Finanțat de
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Planul Național
de Redresare și Reziliență

PNRR: Fonduri pentru România modernă și reformată!

Thermodynamics of rotating fermions

Victor E. Ambruș, A. Gecić

arXiv:2509.17640

West University of Timișoara

MHD Seminar in Frankfurt am Main, $23^{\rm rd}$ Oct 2025

Outline

Introduction

Global thermodynamic equilibrium

- The kinetic theory description of relativistic gases requires that the one-particle distribution function $f_{\mathbf{k}}$ satisfies the Boltzmann equation:

$$k^\mu \partial_\mu f_{\mathbf{k}} = C[f]. \tag{1}$$

- In local equilibrium, $C[f] = 0$ and the fluid is described by

$$f^{(\mathrm{eq})}_{\mathbf{k};\varsigma} = [e^{\beta(E_{\mathbf{k}} - \varsigma\mu)} + a]^{-1}, \tag{2}$$

  where $a = 0$, 1 and -1 for classical, Fermi-Dirac or Bose-Einstein statistics.
- In global equilibrium, $f^{(\mathrm{eq})}_{\mathbf{k};\varsigma}$ satisfies the Boltzmann equation, thus

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \qquad \partial_\mu(\mu/T) = 0, \tag{3}$$

  where $\beta^\mu = \beta u^\mu$ is the temperature four-vector.

Rigid rotation

- States under rigid rotation are described by the following Killing vector:

$$\beta^\mu \partial_\mu = \beta_0(\partial_t + \Omega \partial_\varphi). \tag{4}$$

- u^μ has unit 4-norm and is parallel to β^μ:

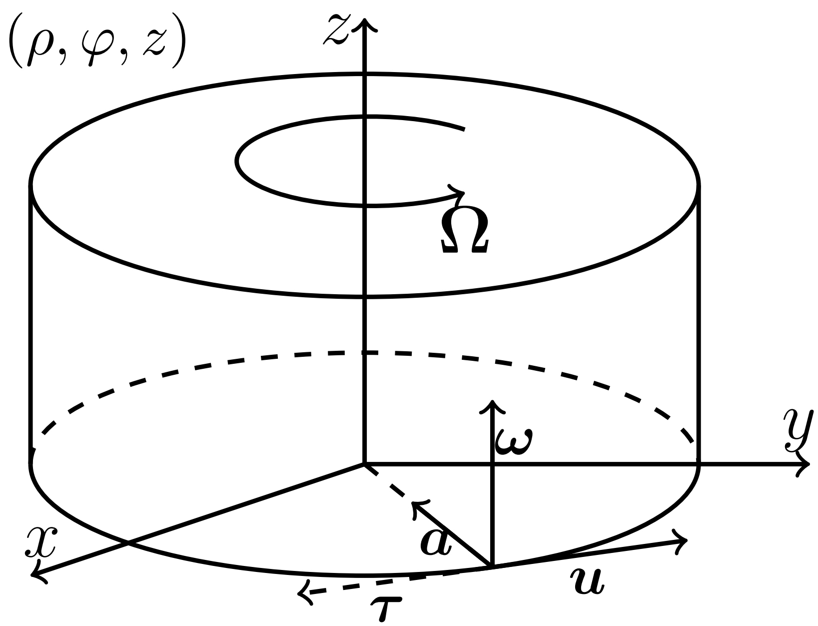
$$u^\mu \partial_\mu = \frac{\beta^\mu \partial_\mu}{\sqrt{\beta_\nu \beta^\nu}} = \Gamma(\partial_t + \Omega_0 \partial_\varphi), \qquad \Gamma = (1 - \rho^2 \Omega^2)^{-1/2}, \tag{5}$$

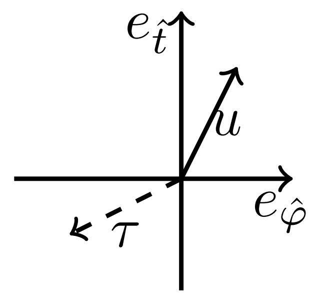
  with $\Gamma \equiv$ the Lorentz factor of a RRO at distance ρ from the r.a.
- The local temperature T represents the inverse norm of β^μ:

$$T = 1/\sqrt{\beta_\mu \beta^\mu} = \Gamma T_0, \tag{6}$$

  with $T_0 \equiv$ the temperature on the r.a.
- In global equilibrium, $\mu/T = \mathrm{const.} \Rightarrow \mu = \Gamma \mu_0$.
- The state diverges at the light cylinder, where a co-rotating fluid element moves at the speed of light, $\rho\Omega = 1$.

Kinematic frame for rigid rotation





A "kinematic" orthogonal tetrad is given by: [Becattini, Grossi, PRD 2015]

$$
\begin{aligned}
&\text{Velocity}: && u^\mu \partial_\mu = \Gamma(\partial_t + \Omega_0 \partial_\varphi), && \Gamma = (1-\rho^2\Omega_0^2)^{-1/2},\\
&\text{Acceleration}: && a^\mu = u^\nu \partial_\nu u^\mu = -\rho\Omega^2\Gamma^2\delta^\mu_\rho, && a^2 = -\mathbf{a}^2 = -\Omega_0^2\Gamma^2(\Gamma^2-1),\\
&\text{Vorticity}: && \omega^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta = \Gamma^2\Omega\delta^\mu_z, && \omega^2 = -\boldsymbol{\omega}^2 = -\Omega_0^2\Gamma^4,\\
&\text{Fourth vector:}^1 && \tau^\mu\partial_\mu = \varepsilon^{\mu\nu\alpha\beta}u_\nu a_\alpha \omega_\beta \partial_\mu = -\Omega^3\Gamma^5(\rho^2\Omega\partial_t + \partial_\varphi), && \tau^2 = -\boldsymbol{\tau}^2 = -\Omega_0^4\Gamma^6(\Gamma^2-1).
\end{aligned}
$$

The vorticity tensor is then

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu}u^\mu\omega^\nu = \Omega_0\Gamma(g_{\alpha x}g_{\beta y} - g_{\alpha y}g_{\beta x}). \quad (7)$$

[1] Aka *another vector*.

[Buzzegoli, Becattini, JHEP**12**(2018)002]

Tolman-Ehrenfest expectation

- The "classical" (cl) Tolman-Ehrenfest law says that in global equilibrium, the charge current and energy-momentum tensor are given by the perfect fluid form,

$$J^\mu_{\rm cl} = Q_{\rm cl} u^\mu, \quad T^{\mu\nu}_{\rm cl} = \epsilon_{\rm cl} u^\mu u^\nu - P_{\rm cl} \Delta^{\mu\nu}, \qquad [\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu] \tag{8}$$

  where $Q_{\rm cl} = u_\mu J^\mu$, $P_{\rm cl} = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$ and $\epsilon_{\rm cl} = u_\mu T^{\mu\nu} u_\nu$.
- For massless fermions, the pressure reads

$$P_{\rm cl} = \frac{7\pi^2 T^4}{180} + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2}. \tag{9}$$

- The pressure can be used to derive the charge and entropy densities:

$$Q_{\rm cl} = \frac{\partial P_{\rm cl}}{\partial \mu} = \frac{\mu T^2}{3} + \frac{\mu^3}{3\pi^2}, \quad s_{\rm cl} = \frac{\partial P_{\rm cl}}{\partial T} = \frac{7\pi^2 T^3}{45} + \frac{\mu^2 T}{3}. \tag{10}$$

- The energy density $\epsilon_{\rm cl}$ satisfies the Euler relation,

$$\epsilon_{\rm cl} = T s_{\rm cl} - P_{\rm cl} + \mu Q_{\rm cl} = 3 P_{\rm cl}. \tag{11}$$

- Goal: Derive similar (local) relations for the quantum case!

Rigidly rotating grand canonical ensemble

- The rigidly rotating state is characterized by [Becattini PRL **108** (2012) 244502]

$$\hat{\rho} = e^{-\beta_0(\widehat{H} - \mu_0 \widehat{Q} - \mathbf{\Omega}_0 \cdot \widehat{\mathbf{J}})}, \tag{12}$$

- The Hamiltonian, charge and angular momentum operators given by

$$\widehat{H} = \int d^3x \, \widehat{\Theta}^{tt}, \quad \widehat{Q} = \int d^3x \, \widehat{J}^t, \quad \widehat{J}^{\mu\nu} = \int d^3x \, \widehat{M}^{t,\mu\nu}. \tag{13}$$

- The charge current $\widehat{J}^{\mu}$, canonical energy-momentum tensor, $\widehat{\Theta}^{\mu\nu}$ and canonical angular momentum tensor $\widehat{M}_C^{\lambda,\mu\nu}$ are given by

$$\begin{gathered}
\widehat{J}_V^{\mu} = \widehat{\overline{\Psi}} \gamma^{\mu} \widehat{\Psi}, \quad \widehat{J}_A^{\mu} = \widehat{\overline{\Psi}} \gamma^{\mu} \gamma^5 \widehat{\Psi}, \quad \widehat{\Theta}^{\mu\nu} = \frac{i}{2} \widehat{\overline{\Psi}} \gamma^{\mu} \overleftrightarrow{\partial}^{\nu} \widehat{\Psi}, \\
\widehat{M}_C^{\lambda,\mu\nu} = \widehat{L}_C^{\lambda,\mu\nu} + \widehat{S}_C^{\lambda,\mu\nu}, \qquad \widehat{L}_C^{\lambda,\mu\nu} = x^{\mu} \widehat{\Theta}^{\lambda\nu} - x^{\nu} \widehat{\Theta}^{\lambda\mu}, \\
\widehat{S}_C^{\lambda,\mu\nu} = \frac{i}{8} \widehat{\overline{\Psi}} \left\{ \gamma^{\lambda}, [\gamma^{\mu}, \gamma^{\nu}] \right\} \widehat{\Psi} = -\frac{1}{2} \varepsilon^{\lambda\mu\nu\alpha} \widehat{J}_{A;\alpha}.
\end{gathered} \tag{14}$$

Mode decomposition

- To facilitate the computation of $\langle \widehat{A} \rangle = Z^{-1}\mathrm{tr}(\hat{\rho}\widehat{A})$, with $Z = \mathrm{tr}(\hat{\rho}) \equiv$ partition function, it is convenient to employ

$$\widehat{\Psi}(x) = \sum_j [\theta(\sigma_j) U_j(x) \hat{a}_j + \theta(-\sigma_j) V_j(x) \hat{b}_j^\dagger], \tag{15}$$

with $U_j \equiv$ particle modes and $V_j(x) = i\gamma^2 U_j^*(x) \equiv$ antiparticle modes.

- The label $j = \{E_j, k_j, m_j, \lambda_j, \sigma_j\}$ signals that

$$\begin{aligned} \sigma_j = 1: \qquad & (H, P^z, J^z, h) U_j = (E_j, k_j, m_j, \lambda_j) U_j, \\ \sigma_j = -1: \qquad & (H, P^z, J^z, h) V_j = (-E_j, -k_j, -m_j, \lambda_j) V_j, \end{aligned} \tag{16}$$

with $H = i\partial_t$, $P^z = -i\partial_z$, $M^z = -i\partial_\varphi + S^z$, $S^z = \frac{i}{2}\gamma^x\gamma^y$ and $h = \mathbf{P} \cdot \mathbf{J}/p$.

- With the above decomposition of $\widehat{\Psi}$, we have

$$\begin{gathered} :\widehat{H} := \sum_j E_j (\hat{a}_j^\dagger \hat{a}_j + \hat{b}_j^\dagger \hat{b}_j), \quad :\widehat{Q} := \sum_j (\hat{a}_j^\dagger \hat{a}_j - \hat{b}_j^\dagger \hat{b}_j), \\ :\widehat{M^z} := \sum_j m_j (\hat{a}_j^\dagger \hat{a}_j + \hat{b}_j^\dagger \hat{b}_j). \end{gathered} \tag{17}$$

Thermal expectation values

- In the cylindrical basis, $\widehat{H}$, $\widehat{Q}$ and $\widehat{J}^z$ are diagonal, i.e.

$$[\widehat{H}, \hat{a}_j^\dagger] = E_j \hat{a}_j^\dagger, \quad [\widehat{Q}, \hat{a}_j^\dagger] = \hat{a}_j^\dagger, \quad [\widehat{J}^z, \hat{a}_j^\dagger] = m_j \hat{a}_j^\dagger. \tag{18}$$

- Using $e^{\widehat{A}}\widehat{B}e^{-\widehat{A}} = \widehat{B} + [\widehat{A}, \widehat{B}] + \frac{1}{2!}[\widehat{A}, [\widehat{A}, \widehat{B}]] + \dots$, it can be shown that

$$\hat{\rho}\hat{a}_j^\dagger \hat{\rho}^{-1} = e^{-\beta_0(E_j - \Omega_0 m_j - \mu_0)} \hat{a}_j^\dagger, \quad \hat{\rho}\hat{b}_j^\dagger \hat{\rho}^{-1} = e^{-\beta_0(E_j - \Omega_0 m_j + \mu_0)} \hat{b}_j^\dagger. \tag{19}$$

- All required expectation values can be expressed w.r.t.

$$\begin{aligned}
\langle \hat{a}_j^\dagger \hat{a}_{j'} \rangle &= Z^{-1} \mathrm{tr}(\hat{\rho}\hat{a}_j^\dagger \hat{\rho}^{-1} \hat{\rho} \hat{a}_{j'}) \\
&= e^{-\beta_0(E_j - \Omega_0 m_j - \mu_0)} Z^{-1} \mathrm{tr}(\hat{\rho}\hat{a}_{j'} \hat{a}_j^\dagger) \\
&= e^{-\beta_0(E_j - \Omega_0 m_j - \mu_0)} [\delta(j, j') - \langle \hat{a}_j^\dagger \hat{a}_{j'} \rangle] \\
&= \delta(j, j') f_j |_{\sigma_j = +1},
\end{aligned} \tag{20}$$

with

$$f_j = \frac{1}{e^{\beta_0(E_j - \Omega_0 m_j - \sigma_j \mu_0)} + 1}. \tag{21}$$

Thermal expectation values

- Since the operators $\widehat{J}^{\mu}_{V}$, $\widehat{J}^{\mu}_{A}$ and $\widehat{\Theta}^{\mu\nu}$ are quadratic in the field operator $\widehat{\Psi}$, it is easy to obtain

$$\langle \widehat{A} \rangle = \frac{1}{8\pi^2} \sum_j f_j \mathcal{A}_j, \quad \sum_j = \overset{\text{(anti/particles)}}{\sum_{\sigma_j = \pm 1}} \; \overset{\text{(helicity)}}{\sum_{\lambda_j = \pm \frac{1}{2}}} \; \overset{\text{(ang.mom.)}}{\sum_{m_j}} \int_M^{\infty} dE_j E_j \int_{-p_j}^{p_j} dk_j, \tag{22}$$

with the sesquilinear form $\mathcal{A}_j$ associated to the operator $\widehat{A}$ computed via the cylindrical modes:

$$U_j \rfloor_{\sigma_j = 1} = \frac{e^{-iE_j t + ik_j z}}{4\pi} \begin{pmatrix} \mathfrak{E}^{+}_j \\ 2\lambda_j \mathfrak{E}^{-}_j \end{pmatrix} \otimes \begin{pmatrix} \mathfrak{p}^{+}_j e^{i(m_j - \frac{1}{2})\varphi} J_{m_j - \frac{1}{2}}(q_j \rho) \\ 2i\lambda_j \mathfrak{p}^{-}_j e^{i(m_j + \frac{1}{2})\varphi} J_{m_j + \frac{1}{2}}(q_j \rho) \end{pmatrix}, \tag{23}$$

where $\mathfrak{E}^{\pm}_j = \sqrt{1 \pm \sigma_j M / E_j}$ and $\mathfrak{p}^{\pm}_j = \sqrt{1 \pm 2\lambda_j k_j / p_j}$.

- Since $V_j = i\gamma^2 U_j$, the antiparticle version of the sesquilinear forms can be trivially obtained:

$$\mathcal{J}^{\mu}_{V;j} \rfloor_{\sigma_j = -1} = -\mathcal{J}^{\mu}_{V;j} \rfloor_{\sigma_j = 1}, \quad \mathcal{J}^{\mu}_{A;j} \rfloor_{\sigma_j = -1} = \mathcal{J}^{\mu}_{A;j} \rfloor_{\sigma_j = 1},$$
$$\Theta^{\mu\nu}_j \rfloor_{\sigma_j = -1} = \Theta^{\mu\nu}_j \rfloor_{\sigma_j = 1}. \tag{24}$$

Thermal expectation values

- For the charge currents, we find

$$\mathcal{J}^t_{V;j} = \sigma_j \left(J^+_j + \frac{2\lambda_j k_j}{p_j} J^-_j \right), \quad \mathcal{J}^\varphi_{V;j} = \frac{\sigma_j q_j}{\rho E_j} J^\times_j, \quad \mathcal{J}^z_{V;j} = \sigma_j \left(\frac{k_j}{E_j} J^+_j + \frac{2\lambda_j p_j}{E_j} J^-_j \right),$$
$$\mathcal{J}^t_{A;j} = \frac{2\lambda_j p_j}{E_j} J^+_j + \frac{k_j}{E_j} J^-_j, \quad \mathcal{J}^\varphi_{A;j} = \frac{2\lambda_j q_j}{p_j} J^\times_j, \quad \mathcal{J}^z_{A;j} = J^-_j + \frac{2\lambda_j k_j}{p_j} J^+_j, \tag{25}$$

where

$$J^\pm_j = J^2_{m_j - \frac{1}{2}}(q_j \rho) \pm J^2_{m_j + \frac{1}{2}}(q_j \rho), \qquad J^\times_j = 2 J_{m_j - \frac{1}{2}}(q_j \rho) J_{m_j + \frac{1}{2}}(q_j \rho). \tag{26}$$

- For the energy-momentum tensor, we have

$$\Theta^{tt}_j = E_j \left(J^+_j + \frac{2\lambda_j k_j}{p_j} J^-_j \right), \quad \Theta^{\varphi t}_j = \frac{q_j}{\rho} J^\times_j, \quad \Theta^{zt}_j = k_j J^+_j + 2\lambda_j p_j J^-_j,$$
$$\Theta^{tz}_j = k_j \left(J^+_j + \frac{2\lambda_j k_j}{p_j} J^-_j \right), \quad \Theta^{\varphi z}_j = \frac{q_j k_j}{\rho E_j} J^\times_j, \quad \Theta^{zz}_j = \frac{k_j^2}{E_j} J^+_j + \frac{2\lambda_j p_j k_j}{E_j} J^-_j,$$
$$\Theta^{\rho\rho}_j = \frac{q_j^2}{E_j} J^+_j - \frac{q_j m_j}{\rho E_j} J^\times_j, \quad \Theta^{\varphi\varphi}_j = \frac{m_j q_j}{\rho^3 E_j} J^\times_j,$$
$$\rho^2 \Theta^{t\varphi} = m_j \left(J^+_j + \frac{2\lambda_j k_j}{p_j} J^-_j \right) - \frac{1}{2} \mathcal{J}^z_{A;j},$$
$$\rho^2 \Theta^{z\varphi} = m_j \left(\frac{k_j}{E_j} J^+_j + \frac{2\lambda_j p_j}{E_j} J^-_j \right) - \frac{1}{2} \mathcal{J}^t_{A;j}. \tag{27}$$

- Noteworthy relations:

$$\frac{\sigma_j}{E_j} \Theta^{\mu t}_j = \mathcal{J}^\mu_{V;j}, \qquad \frac{m_j}{E_j} \Theta^{tt}_j = \rho^2 \Theta^{t\varphi}_j + \frac{1}{2} \mathcal{J}^z_{A;j}. \tag{28}$$

Grand canonical ensemble

Thermodynamics of static systems

- In the case of a static system ($\Omega_0 = 0$), we have

$$\hat{\rho} = e^{-\beta_0(\widehat{H} - \mu_0 \widehat{Q})}, \tag{29}$$

where the operators $\widehat{H}$ and $\widehat{Q}$ are obtained via Noether's theorem, by $\mathbb{R}^3$ integration:

$$\widehat{H} = \int d^3x\, \widehat{\Theta}^{tt}, \quad \widehat{Q} = \int d^3x\, \widehat{J}^z_V. \tag{30}$$

- The partition function, $Z = \mathrm{tr}(\hat{\rho})$, can be used to define the total thermodynamic potential as:

$$\Phi = -T_0 \ln Z = \phi V, \tag{31}$$

where $V = \int d^3x$ is the (infinite) volume of Space.

- The grand potential (density) satisfies:

$$P = -\frac{d\Phi}{dV} = -\phi, \quad Q_V = -\frac{\partial \phi}{\partial \mu_0}, \quad s = -\frac{\partial \phi}{\partial T_0}, \tag{32}$$

while $\epsilon = sT + \phi + \mu_0 Q$.

Rotating system: Grand potential

- In the rotating system, $\hat{\rho}$ is again constructed with the Noether operators:

$$(\widehat{H}, \widehat{Q}, \widehat{M}^z) = \int d^3x (\widehat{\Theta}^{tt}, \widehat{J}^t_V, \mathcal{J}_C^{0,xy}), \tag{33}$$

  where the integral runs over $\mathbb{R}^3$.
- The rotating system is inhomogeneous (things depend on ρ) and ill defined at distances $\rho \geq \Omega^{-1}$, hence $\Phi \neq \phi V$.
- In the rotating case, we introduce the grand canonical potential as

$$\Phi = \int_V d^3x \phi(x) = \mathcal{E} - T_0 \mathcal{S} - \mu_0 \mathcal{Q} - \Omega_0 \mathcal{M}^z, \tag{34}$$

  where we demand

$$\mathcal{S} = -\frac{\partial \Phi}{\partial T_0}, \quad \mathcal{Q} = -\frac{\partial \Phi}{\partial \mu_0}, \quad \mathcal{M}^z = -\frac{\partial \Phi}{\partial \Omega_0}, \tag{35}$$

  thus ensuring the Gibbs-Duhem relation:

$$d\Phi = -\mathcal{S} dT_0 - \mathcal{Q} d\mu_0 - P dV - \boldsymbol{\mathcal{M}} \cdot d\boldsymbol{\Omega}_0. \tag{36}$$

Rotating system: Grand potential

- We can express $\mathcal{E}$ in terms of Φ as

$$\begin{aligned}\mathcal{E} &= \Phi - T_0 \left(\frac{\partial \Phi}{\partial T_0}\right)_{\mu_0,\Omega_0} - \mu_0 \left(\frac{\partial \Phi}{\partial \mu_0}\right)_{T_0,\Omega_0} - \Omega_0 \left(\frac{\partial \Phi}{\partial \Omega_0}\right)_{T_0,\mu_0} \\ &= -T_0^2 \left[\frac{\partial}{\partial T_0} \left(\beta_0 \Phi[T_0, (\beta_0\mu_0)T_0, (\beta_0\Omega_0)T_0]\right)\right]_{\beta_0\mu_0,\beta_0\Omega_0}. \end{aligned} \tag{37}$$

- Then, Φ can be obtained from $\mathcal{E}$ by ensemble integration:

$$\Phi = \frac{1}{\beta_0} \int d\beta_0 (\mathcal{E})_{\beta_0\mu_0,\beta_0\Omega_0}. \tag{38}$$

- The most natural candidate for the total energy is

$$\mathcal{E} = \int d^3x \Theta^{tt}, \tag{39}$$

which gives

$$\phi(x) = \frac{1}{\beta_0} \int d\beta_0 (\Theta^{tt})_{\beta_0\mu_0,\beta_0\Omega_0}. \tag{40}$$

Rotating system: Grand potential

- The β_0 integration involves only the distribution function:

$$\phi(x) = \frac{1}{8\pi^2} \sum_j \Theta_j^{tt} \times \frac{1}{\beta_0 E_j} \int \frac{d(\beta_0 E_j)}{e^{\beta_0 E_j - (\beta_0 \Omega_0) m_j - (\beta_0 \mu_0) \sigma_j} + 1}$$
$$= -\frac{1}{8\pi^2 \beta_0} \sum_j F_j E_j^{-1} \Theta_j^{tt}, \tag{41}$$

where

$$F_j = \ln(1 + e^{-\beta_0 (E_j - \Omega_0 m_j - \sigma_j \mu_0)}). \tag{42}$$

- The expression for $\phi(x)$ can be further processed using integration by parts:

$$\phi(x) = -\frac{1}{8\pi^2 \beta_0} \sum_{\sigma_j, \lambda_j, m_j} \int dq_j q_j \int dk_j F_j \left(J_j^+ \frac{dk_j}{dk_j} + 2\lambda_j J_j^- \frac{dp_j}{dk_j} \right)$$
$$= -\frac{1}{8\pi^2} \sum_j f_j \left(\frac{k_j^2}{E_j} J_j^+ + \frac{2\lambda_j p_j k_j}{E_j} J_j^- \right)$$
$$= -\Theta^{zz}. \tag{43}$$

- The (relativistically non-covariant) thermodynamic pressure is $P = \Theta^{zz}$.

Rotating system: Thermodynamic consistency

- So far, we *made a choice* when constructing $\mathcal{E}$ from Θ^{tt}.
- The charge density can be obtained via

$$-\frac{\partial \phi}{\partial \mu_0} = \frac{1}{8\pi^2 \beta_0} \sum_j E_j^{-1} \Theta_j^{tt} \frac{\partial F_j}{\partial \mu_0}$$
$$= \frac{1}{8\pi^2} \sum_j f_j \frac{\sigma_j}{E_j} \Theta_j^{tt} = J_V^t, \qquad (44)$$

where we used $\partial F_j / \partial \mu_0 = \sigma_j \beta_0 f_j$ and $\Theta^{tt} = \sigma_j E_j J^t_{V;j}$.

- Using $\partial F_j / \partial \Omega_0 = \beta_0 m_j f_j$, the angular momentum density can be obtained as

$$-\frac{\partial \phi}{\partial \Omega_0} = \frac{1}{8\pi^2 \beta_0} \sum_j E_j^{-1} \Theta_j^{tt} \frac{\partial F_j}{\partial \Omega_0}$$
$$= \rho^2 \Theta^{t\varphi} + \frac{1}{2} J_A^z = M_C^{0,xy}, \qquad (45)$$

with $M_C^{0,xy} = x\Theta^{ty} - y\Theta^{tx} - \frac{1}{2}\varepsilon^{0xyz} J_{A;z}$ chosen automatically in the canonical pseudogauge.

Rotating system: Thermodynamic consistency

- The entropy density can be obtained similarly,

$$S = -\frac{\partial \phi}{\partial T_0} = -\frac{\phi}{T_0} + \frac{1}{8\pi^2 T_0} \sum_j f_j E_j^{-1} \Theta_j^{tt} (E_j - \Omega_0 m_j - \sigma_j \mu_0)$$

$$= \frac{1}{T_0} \left(\Theta^{tt} - \phi - \rho^2 \Omega_0 \Theta_j^{t\varphi} - \mu_0 J_V^t - \frac{1}{2} \Omega_0 J_A^z \right). \tag{46}$$

where we used $\partial F_j / \partial T_0 = f_j (E_j - \Omega_0 m_j - \sigma_j \mu_0)/T_0^2$.

- Taking into account the four-velocity $u^\mu \partial_\mu = \Gamma(\partial_t + \Omega_0 \partial_\varphi)$, we see that

$$\Theta^{tt} - \rho^2 \Omega_0 \Theta_j^{t\varphi} = \frac{1}{\Gamma} \Theta^{t\mu} u_\mu. \tag{47}$$

- Writing the last term as $\frac{1}{2} \Omega_0 J_A^z = \Omega_0 S_C^{t,xy}$, and using the vorticity tensor

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu} u^\mu \omega^\nu = \Omega_0 \Gamma (g_{\alpha x} g_{\beta y} - g_{\alpha y} g_{\beta x}), \tag{48}$$

we arrive at

$$S \equiv s^t = \frac{1}{T} \left(\Theta^{t\mu} u_\mu - \tilde{\phi}^t - \mu J_V^t - \frac{1}{2} S_C^{t,\alpha\beta} \omega_{\alpha\beta} \right), \tag{49}$$

with $T = \Gamma T_0$, $\mu = \Gamma \mu_0$ and $\tilde{\phi}^t = \Gamma \phi$.

Thermodynamic potential current

Grand potential current

- The emergence $S \to s^t$ prompts the construction of the function

$$\phi^\mu = \frac{1}{\beta_0} \int d\beta_0 (\Theta^{\mu t})_{\beta_0 \mu_0, \beta_0 \Omega_0} = -\frac{1}{8\pi^2 \beta_0} \sum_j F_j E_j^{-1} \Theta_j^{\mu t}. \tag{50}$$

- Using $\frac{\sigma_j}{E_j} \Theta_j^{\mu t} \mathcal{J}^\mu_{V;j}$ and $\partial F_j / \partial \mu_0 = \sigma_j \beta_0 f_j$, we find

$$-\frac{\partial \phi^\mu}{\partial \mu_0} = \frac{1}{8\pi^2} \sum_j f_j \frac{\sigma_j}{E_j} \Theta_j^{\mu t} = J_V^\mu. \tag{51}$$

- Using $\partial F_j / \partial \Omega_0 = \beta_0 m_j f_j$

$$-\frac{\partial \phi^\mu}{\partial \Omega_0} = \frac{1}{8\pi^2} \sum_j f_j \frac{m_j}{E_j} \Theta_j^{\mu t} = M_C^{\mu, xy}, \tag{52}$$

with $M^{\varphi, xy} = \rho^2 \Theta^{\varphi\varphi}$ and $M^{z,xy} = \rho^2 \Theta^{z\varphi} + \frac{1}{2} J_A^t$.

Grand potential current

- Similarly, the entropy current comes out to be

$$s^\mu = -\frac{\partial \phi^\mu}{\partial T_0} = \frac{1}{T_0}(\Theta^{\mu t} - \phi^\mu - \mu_0 J_V^\mu - \rho^2 \Omega \Theta^{\mu\varphi} + \tfrac{1}{2}\Omega_0 \varepsilon^{\mu x y \nu} J_{A;\nu}). \tag{53}$$

- Using $\Theta^{\mu\nu} u_\nu = \Gamma(\Theta^{\mu t} - \rho^2 \Omega_0 \Theta^{\mu\varphi})$ and $\omega_{\alpha\beta} S_C^{\mu,\alpha\beta} = -\Gamma \Omega_0 \varepsilon^{\mu x y \nu} J_{A;\nu}$, we arrive at

$$s^\mu = \frac{1}{T}\left(\Theta^{\mu\nu} u_\nu - \tilde{\phi}^\mu - \mu J_V^\mu - \frac{1}{2} S_C^{\mu,\alpha\beta} \omega_{\alpha\beta}\right), \tag{54}$$

with $\tilde{\phi}^\mu = \Gamma \phi^\mu$.
- Contracting now with u_μ, we find

$$s = \frac{1}{T}\left(\epsilon + P - \mu Q_V - \frac{1}{2} S_C^{\alpha\beta} \omega_{\alpha\beta}\right), \tag{55}$$

with the usual definitions:

$$\epsilon = u_\mu \Theta^{\mu\nu} u_\nu, \quad Q_V = u_\mu J_V^\mu, \quad S^{\alpha\beta} = u_\mu S^{\mu,\alpha\beta}, \tag{56}$$

while the thermodynamic pressure reads now $P = -\tilde{\phi}^\mu u_\mu \neq \Theta^{zz}$.

From global to local state

- So far, we considered thermodynamic derivatives with respect to the global ensemble parameters: T_0, μ_0 and Ω_0.
- The transition to the local parameters: $T = \Gamma T_0$, $\mu = \Gamma \mu_0$ and $\omega_{xy} = \Gamma \Omega_0$, can be understood at the level of the distribution function:

$$f_j = \left[e^{\beta_0(E_j - \sigma_j \mu_0 - \Omega_0 m_j)} + 1\right]^{-1} = \left[e^{\beta(\Gamma E_j - \sigma_j \mu - \omega_{xy} m_j)} + 1\right]^{-1}. \tag{57}$$

- In particular, we aim to regard $\tilde{\phi}^\mu = \Gamma \phi^\mu \equiv \tilde{\phi}^\mu(T, \mu, \omega_{xy})$.
- The transition $(T_0, \mu_0) \to (T, \mu)$ is straightforward, since:

$$\frac{\partial \tilde{\phi}^\alpha}{\partial T} = \frac{\partial \phi^\alpha}{\partial T_0} = -s^\alpha, \quad \frac{\partial \tilde{\phi}^\alpha}{\partial \mu} = \frac{\partial \phi^\alpha}{\partial \mu_0} = -J_V^\alpha. \tag{58}$$

- However, $\Omega_0 \to \omega_{xy}$ is less clear, since

$$\frac{\partial \phi^\mu}{\partial \Omega_0} = \Gamma^2 \frac{\partial \tilde{\phi}^\mu}{\partial \omega_{xy}} - \rho^2 \Omega_0 \Gamma (\tilde{\phi}^\mu - T s^\mu - \mu J_V^\mu). \tag{59}$$

- A little rearrangement reveals that

$$-\frac{\partial \tilde{\phi}^\mu}{\partial \omega_{xy}} = S_C^{\mu, xy} + \Theta^{\mu\nu} \tilde{\tau}_\nu, \quad \tilde{\tau}^\mu \partial_\mu = -\rho^2 \Omega \partial_t - \partial_\varphi. \tag{60}$$

Local state thermodynamic

Spin potential / vorticity duality

- The derivative w.r.t. vorticity tensor gives the expected spin term, plus an undesirable orbital term.
- **Idea:** In equilibrium, $\Omega_{\alpha\beta} = \omega_{\alpha\beta}$. [Nora, David, Dirk,...]
- $\omega_{\alpha\beta}$ is kinematic in nature, while $\Omega_{\alpha\beta}$ is thermodynamic.
- We wish to disentangle the vortical = kinematic + thermodynamic, s.t.:

$$-\frac{\partial \tilde{\phi}^\mu}{\partial \omega_{xy}} = \Theta^{\mu\nu} \tilde{\tau}_\nu, \quad -\frac{\partial \tilde{\phi}^\mu}{\partial \Omega_{xy}} = S_C^{\mu,xy}. \tag{61}$$

- The kinematic part is comprised of u^μ, a^μ, ω^μ and $\tau^\mu = \varepsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha \omega_\beta$ and

$$\omega_{\alpha\beta} = a_\alpha u_\beta - a_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu} u^\mu \omega^\nu. \tag{62}$$

- Similarly, we introduce

$$\Omega_{\alpha\beta} = \kappa_\alpha u_\beta - \kappa_\beta u_\alpha - \varepsilon_{\alpha\beta\mu\nu} u^\mu \Omega^\nu,$$
$$\kappa^\mu = \Omega^{\mu\nu} u_\nu, \quad \Omega^\mu = -\frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} u_\nu \Omega_{\alpha\beta}. \tag{63}$$

- To implement this programme, we consider massless (free) fermions.

Massless fermions: exact results

- In the case of massless fermions, we have the following beta-frame decomposition:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - P_{\rm eff}\Delta^{\mu\nu} + \pi^{\mu\nu} + \sigma^\tau_\varepsilon(\tau^\mu u^\nu + \tau^\nu u^\mu),$$
$$J^\mu_V = Q_V u^\mu + \sigma^\tau_V \tau^\mu, \quad J^\mu_A = \sigma^\omega_A \omega^\mu, \tag{64}$$

where $T^{\mu\nu} = \Theta^{(\mu\nu)}$ is in the Belinfante pseudogauge, while

$$\pi^{\mu\nu} = -\frac{2}{27\pi^2}\left(\tau^\mu\tau^\nu + \frac{\omega^2}{2}a^\mu a^\nu + \frac{a^2}{2}\omega^\mu\omega^\nu\right), \tag{65}$$

with $\omega^2 = \omega_\mu\omega^\mu = -\Omega^2\Gamma^4$ and $a^2 = a_\mu a^\mu = -\Omega^2\Gamma^2(\Gamma^2-1)$.

- Furthermore,

$$P_{\rm eff} = P_{\rm cl} - \frac{3\omega^2 + a^2}{12}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{46}{45}\omega^2 a^2 - \frac{17}{15}a^4}{192\pi^2},$$
$$\sigma^\tau_\varepsilon = -\frac{1}{3}\sigma^\omega_{A;{\rm cl}} + \frac{39\omega^2 + 31a^2}{360\pi^2}, \quad \sigma^\omega_A = \sigma^\omega_{A;{\rm cl}} - \frac{\omega^2 + 3a^2}{24\pi^2},$$
$$Q_V = Q_{V;{\rm cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2}, \quad \sigma^\tau_V = \frac{\mu}{6\pi^2}, \tag{66}$$

while $\epsilon = 3P_{\rm eff}$ and $P_{\rm eff} = P + \Pi$.

Thermodynamic potential current

- To compute $\phi^\mu = \beta_0^{-1} \int d\beta_0 (\Theta^{\mu t})_{\beta_0\mu_0, \beta_0\Omega_0}$, we require $\Theta^{\mu\nu}$, satisfying

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\partial_\lambda (S_C^{\lambda,\mu\nu} + S_C^{\mu,\nu\lambda} - S_C^{\nu,\lambda\mu}), \tag{67}$$

  which leads to $\Theta^{t\varphi} = T^{t\varphi} + \frac{1}{4\rho}\partial_\rho J_A^z$ and $\Theta^{\varphi t} = T^{\varphi t} - \frac{1}{4\rho}\partial_\rho J_A^z$.
- After performing the integration, we arrive at

$$\begin{aligned}
\tilde{\phi}^t &= -\Gamma \left[P_{\rm cl} - \frac{3\omega^2 + a^2}{12} \sigma^\omega_{A;\rm cl} + \frac{\omega^4 + \frac{122}{15}\omega^2 a^2 - \frac{17}{15} a^4}{192\pi^2} \right], \\
\tilde{\phi}^\varphi &= -\Omega\Gamma \left[P_{\rm cl} - \frac{\omega^2 + 3a^2}{12} \sigma^\omega_{A;\rm cl} + \frac{\omega^4 + 22\omega^2 a^2 + 17 a^4}{960\pi^2} \right].
\end{aligned} \tag{68}$$

Entropy, charge and dynamic pressure

- Since $\tilde{\phi}^\rho = \tilde{\phi}^\varphi = 0$, we can decompose $\tilde{\phi}^\mu$ w.r.t. u^μ and τ^μ:

$$\tilde{\phi}^\mu = -Pu^\mu - \sigma_\phi^\tau \tau^\mu, \tag{69}$$

where

$$\begin{aligned} P &= P_{\rm cl} - \frac{\omega^2 + a^2}{4}\sigma^\omega_{A;{\rm cl}} + \frac{\omega^4 + \frac{134}{15}\omega^2 a^2 + \frac{17}{5}a^4}{192\pi^2}, \\ \sigma_\phi^\tau &= \frac{1}{6}\sigma^\omega_{A;{\rm cl}} - \frac{3\omega^2 + 17a^2}{720\pi^2}. \end{aligned} \tag{70}$$

- The entropy and charge densities read

$$s = \frac{\partial P}{\partial T} = s_{\rm cl} - \frac{T(\omega^2 + a^2)}{12}, \quad Q_V = \frac{\partial P}{\partial \mu} = Q_{V;{\rm cl}} - \frac{\mu(\omega^2 + a^2)}{4\pi^2}. \tag{71}$$

- The dynamic pressure then reads:

$$\Pi = \frac{1}{3}\epsilon - P = \frac{a^2}{6}\sigma^\omega_{A;{\rm cl}} - \frac{a^2(89\omega^2 + 51a^2)}{2160\pi^2}. \tag{72}$$

Spin tensor

- The spin tensor reads
$$S_C^{\alpha\beta} = -\frac{1}{2} u_\mu \varepsilon^{\mu\alpha\beta\nu} J_{A;\nu}. \tag{73}$$
- In equilibrium, $J_A^\mu = \sigma_A^\omega \omega^\mu$ and
$$S_C^{\alpha\beta} = \frac{1}{2}(\omega^{\alpha\beta} + u^\alpha a^\beta - u^\beta a^\alpha)\sigma_A^\omega. \tag{74}$$
- We now assume that the tensor structure of $S^{\alpha\beta}$ is actually given by the spin tensor, instead of the vorticity tensor:
$$S_C^{\alpha\beta} = \frac{1}{2}(\Omega^{\alpha\beta} + u^\alpha \kappa^\beta - u^\beta \kappa^\alpha)\sigma_A^\omega \equiv \frac{\partial P}{\partial \Omega_{\alpha\beta}}. \tag{75}$$
- The only dependence of P on $\Omega_{\alpha\beta}$ can come by replacing $\omega^2 \to \Omega^2$ or $a^2 \to \kappa^2$. Noting that
$$\frac{\partial \Omega^2}{\partial \Omega_{\alpha\beta}} = 2(u^\beta \kappa^\alpha - u^\alpha \kappa^\beta - \Omega^{\alpha\beta}), \quad \frac{\partial \kappa^2}{\partial \Omega_{\alpha\beta}} = 2(u^\beta \kappa^\alpha - u^\alpha \kappa^\beta), \tag{76}$$
it is clear that $\partial\kappa^2/\partial\Omega_{\alpha\beta}$ is incompatible with $S_C^{\alpha\beta}$.

Local pressure

- The solution that reduces to (70) when $\Omega_{\alpha\beta} \to \omega_{\alpha\beta}$ reads

$$P = P_{\rm cl} - \frac{\Omega^2 + a^2}{4} \sigma^\omega_{A;{\rm cl}} + \frac{\Omega^4 + 6\Omega^2 a^2}{192\pi^2} + \frac{44\omega^2 a^2 + 51 a^4}{2880\pi^2}, \qquad (77)$$

which depends on both the spin potential and the vorticity tensor.
- The local quantities read

$$s = s_{\rm cl} - \frac{T(\Omega^2 + a^2)}{12}, \quad Q_V = Q_{V;{\rm cl}} - \frac{\mu(\Omega^2 + a^2)}{4\pi^2},$$
$$S_C^{\alpha\beta} = \frac{1}{2}(\Omega^{\alpha\beta} + u^\alpha \kappa^\beta - u^\beta \kappa^\alpha) \left(\sigma^\omega_{A;{\rm cl}} - \frac{\Omega^2 + 3a^2}{24\pi^2}\right),$$
$$\epsilon = \epsilon_{\rm cl} - \frac{3\Omega^2 + a^2}{4} \sigma^\omega_{A;{\rm cl}} + \frac{\Omega^4 + 2\Omega^2 a^2 - \frac{44}{45}\omega^2 a^2 - \frac{17}{15} a^4}{64\pi^2}. \qquad (78)$$

where $\epsilon = sT - P + \mu Q_V + \frac{1}{2}\omega_{\alpha\beta} S_C^{\alpha\beta}$ was employed.
- Knowing that $P_{\rm eff} = \epsilon/3 = P + \Pi$, we can identify the dynamic pressure as

$$\Pi = \frac{a^2}{6} \sigma^\omega_{A;{\rm cl}} - \frac{\Omega^2 a^2 + \frac{44}{45}\omega^2 a^2 + \frac{17}{5} a^4}{48\pi^2}, \qquad (79)$$

which is compatible with the relation $\Pi = -\frac{1}{6}\omega_{\alpha\beta}(\partial P/\partial \omega_{\alpha\beta})$.

Bonus: Unruh effect

- Consider now a fluid undergoing accelerated motion (no vorticity):

$$\epsilon = \epsilon_{\rm cl} - \frac{a^2}{4}\sigma^\omega_{A;\rm cl} - \frac{17a^4}{960\pi^2}. \tag{80}$$

- The solution $\epsilon = 0$ gives

$$\mathbf{a}^2 = \frac{120\pi^2}{17}\left(\sigma^\omega_{A;\rm cl} + \sqrt{(\sigma^\omega_{A;\rm cl})^2 + \frac{17}{15\pi^2}\epsilon_{\rm cl}}\right). \tag{81}$$

- When $\mu = 0$, we have $\sigma^\omega_{A;\rm cl} = T^2/6$ and $\epsilon_{\rm cl} = 7\pi^2 T^4/60$, such that

$$\mathbf{a}^2 = (2\pi T)^2. \tag{82}$$

- The above relation confirms our thermodynamic relations are compatible with the Unruh effect:

$$T > T_U = \frac{|\mathbf{a}|}{2\pi}. \tag{83}$$

- When $T < T_U$, we have $\epsilon < 0$ and consequently, $P_{\rm eff} = P + \Pi < 0 \Rightarrow$ instability.

Conclusion

Conclusion

- We explored the thermodynamics of a quantum fluid exhibiting vorticity and acceleration.
- In the rigidly rotating ensemble, we constructed the thermodynamic potential within a fictitious cylinder of radius $R < \Omega^{-1}$ (within the causal cylinder).
- We constructed its associated current and obtained a thermodynamically consistent framework.
- Locally, we revealed the requirement of disentangling between kinematic (derived from vorticity tensor) and thermodynamic (derived from spin potential) vortical contributions.
- In the frame of massless fermions, we obtained the thermodynamic pressure of a fluid at finite spin potential and under finite vorticity.

- This work was supported by the European Union–NextGenerationEU through Grant No. 760079/23.05.2023, funded by the Romanian ministry of research, innovation and digitization through Romania's National Recovery and Resilience Plan, Call No. PNRR-III-C9-2022-I8.