criterion: Equilibrium state ($\lambda = 0$) is the state where the entropy is maximum compared to all states ($\lambda \neq 0$) with the same values of conserved charges



- Solving EOM
  $\rightarrow \psi_i(\lambda) \in \{\varepsilon, P, u^\mu, \cdots\}$
- Slightly away from equilibrium

$$\psi_i(\lambda) = \psi_i(0) + \frac{\mathrm{d}\psi_i}{\mathrm{d}\lambda}\delta\lambda$$

Gibbs stability

- A conserved charge arises from $D_\mu J^\mu_I = 0$[8]

$$Q_I = \int_\Sigma \mathrm{d}\Sigma_\mu \, J^\mu_I$$

- q^{IA} quantum charge I carried by species A
- Find the stationary point of the following in the solutions space

$$\phi^\mu = S^\mu + \underbrace{\overbrace{\alpha^A_\star}^{\text{constants}} N^\mu_A - T^\mu{}_\nu \overbrace{\beta^\nu_\star}^{\text{Timelike killing vector}}}_{\text{Our knowledge}}$$

[8] D is the covariant derivative

Equilibrium without EM fields

The stationary point $\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0$ at $\lambda = 0$

$$\frac{\mu^A}{T} = \alpha^{A\star} \qquad T = \frac{1}{\sqrt{\beta^\star \cdot \beta^\star}} \qquad u^\mu = \frac{\beta^{\star\mu}}{\sqrt{\beta^\star \cdot \beta^\star}}$$

$$V^A_\mu = \Pi = \pi^{\mu\nu} = 0 \qquad \mathcal{L}_{\beta^\star}\text{Physics} = 0$$

- The information current must be future-directed non-spacelike

$$\mathcal{I}^\mu \equiv -\frac{1}{2}\frac{\mathrm{d}^2\phi^\mu}{\mathrm{d}\lambda^2}$$

- $\implies$ (linear) stability and causality conditions

Example of SC conditions

A single charge diffusive fluid with $\eta = \zeta = 0$

$$h = \varepsilon + P > 0 \qquad c_s^2 = \left.\frac{\partial P}{\partial \varepsilon}\right|_{n/s} \in (0,1] \qquad c_v = T\left.\frac{\partial s}{\partial T}\right|_n > 0$$

$$\tau_V > \frac{h\sigma}{qn^2}\left[\frac{1}{1-c_s^2}\left(1 - \frac{n}{T}\frac{\partial^2 \varepsilon}{\partial n \partial s} - \frac{s}{T}\frac{\partial^2 \varepsilon}{\partial s^2}\right)^2 + \frac{h}{T^2}\frac{\partial^2 \varepsilon}{\partial s^2} - 1\right]$$

- The case of multiple charges is derived in [Gavassino and Shokri (2023)]
- How do EM fields change the equilibrium and stability conditions?

Please stay with me, I have a surprise for you.

Equilibrium in background EM fields

- Energy-momentum is not conserved anymore $\nabla_\mu T^{\mu\nu} = -F^{\nu\lambda} J_\lambda$
- The external fields must be stationary
  $\mathcal{L}_{\beta^\star} F = 0 \implies \beta^\star_\nu F^\nu{}_\mu = -\partial_\mu \psi^\star$
- The external electrostatic potential $\psi^\star$ appears in ϕ^μ

$$\phi^\mu = S^\mu + (\alpha^\star_I q^{IA} + \psi^\star q^{eA}) N^\mu_A - \beta^\star_\nu T^{\nu\mu}$$

- ... and the equilibrium condition

$$(\star)\ \frac{\mu^A}{T} = \alpha^\star_I q^{IA} + \psi^\star q^{eA} \qquad E_\mu = -T\partial_\mu \psi^\star$$

- The fluid counterbalances the electric force

$$T\partial_\mu(\mu^A/T) + q^{eA} E_\mu = 0$$

- The stability conditions for these equilibrium states are found by replacing μ/T from $(\star)$ in the previous ones!

Equilibrium in dynamic EM fields

- Now $\phi^\mu = \phi^\mu_{\rm f} + \phi^\mu_{\rm em}$

$$\phi^\mu_{\textsf{em}} = \beta^\star_\nu F^\mu_{\ \alpha} F^{\nu\alpha} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \beta^{\star\mu}$$

- Using some mathematical tricks

$$\delta\phi^\mu_{\textsf{em}} = \beta^\star_\nu A^\nu \delta J^\mu + 2\delta A_\nu \beta^{\star[\nu} J^{e\mu]} + \delta F^{\mu\alpha} (\mathcal{L}_{\beta^\star} A)_\alpha - \delta A_\alpha (\mathcal{L}_{\beta^\star} F)^{\mu\alpha} + D_\alpha Z^{[\alpha\mu]}_{(1)}$$

- Clearly $\mathcal{L}_{\beta^\star} F^{\mu\alpha} = 0 \implies \mathcal{L}_{\beta^\star} A_\mu = -\partial_\mu \Lambda$
- ... partially absorbed into the last term $(Z_{(1)})$
- Stokes theorem is applicable because $Z_{(1)}$ is antisymmetric

$$\int_\Sigma \mathrm{d}\Sigma_\mu \, D_\alpha Z^{[\alpha\mu]}_{(1)} = \oint_{\partial\Sigma} \overbrace{Z^{[\alpha\mu]}_{(1)} \, \mathrm{d}S_{\alpha\mu}}^{\textsf{pertubations on } \partial\Sigma \to 0} = 0$$

Stationary point

$$(\star\star)\ \frac{\mu^A}{T} = q^{IA}\alpha_I^\star + q^{eA}(\beta_\nu^\star A^\nu + \Lambda)$$

- Using $(\star\star)$ in $\frac{\mathrm{d}^2\phi^\mu}{\mathrm{d}\lambda^2}$ the current terms are absorbed into the fluid sector
- The Maxwell energy-momentum tensor with $F \to \delta F$

$$T\mathcal{I}^\mu_{\mathsf{em}} = u_\nu\left[\delta F^\mu_\alpha \delta F^{\nu\alpha} - \frac{1}{4} g^{\mu\nu}\delta F^{\alpha\beta}\delta F_{\alpha\beta}\right]$$

- Dominant energy condition $T^{\mu\nu}u_\nu$ for a nonspacelike u is a future-pointing causal vector (for any observer energy density can never flow faster than light)

The electromagnetic part of the information current is stable and causal by construction and, therefore, the stability criteria found for Israel-Stewart-type theories of hydrodynamics automatically extend to similar formulations of magnetohydrodynamics.

- A simple form of the multi-component diffusion equation

$$\tau_A^B u^\nu D_\nu V^{B\langle\mu\rangle} + V^{A\mu} = \kappa_{AB} D^{\langle\mu\rangle}\alpha^B + \sigma_A E^\mu$$

- The multi-component form of the Wiedemann-Franz law

$$\sigma_A = \frac{\kappa_{AC} q^{eC}}{T}$$

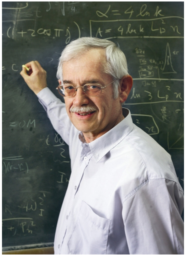
Solutions

Examples of analytical and numerical solutions

- Analytical solutions
  - Bjorken: iMHD [Roy et al. (2015)], iMHD with magnetization [Pu et al. (2016)], resistive inviscid MHD [Shokri and Sadooghi (2017)]
  - Perturbative solutions [Pu and Yang (2016)]
  - Gubser MHD [Shokri and Sadooghi (2018)]
  - Bantilan-Ishi-Romatschke (BIR) MHD [Shokri (2020)]
  - Static fluid solutions [Gursoy et al. (2014)]
- Numerical solutions
  - iMHD solutions: ECHO-QGP [Inghirami et al. (2018)], BHAC-QGP [Mayer et al. (2024)]
  - Charge diffusion in relativistic resistive second-order dissipative magnetohydrodynamics (1+1 dimensions) [Dash et al. (2023)]
  - Resistive MHD in 3+1 dimensions [Takamoto and Inoue (2011)]

Bjorken iMHD

> Symmetries are the poor man's best friends. But one cannot spend too much time with all of one's friends.
> The method used here is an extension of [Bekenstein and Oron (1978)].



Bjorken iMHD

Milne metric

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \tau^2\,\mathrm{d}\eta^2$$

Bjorken symmetries

$$\xi_i^\mu = \delta_i^\mu \qquad i = \overbrace{1,2}^{\text{trans.}}, \overbrace{3}^{\text{boost}} \qquad \overbrace{\xi_\phi^\mu = (0,-y,x,0)}^{\text{rotation}}$$

- Bjorken symmetries
  - Translational invariance in the transverse (xy) plane + homogeneous Maxwell equations[9] + $T^{\mu\nu}(\infty) \to 0 \implies F_{12} = 0$[10] (No longitudinal B)
  - Also homogeneous Maxwell equations $\implies \partial_\tau F_{\mu\nu} = 0$
  - If we apply the rotational symmetry $\to F_{\mu\nu} = 0$!

[9] Used in the form $\partial_{[\alpha}F_{\beta\gamma]} = 0$.
[10] For example $2u_\mu u_\nu T^{\mu\nu}_{\mathrm{em}} = F_{12}^2 + (F_{13}^2 + F_{23}^2)/\tau^2$.

Breaking the rotational symmetry and assuming the boost invariance $\Longrightarrow$

$$F_{\mu\nu} = \overbrace{B_0 \tau_0}^{\text{from dim.}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\phi_B \\ 0 & 0 & 0 & \cos\phi_B \\ 0 & \sin\phi_B & -\cos\phi_B & 0 \end{bmatrix}$$

- Bjorken MHD solution

$$B^\mu = B_0 \overbrace{\frac{\tau_0}{\tau}(0, \cos\phi_B, \sin\phi_B, 0)}^{\tau\text{-dependecy from } \epsilon^{\mu\nu\alpha\beta}} \Longrightarrow B = B_0 \frac{\tau_0}{\tau} \Longrightarrow \frac{B}{s} = \text{const.}$$

- Found first in **[Roy et al. (2015)]** and extended to magnetized fluid in **[Pu et al. (2016)]**

Bjorken resistive MHD

- Again start with symmetries (still $F_{12} = 0$)

$$\mathcal{L}_{\xi_i} F = 0 \implies F_{i0} = -\partial_\tau \psi_i \qquad F_{12} = 0 \qquad F_{i3} \propto B_0 \tau_0$$

- Maxwell equations give rise to

$$\psi'_{1,2}(\tau) = c_{1,2} \frac{\tau_0}{\tau} e^{-\sigma(\tau - \tau_0)} \qquad \psi'_3(\tau) = c_3 \frac{\tau}{\tau_0} e^{-\sigma(\tau - \tau_0)}$$

- $\nabla_\mu T^{\mu\nu} = 0$ then requires

$$c_1 \sin\phi_B - c_2 \cos\phi_B = 0 \qquad c_3 = 0$$

- Namely, the Poynting vector must vanish

$$\epsilon^{\mu\nu\alpha\beta} u_\nu E_\alpha B_\beta = 0$$

Bjorken resistive MHD

$\mathbf{E}$ is anti-parallel to $\mathbf{B}$ and suppressed by an exponential factor

$$E^\mu = E_0 \frac{\tau_0}{\tau} e^{-\sigma(\tau-\tau_0)} \left(0, -\cos\phi_B, -\sin\phi_B, 0\right)$$

While the magnetic field remains similar to iMHD

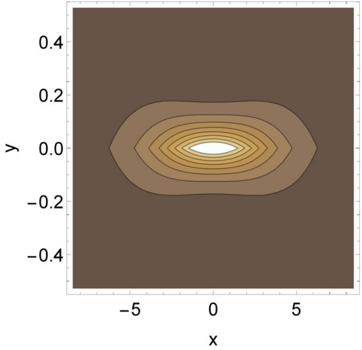
$$B^\mu = B_0 \frac{\tau_0}{\tau} \left(0, \cos\phi_B, \sin\phi_B, 0\right)$$

Joule heating

$$\tau \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + (\varepsilon + P) = (\sigma\tau) E_0^2 \frac{\tau_0^2}{\tau^2}$$

- Electric field is suppressed if $\sigma\tau \gg 1$
- At the same time strong Joule heating requires $\sigma\tau \gg 1$
- This is not the case for the LQCD results

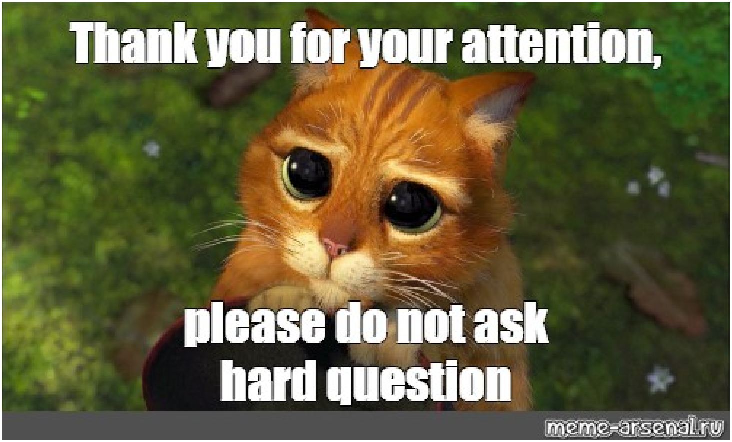
Further analytical solutions



- Bjorken resistive MHD with partially relaxed boost-invariance $\phi_B = \phi_{B,0} + \omega_0 \eta$ and modified evolution of B and E **[Shokri and Sadooghi (2017)]**
- Gubser MHD: only B_z is fully consistent with the symmetries **[Shokri and Sadooghi (2018)]**
- BIR MHD: the neutrality condition leads to singularities **[Shokri (2020)]**

Summary

- Ideal MHD is a poor approximation for the QGP
- Resistive MHD is required for understanding the evolution of strong electromagnetic fields produced in heavy ion collisions after early times
- This will let us investigate the existence of possible effects from EM fields
- EM fields modify the equilibrium conditions of a nonpolarizable fluid but the Maxwell sector is stable and causal by construction
- We are still waiting for a realistic numerical simulation of resistive and diffusive MHD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Backup

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